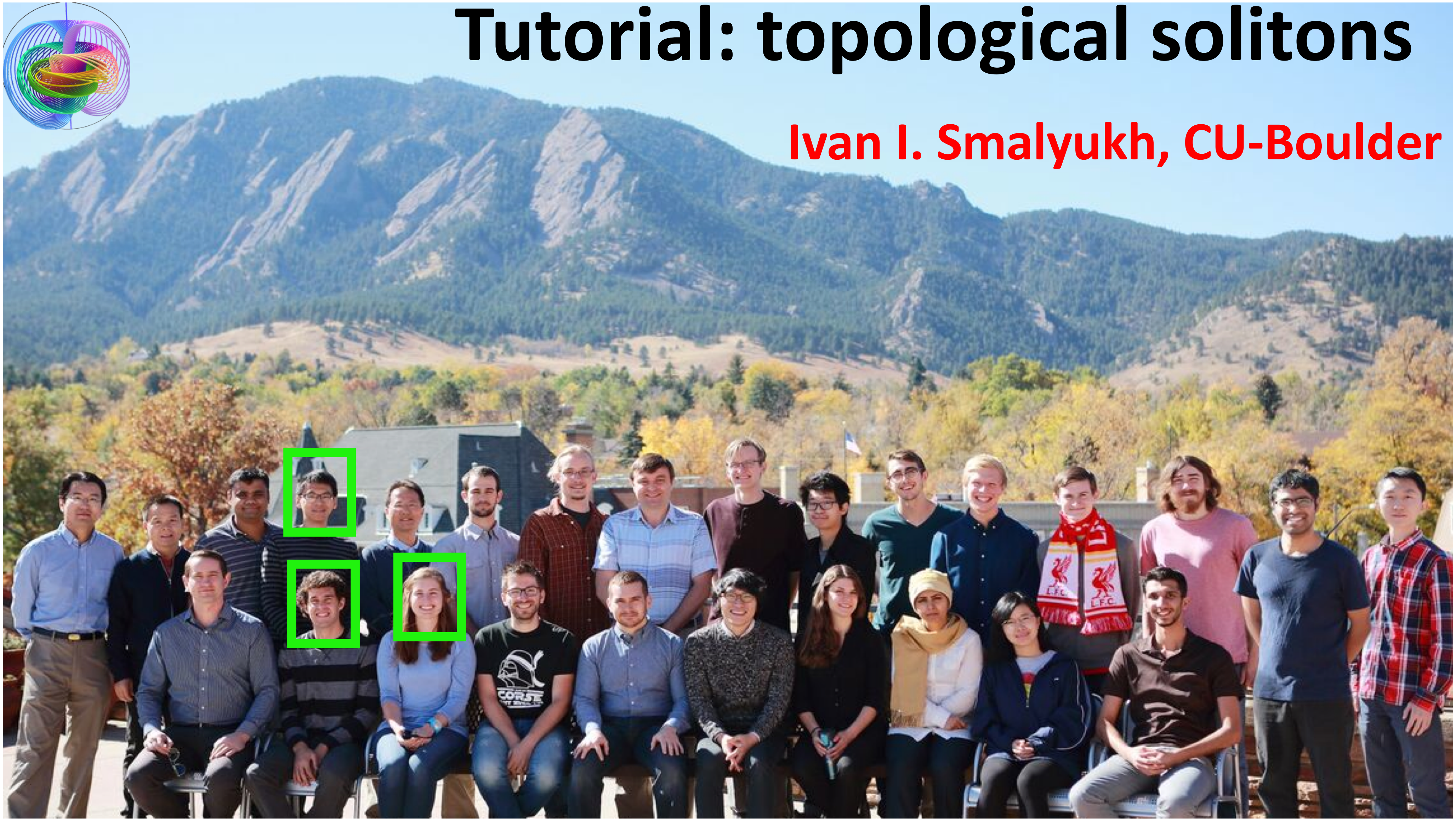
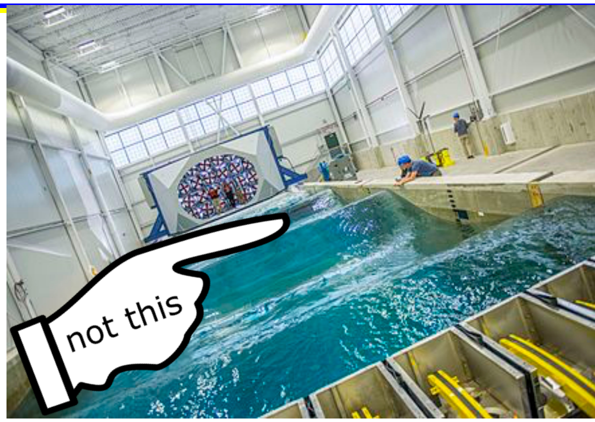
Tutorial: topological solitons

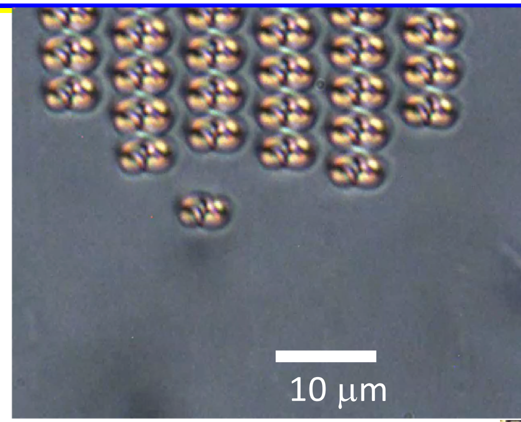
Ivan I. Smalyukh, CU-Boulder



Topological solitons …



U. Of Maine wave basin

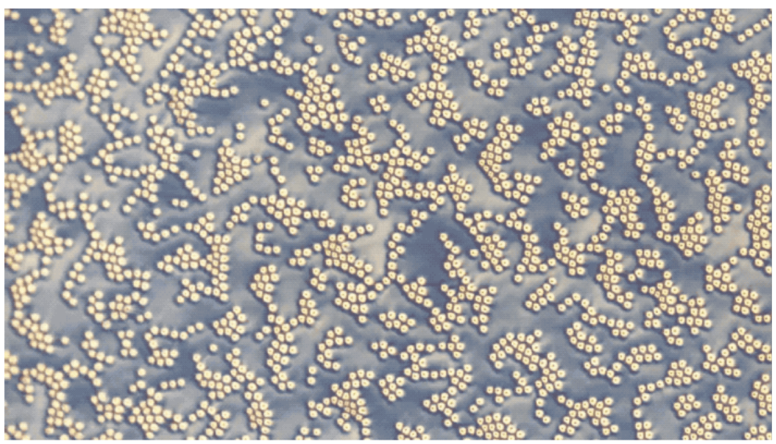


Micro-sized topological structures

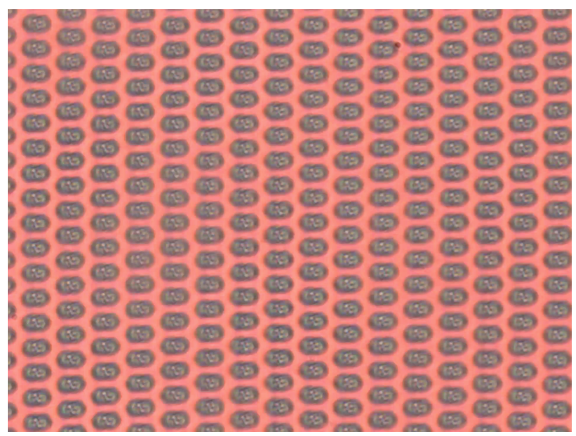
→Different solitons

→Self-assembly into crystals

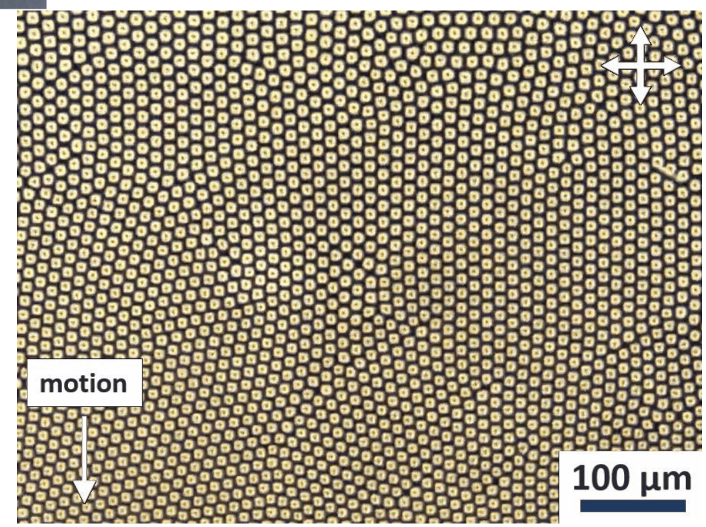
→Out of equilibrium



Sohn, Liu & Smalyukh, Nature Comm 10, 4744 (2019).



Tai & Smalyukh, Science 365, 1449-1453 (2019).



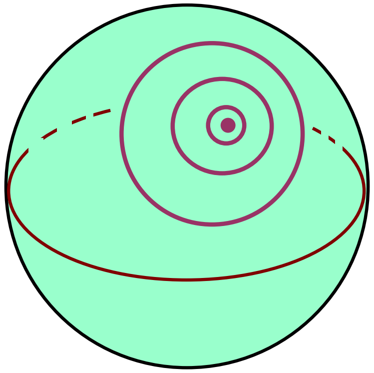
Sohn & Smalyukh, PNAS 117, 6437-6445 (2020).

Topology, Poincare Theorem

J. Henri Poincaré (1854–1912)

Every closed simply connected (no holes) 2D surface is homeomorphic to a sphere

Simply connected surfaces (no holes)



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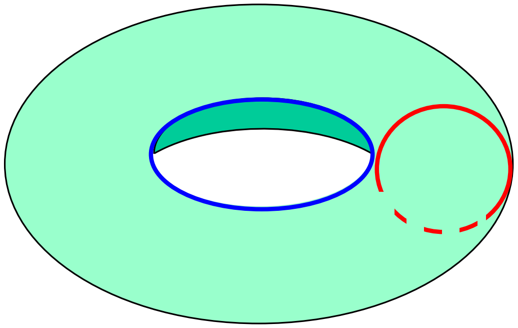
Not simply connected



Topology & topological soliton

- Topology – properties preserved under **continuous** deformations.

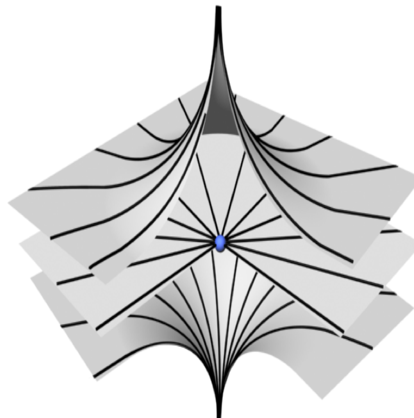
Surfaces

genus 1

≠

genus 0

https://en.wikipedia.org/wiki/Topology



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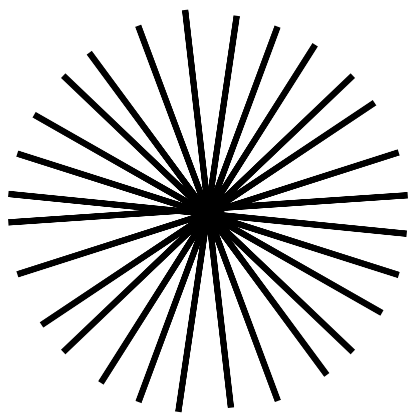
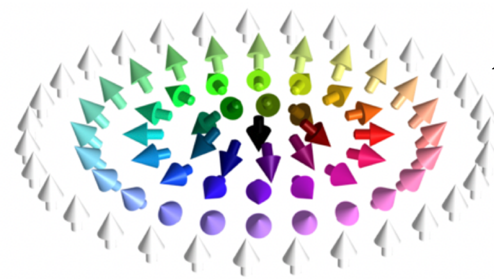


Nonsingular fields



$N_{\mathrm{sk}} = 1$

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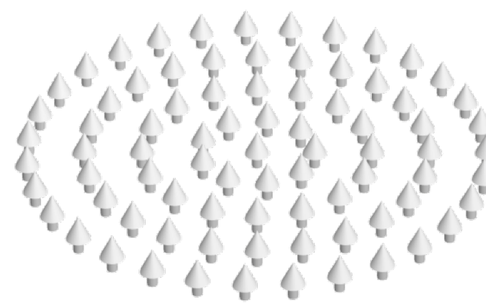


$N_{\mathrm{sk}} = 0$

Sphere-sphere maps are well understood

	π_1	π_2	π_3	π_4	π_5
S^0	0	0	0	0	0
S^1	**Z**	0	0	0	0
S^2	0	**Z**	**Z**	$\mathbf{Z}_2$	$\mathbf{Z}_2$
S^3	0	0	**Z**	$\mathbf{Z}_2$	$\mathbf{Z}_2$
S^4	0	0	0	**Z**	$\mathbf{Z}_2$
S^5	0	0	0	0	**Z**

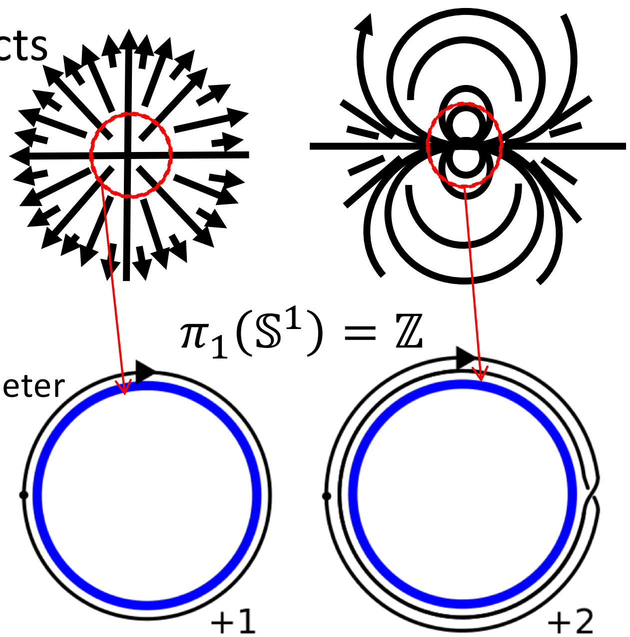
two maps connected by continuous path are said to be homotopic

- Spheres as OP Spaces and defect-surrounding surfaces
- i-th homotopy group $\pi_i(S^n)$ – ways the *i*-dimensional sphere S^i can be mapped into *n*-dimensional sphere S^n

Homotopy theory (sphere-sphere maps) of defects

- Singular defects

2D defects

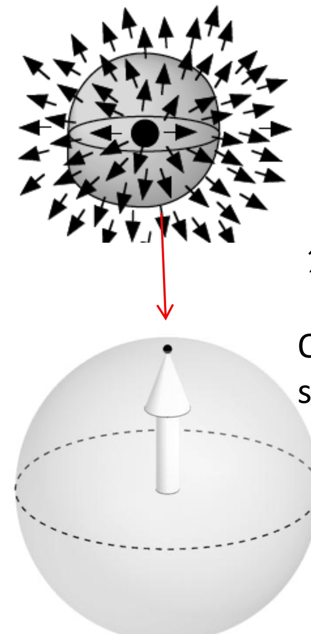


$\pi_1(\mathbb{S}^1) = \mathbb{Z}$

Order-parameter space: $\mathbb{S}^1$

	π_1	π_2	π_3
$\mathbb{S}^0$	0	0	0
$\mathbb{S}^1$	$\mathbb{Z}$	0	0
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{S}^3$	0	0	$\mathbb{Z}$

3D defects



$\pi_2(\mathbb{S}^2) = \mathbb{Z}$

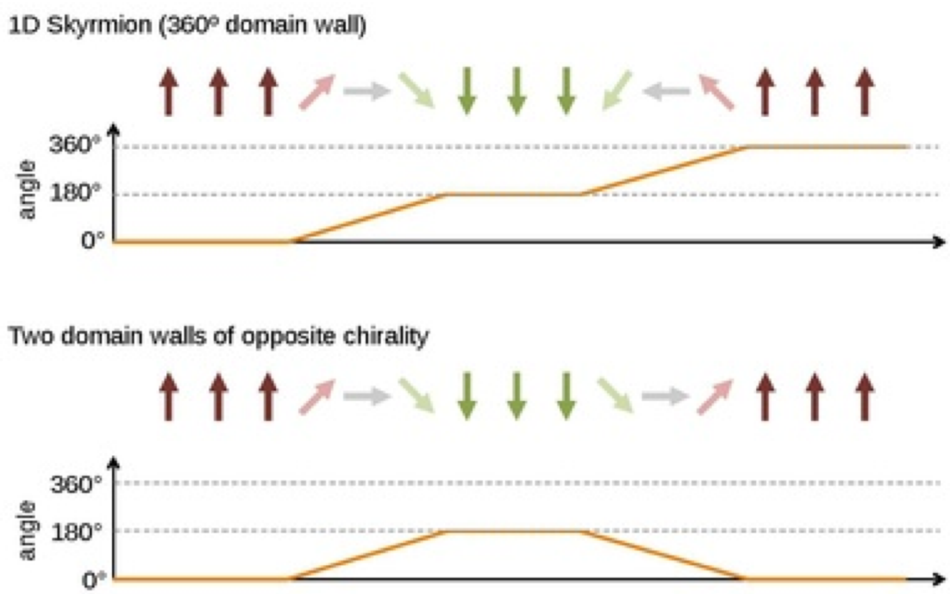
Order-parameter space: $\mathbb{S}^2$

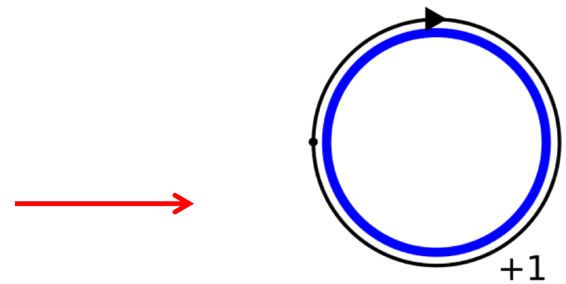
→Spheres as order-parameter Spaces and defect-surrounding surfaces

→i-th homotopy group $\pi_i(S^n)$ – ways the field on S^i can be mapped to S^n

Topological solitons – continuous but topologically nontrivial field configurations

Example – 1D soliton in a ferromagnet



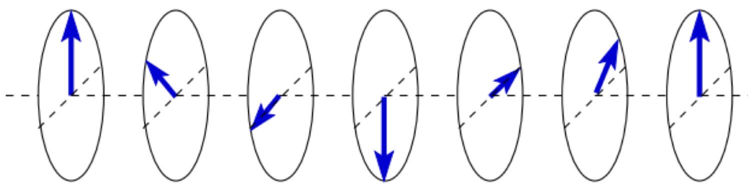


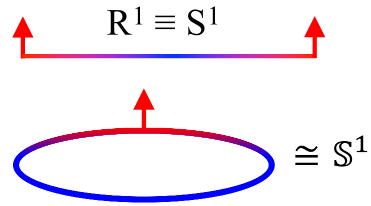
Homotopy maps
$\pi_1(S^1) = Z$

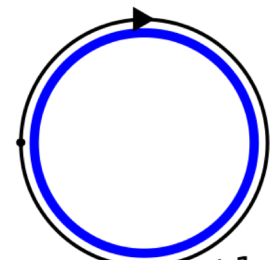
- **Mapping the field along x into S^1 fully covers it!**
- **equivalent to the S^1 surrounding space when the far-field is uniform!**

Directors versus vectors & twist walls as topological 1D solitons

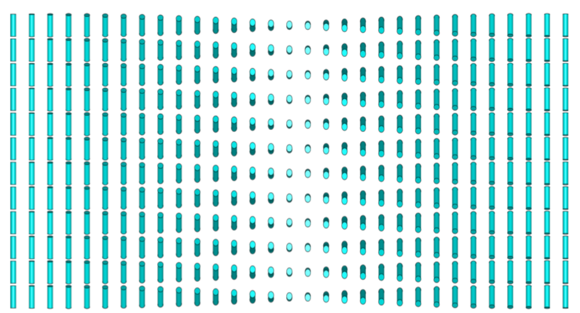
Vector field

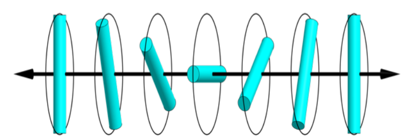


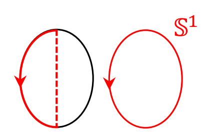




Director field (nonpolar)



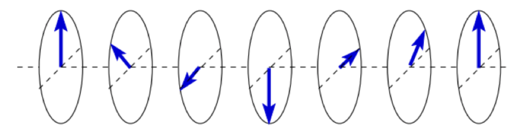




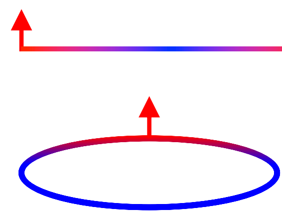
Homotopy theory and topological solitons

$\boldsymbol{m}(\boldsymbol{r})$: $\boldsymbol{r} \in$ configuration space $\longrightarrow \boldsymbol{m} \in$ order-parameter space

- 1D kink or wall ($\pi_1(\mathbb{S}^1) = \mathbb{Z}$)
  - Configuration space $\mathbb{R}^1$
  - $\mathbb{R}^1 \cong \mathbb{S}^1$ if far-field uniform
  - Examples: domain walls in ferromagnets with ***M*** constrained in a plane ($\mathbb{S}^1$)

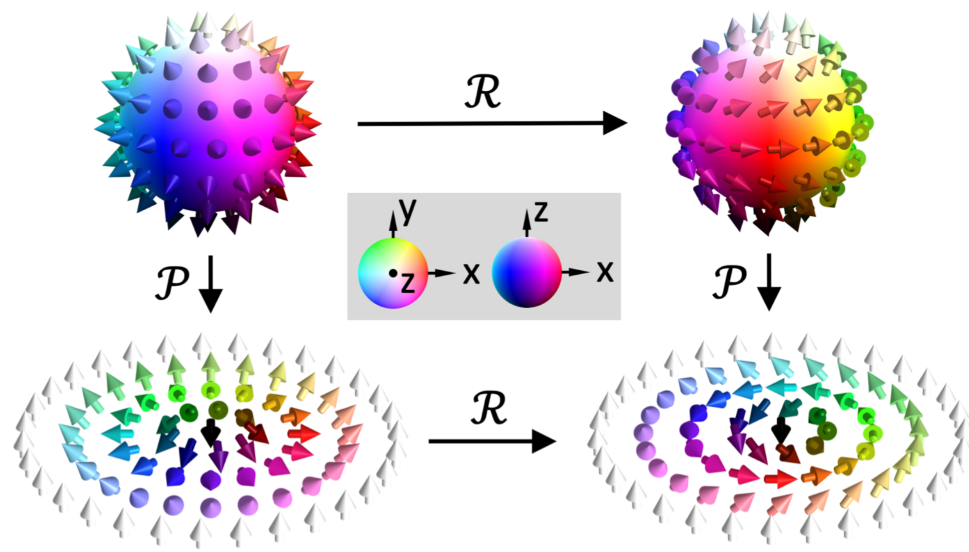


http://iht.univ.kiev.ua/Kolezhuk



- 2D skyrmion ($\pi_2(\mathbb{S}^2) = \mathbb{Z}$)
  - $\mathbb{R}^2 \cong \mathbb{S}^2$ when the far-field is uniform
  - 2D soliton in a ferromagnet with ***M*** taking all possible orientations on $\mathbb{S}^2$

$$N_{\rm sk} = \frac{1}{4\pi}\int dxdy\,\boldsymbol{m}(\boldsymbol{r})\cdot\left(\partial_x \boldsymbol{m}(\boldsymbol{r})\times\partial_y \boldsymbol{m}(\boldsymbol{r})\right)$$

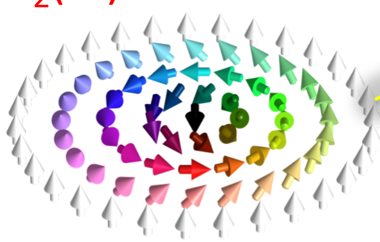


Sphere-sphere maps, homotopy theory

Growing up ("teenager"?) skyrmions?
→$\pi_3(S^2)$ = Z & $\pi_3(S^2/Z_2)$ = Z (hopfions)
→$\pi_3(S^3/Z_2)$= Z solitons

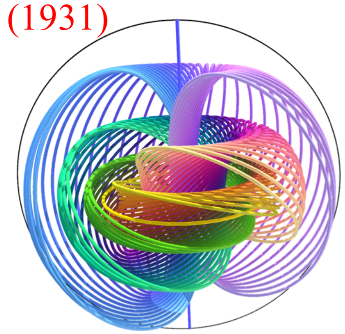
1D wall ("infant"?) solitons $\pi_1(S^1)$ = Z

2D "baby" skyrmions $\pi_2(S^2)$ = Z

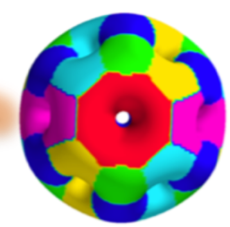


	π_1	π_2	π_3	π_4	π_5
$\mathbb{S}^0$	0	0	0	0	0
$\mathbb{S}^1$	$\mathbb{Z}$	0	0	0	0
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$\mathbb{S}^5$	0	0	0	0	0

Hopf fibration (1931)



Skyrme solitons in high energy physics $\pi_3(S^3)$=Z

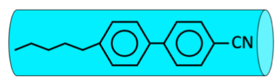


$\mathbb{R}^1 \cong \mathbb{S}^1$, $\mathbb{R}^2 \cong \mathbb{S}^2$ & $\mathbb{R}^3 \cong \mathbb{S}^3$ when the far-field is uniform

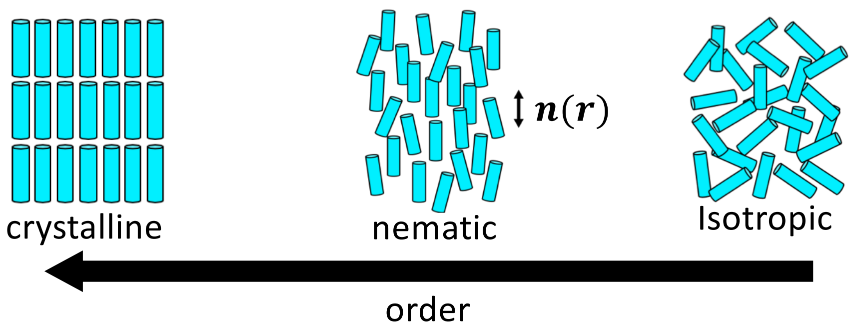
→Derrick-Hobart (theorem): 2D, 3D solitons cannot be stable (within a simplest model)
→Skyrme, high energy physicis: stabilization by adding nonlinear terms (nonlinear sigma model)

Liquid crystals (LCs)

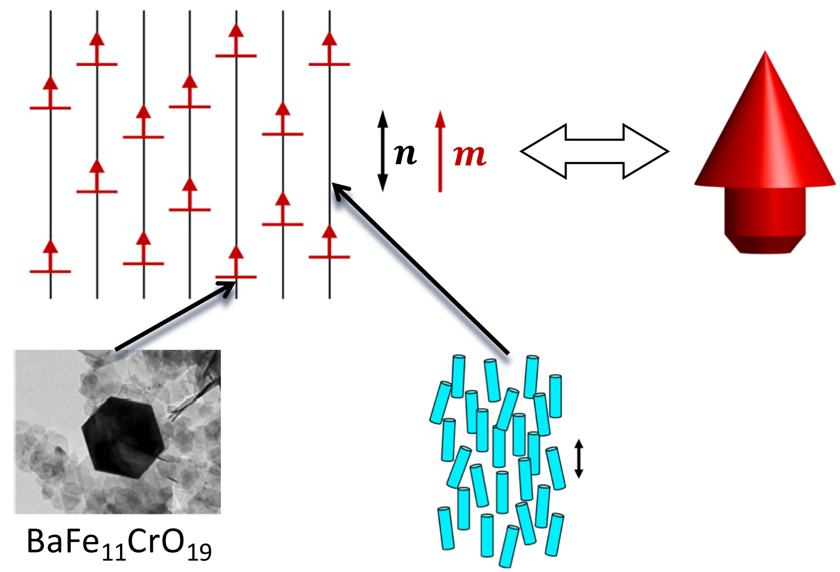
- Ordered fluid with long-range orientational order



- Director $\boldsymbol{n}(\boldsymbol{r})$



- LC ferromagnets



- Vector $\boldsymbol{m}(\boldsymbol{r})$
- Polar switching by weak fields

Free energy minimization

→Overcoming constrains of Derrick-Hobart (theorem): chiral terms

- Frank-Oseen free energy of chiral ferromagnetic LCs

$$F_{\mathrm{LCF}} = \int_{\Omega} d^3\boldsymbol{r} \left\{ \frac{K_{11}}{2}(\nabla\cdot\boldsymbol{m})^2 + \frac{K_{22}}{2}\left[\boldsymbol{m}\cdot(\nabla\times\boldsymbol{m}) + \frac{2\pi}{p}\right]^2 + \frac{K_{33}}{2}[\boldsymbol{m}\times(\nabla\times\boldsymbol{m})]^2 \right\} - \frac{W}{2}\int_{\partial\Omega} d^2\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{m}_0)^2 .$$

splay — twist — bend — surface anchoring

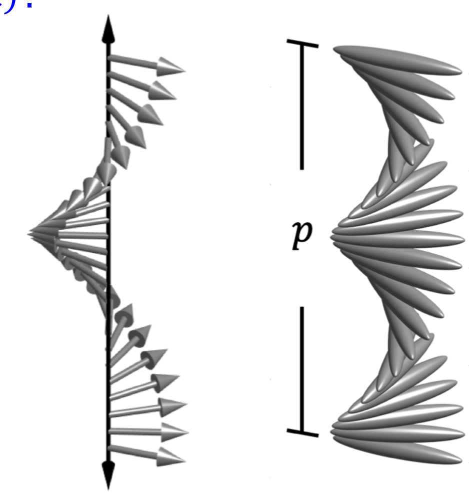
$$F_{\mathrm{dielectric}} = -\frac{\varepsilon_0\Delta\varepsilon}{2}\int_{\Omega} d^3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{E})^2 .$$

$$F_{\mathrm{magnetic}} = -\mu_0 M \int_{\Omega} d^3\boldsymbol{r}(\boldsymbol{H}\cdot\boldsymbol{m}) .$$

- Hamiltonian for chiral ferromagnets

$$H = \int d^3\boldsymbol{r}\left\{\frac{J}{2}(\nabla\boldsymbol{m})^2 + D\boldsymbol{m}\cdot(\nabla\times\boldsymbol{m}) - \mu_0 M_s(\boldsymbol{H}\cdot\boldsymbol{m})\right\}$$

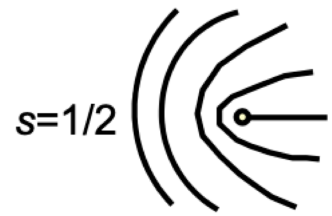
- Derrick's theorem: 2D, 3D solitons cannot be stable
- Stabilization by
  - Chirality
  - Boundary effects
  - Elastic anisotropy
  - External fields



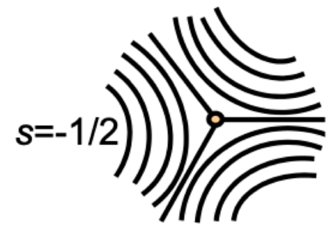
Landau-de Gennes free energy

- Tensor order-parameter

$$Q_{ij} = \frac{S}{2}\left(3n_i n_j - \delta_{ij}\right) + \frac{P}{2}\left(e_i^{(1)} e_j^{(1)} - e_i^{(2)} e_j^{(2)}\right)$$



- Defect and disorder requires Q-tensor description of nematics.

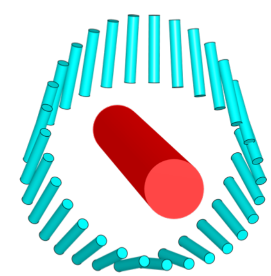


$$F = F_t + F_e + F_s$$

$$= \int_{\Omega} d^3\boldsymbol{r}\, \frac{1}{2} a(T - T^*)\mathrm{Tr}(Q^2) + \frac{1}{3} B\,\mathrm{Tr}(Q^3) + \frac{1}{4} C\left[\mathrm{Tr}(Q^2)\right]^2$$

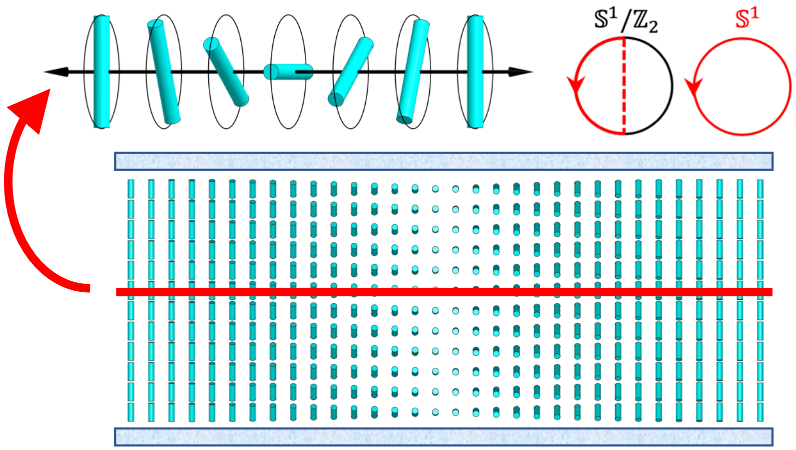
$$+ \int_{\Omega} d^3\boldsymbol{r}\, \frac{L}{2} \frac{\partial Q_{ij}}{\partial x_k} \frac{\partial Q_{ij}}{\partial x_k} + 2q_0 L \varepsilon_{ikl} Q_{ij} \frac{\partial Q_{ij}}{\partial x_k}$$

$$+ \int_{\partial\Omega} d^2 r\, \frac{W}{2}\left(Q_{ij} - Q_{ij}^0\right)^2$$

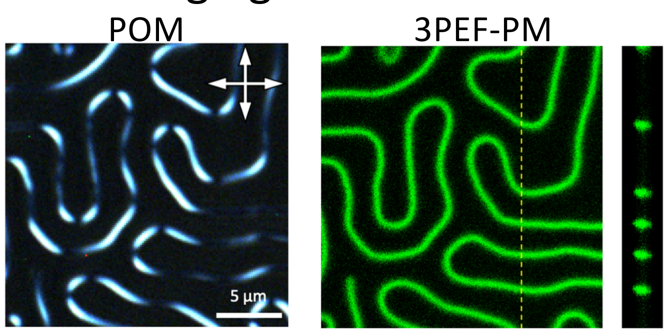


1D walls ($\pi_1(\mathbb{S}^1) = \mathbb{Z}$) in LCs

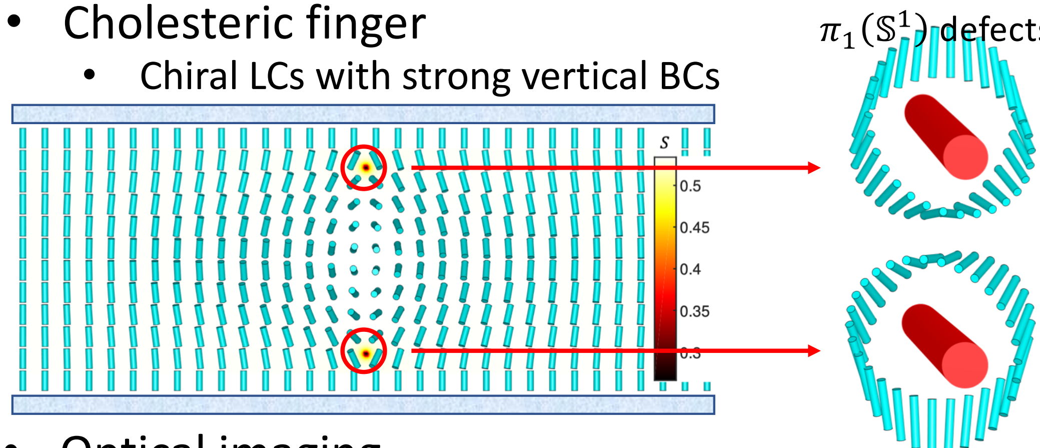
- 1D twisted walls
  - Chiral LCs with vertical BCs



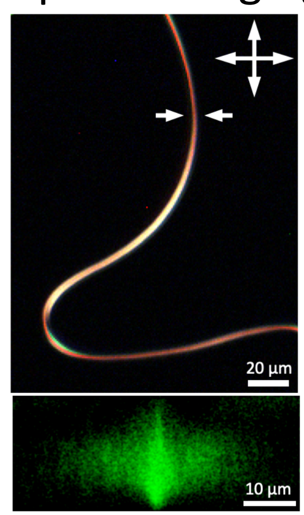
- Optical imaging

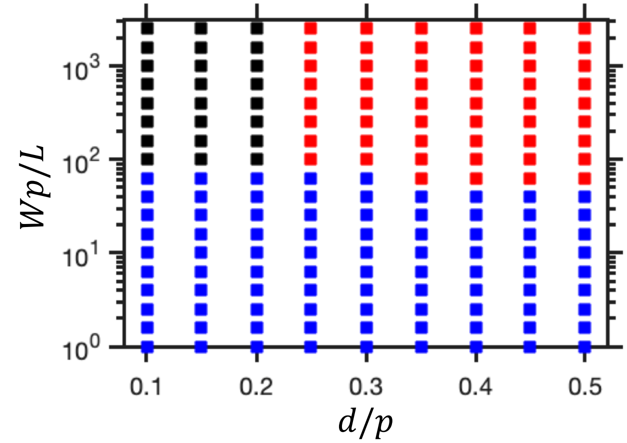


- Cholesteric finger
  - Chiral LCs with strong vertical BCs



- Optical imaging

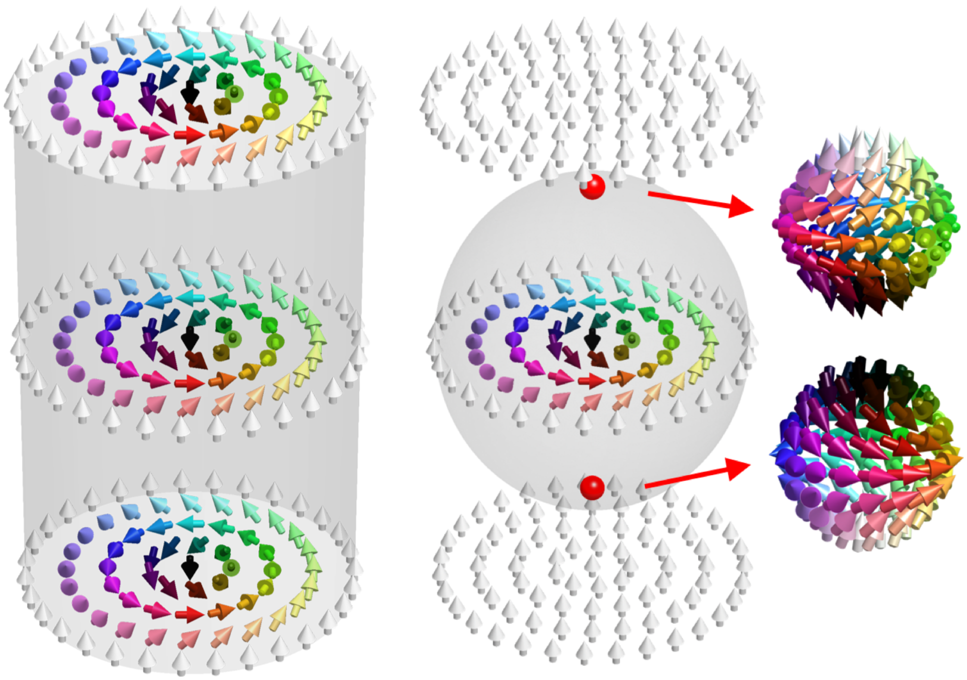




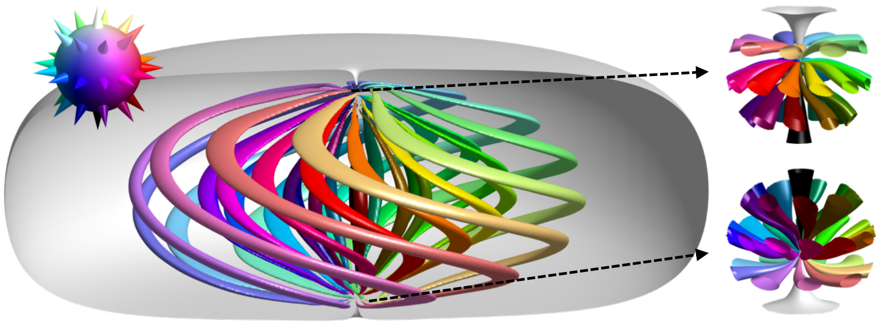
JSB Tai, II Smalyukh, accepted by *Phys. Rev. E*

2D skyrmions ($\pi_2(\mathbb{S}^2) = \mathbb{Z}$) in LCs

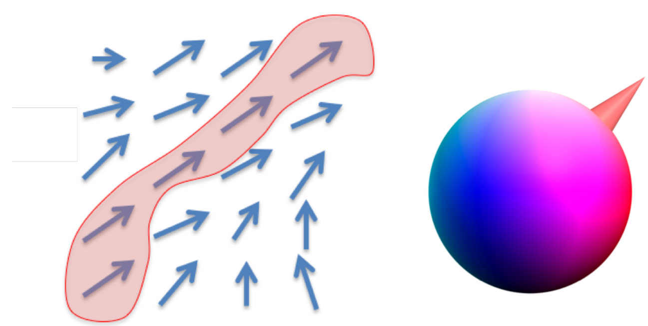
- 2D skyrmion & Toron



- Preimages

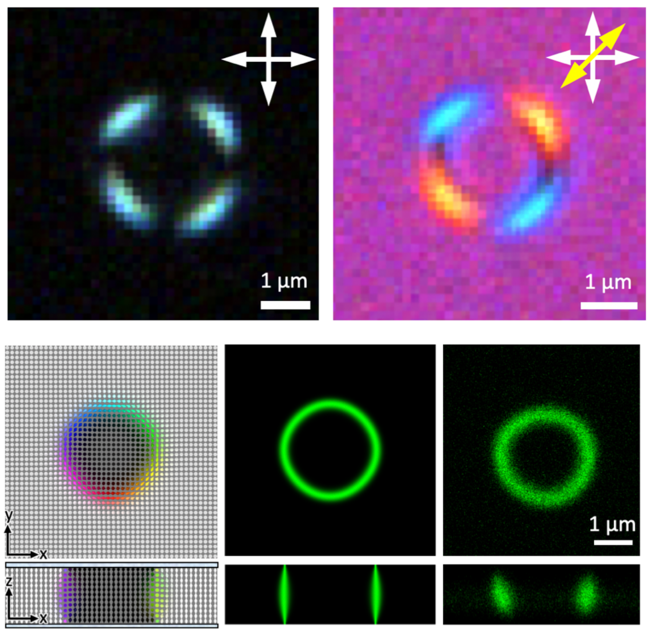


Preimage (inverse image) of $\boldsymbol{m}(\boldsymbol{r})$: $\boldsymbol{r} \longrightarrow \boldsymbol{m}$ map
Region of a constant $\boldsymbol{m}$

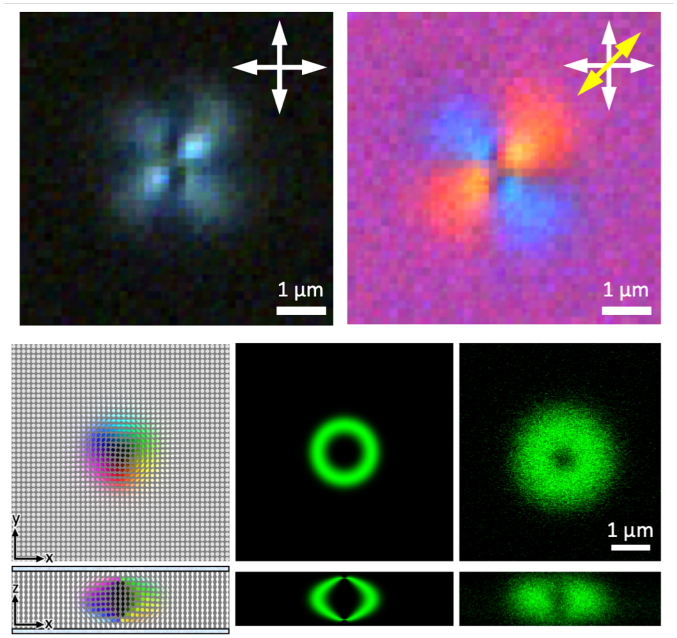


2D skyrmions in LCs (cont.)

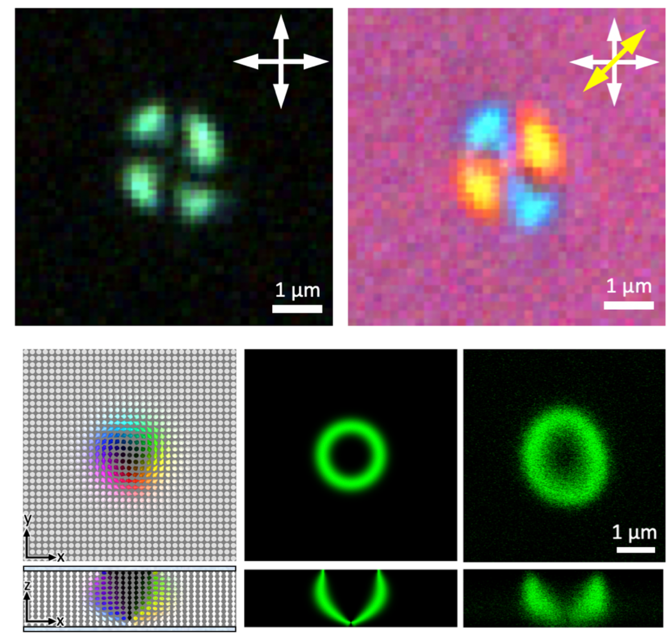
- 2D skyrmion



- Elementary toron



- Skyrmion/toron hybrid



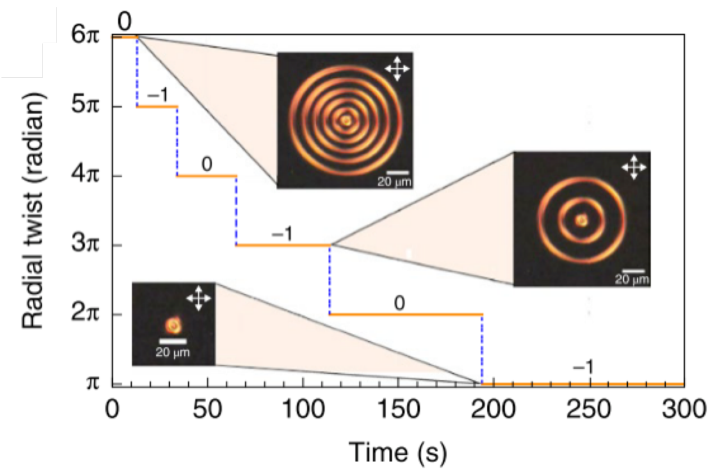
JSB Tai, II Smalyukh, *Phys. Rev. E.* (2020)

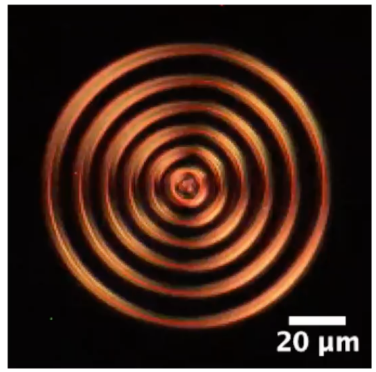
High-charge skyrmions in LCs

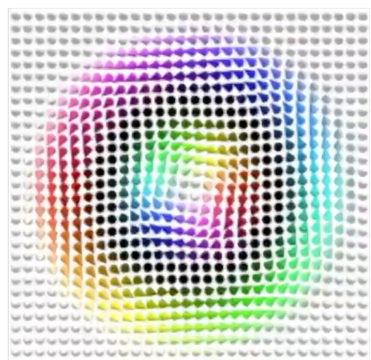
- Can we have $|N_{sk}| > 1$?

1. Multiple π-twist

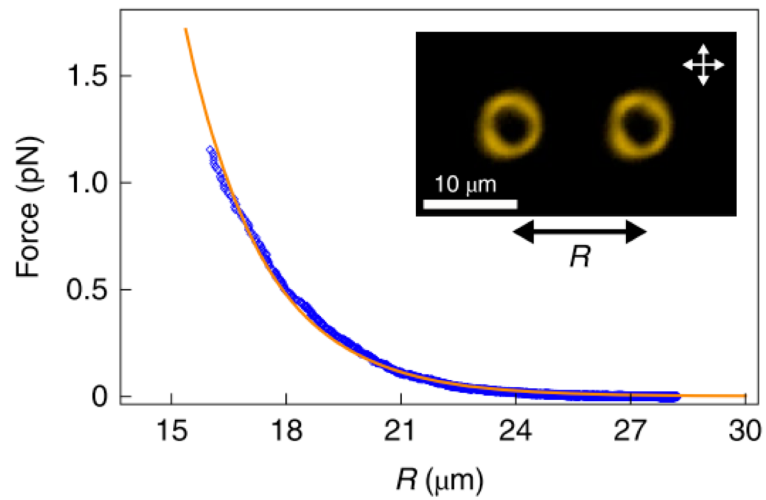
$$N_{sk} = \frac{1}{4\pi}\int dxdy\boldsymbol{m}\cdot(\partial_x\boldsymbol{m}\times\partial_y\boldsymbol{m})$$





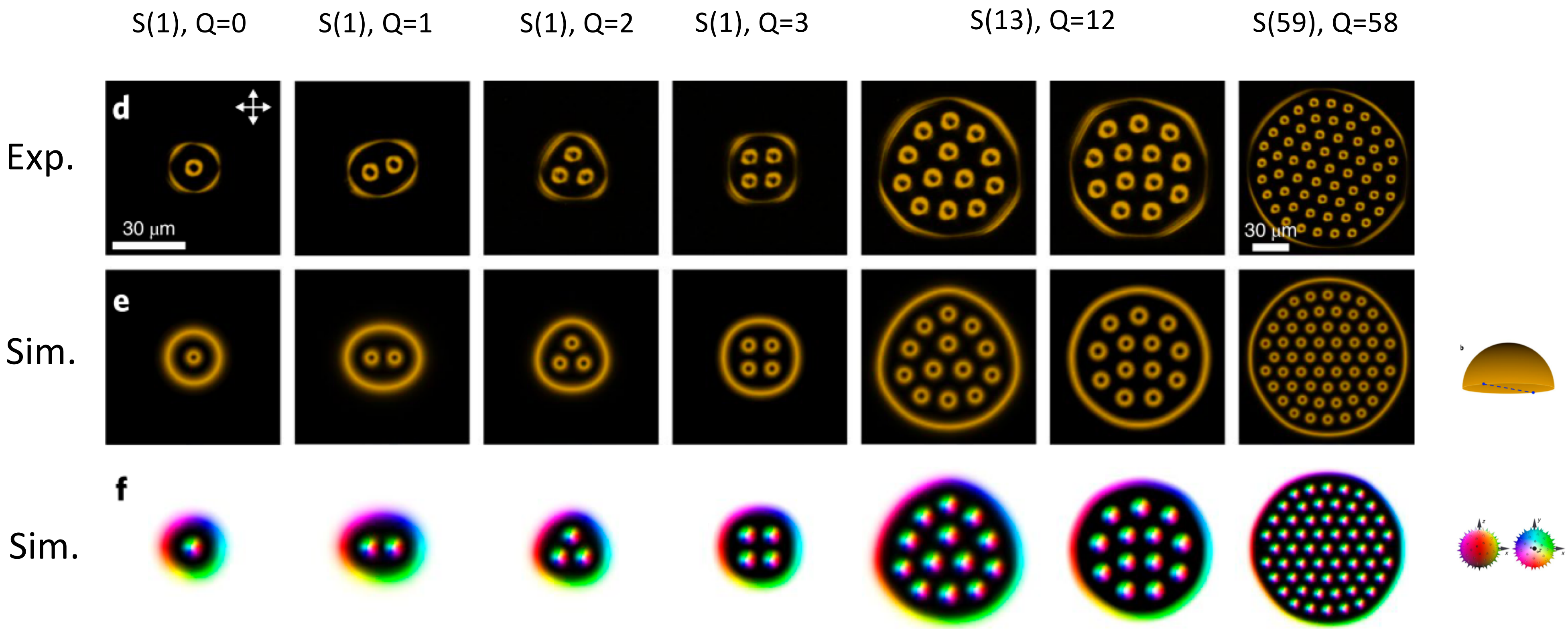


2. Clusters of skyrmions
   skyrmions repel



D Foster, C Kind, PJ Ackerman, JSB Tai, MR Dennis, II Smalyukh, *Nature Physics* **15**, 655 (2019).

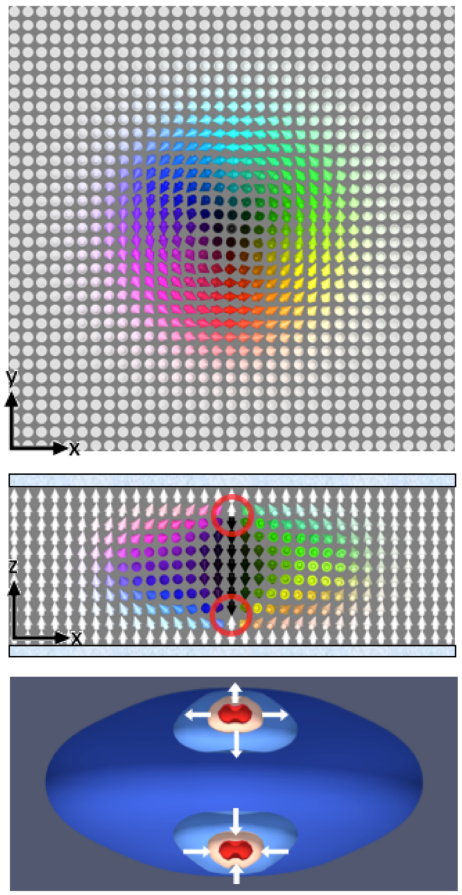
High-chrage “skyrmion bags” in LCs



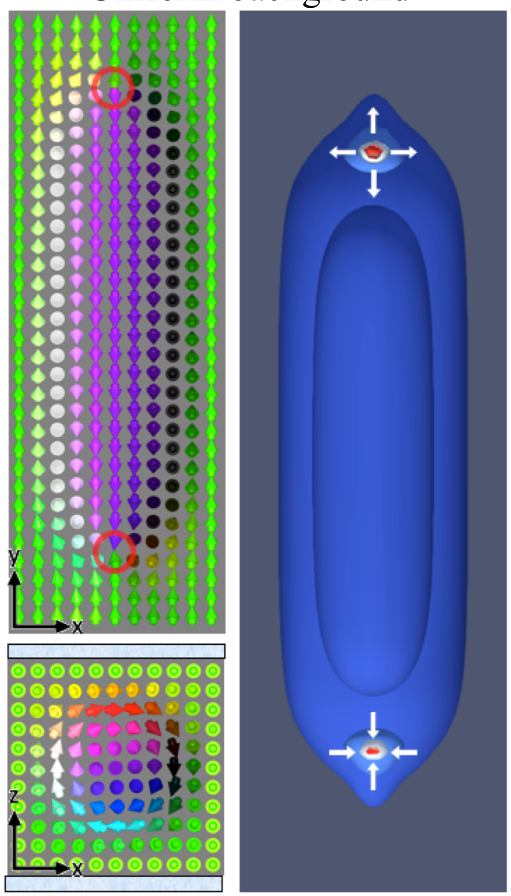
D Foster, C Kind, PJ Ackerman, JSB Tai, MR Dennis, II Smalyukh, *Nature Physics* **15**, 655 (2019).

Field-controlled dynamics of skyrmions & monopoles (cont.)

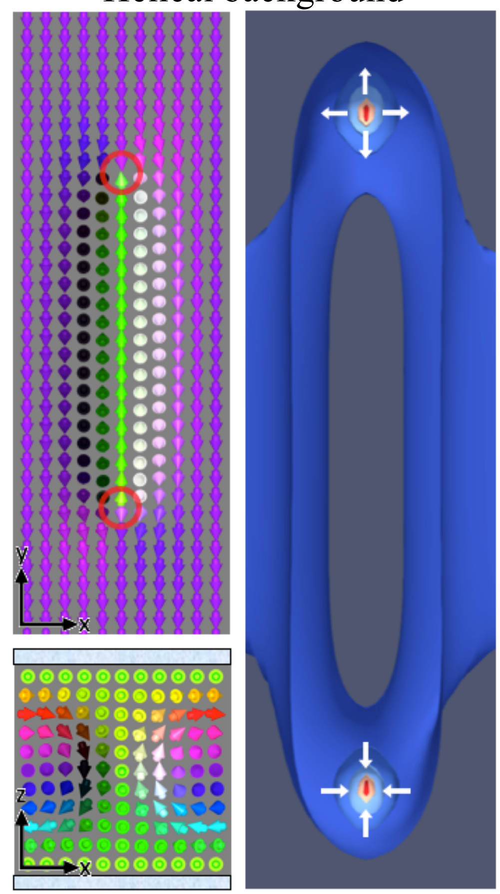
Homeotropic anchoring



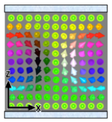
Planer anchoring
Uniform background



Planar anchoring
Helical background

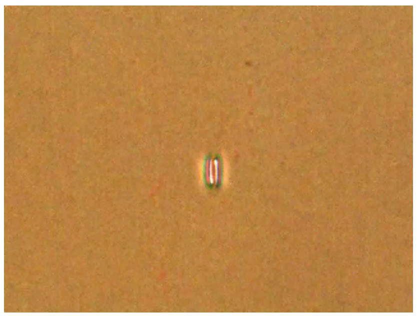


Field-controlled dynamics of skyrmions & monopoles (cont.)

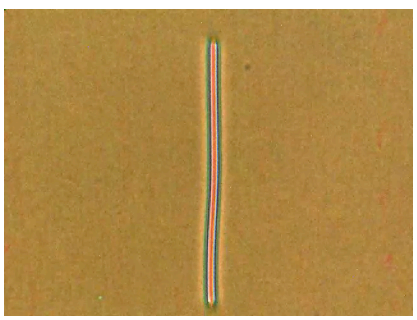


Dynamics by ***E*** fields.

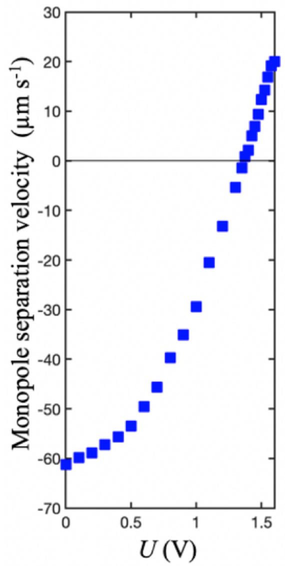
U = 1.5 V

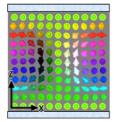


U = 0 V



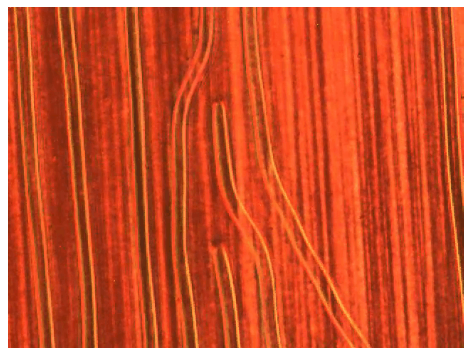
$$F_{\mathrm{electric}} = -\frac{\epsilon_0 \Delta\epsilon}{2} \int d^3\boldsymbol{r} (\boldsymbol{n} \cdot \boldsymbol{E})^2$$

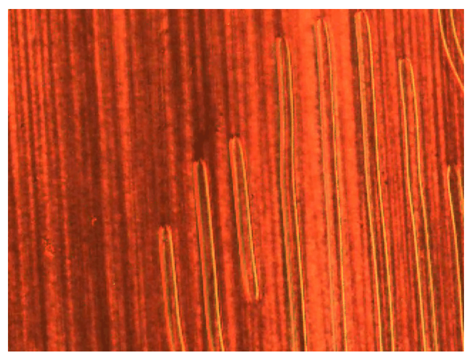


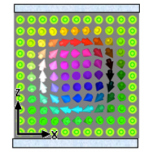


Dynamics by ***H*** fields.

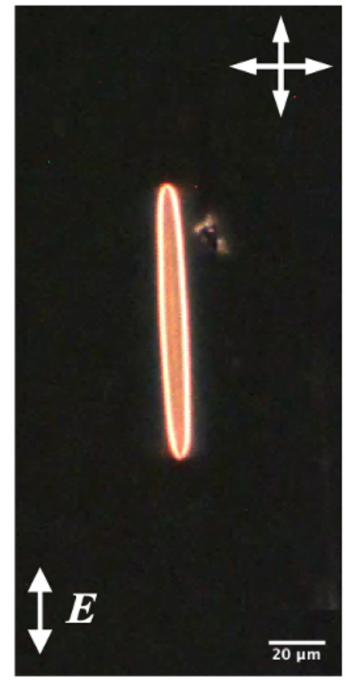
$$F_{\mathrm{magnetic}} = -\mu_0 \int d^3\boldsymbol{r}\, (\boldsymbol{H} \cdot \boldsymbol{M})$$







Dynamics by in-plane ***E*** fields.

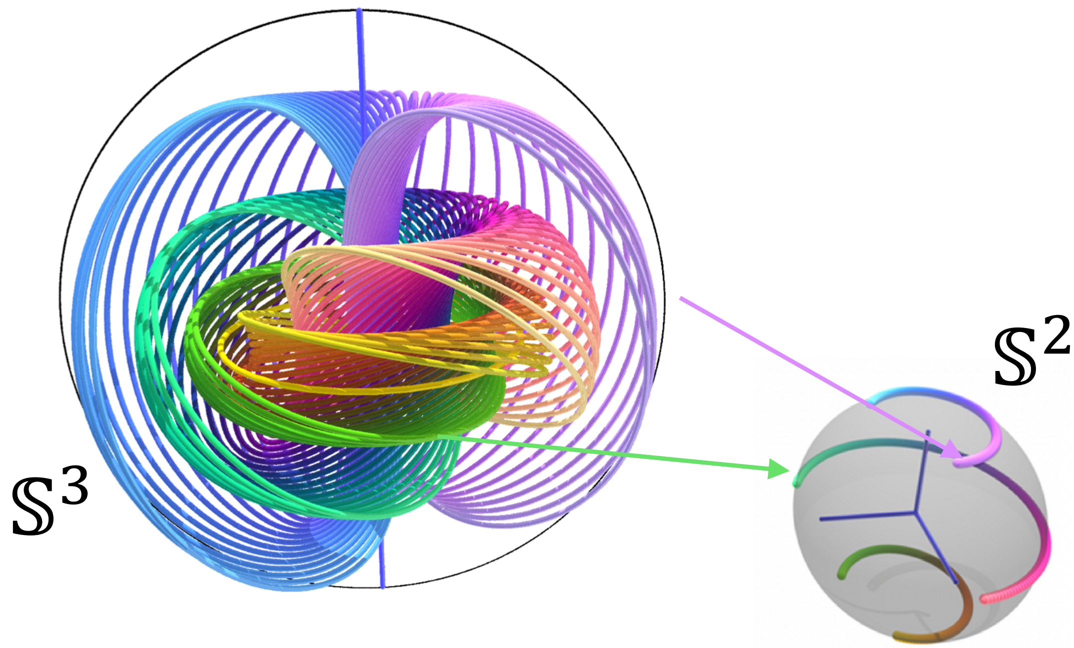


3D Hopf solitons

??

	π_1	π_2	π_3	π_4	π_5
$\mathbb{S}^0$	0	0	0	0	0
$\mathbb{S}^1$	$\mathbb{Z}$	0	0	0	0
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

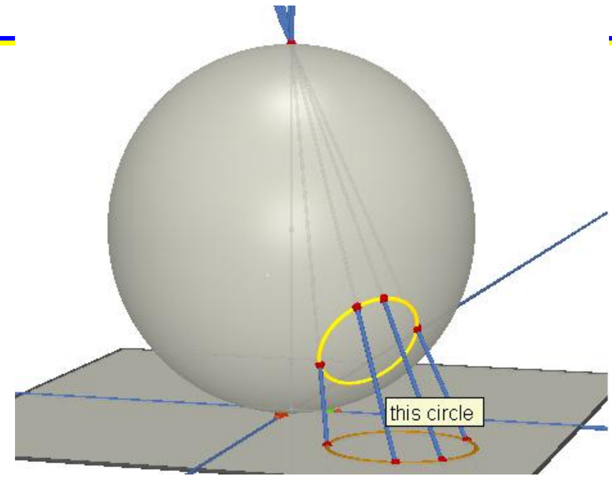
- Hopf fibration (Heinz Hopf 1931)
  - A map $\mathbb{S}^3 \to \mathbb{S}^2$
  - $\mathbb{R}^3 \cong \mathbb{S}^3$ when far-field uniform
  - A circle in $\mathbb{S}^3 \to$ A point on $\mathbb{S}^2$
  - Circles are linked (topology)



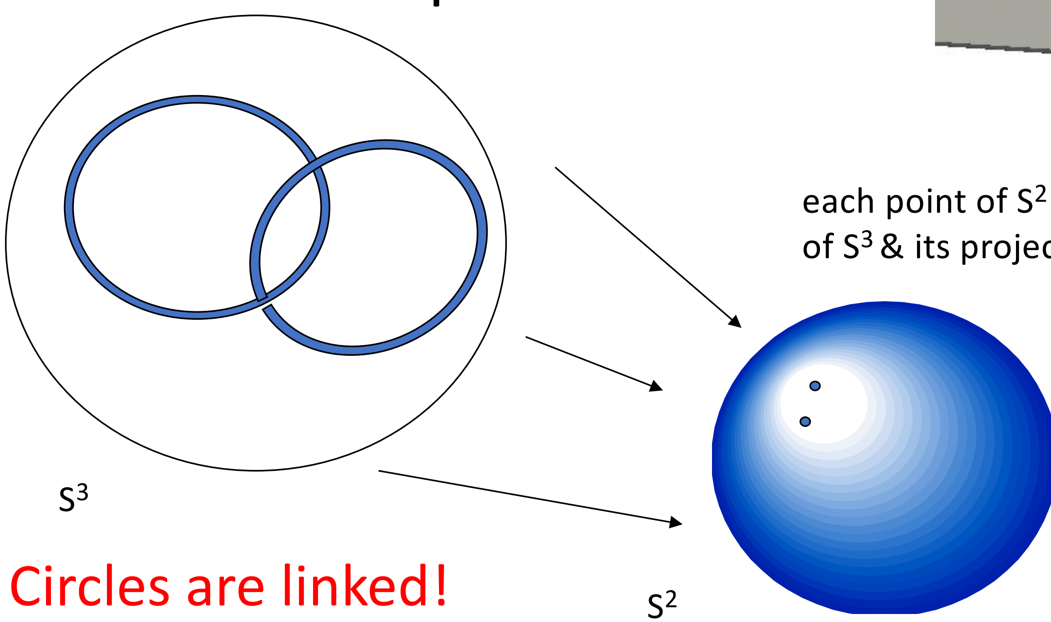
Stereographic projection

- Stereographic Projection of S^2 onto a 2D plane

- Stereographic Projection of S^3 into the 3D space



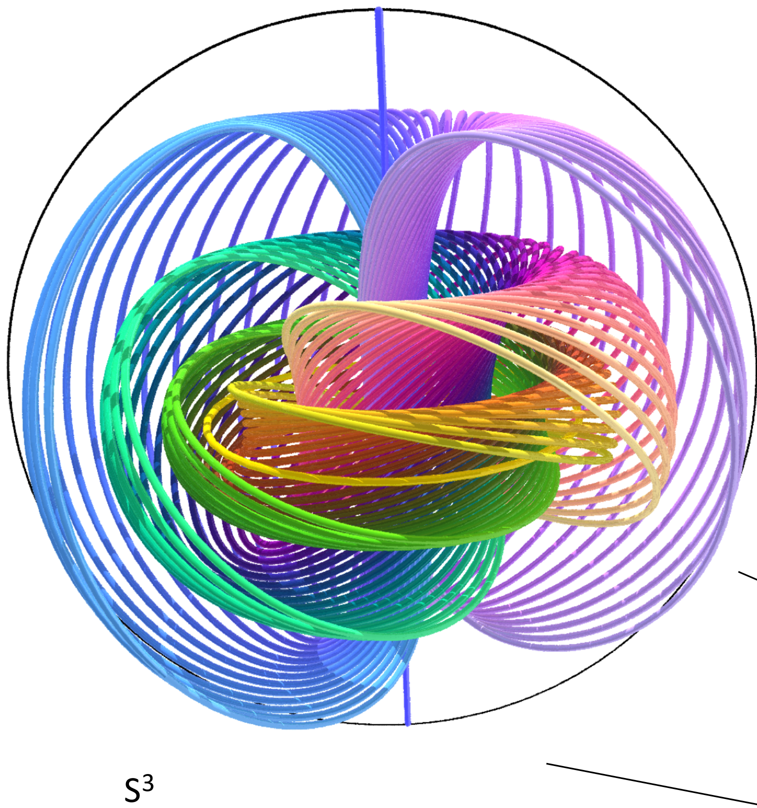
each point of S^2 corresponds to a distinct circle of S^3 & its projection

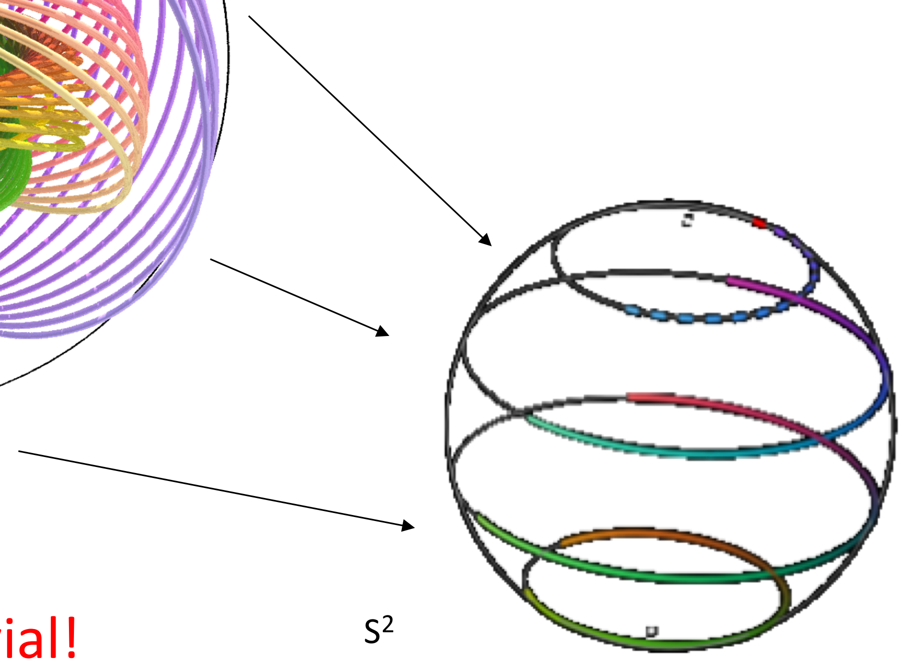


Circles are linked!

Hopf fibration

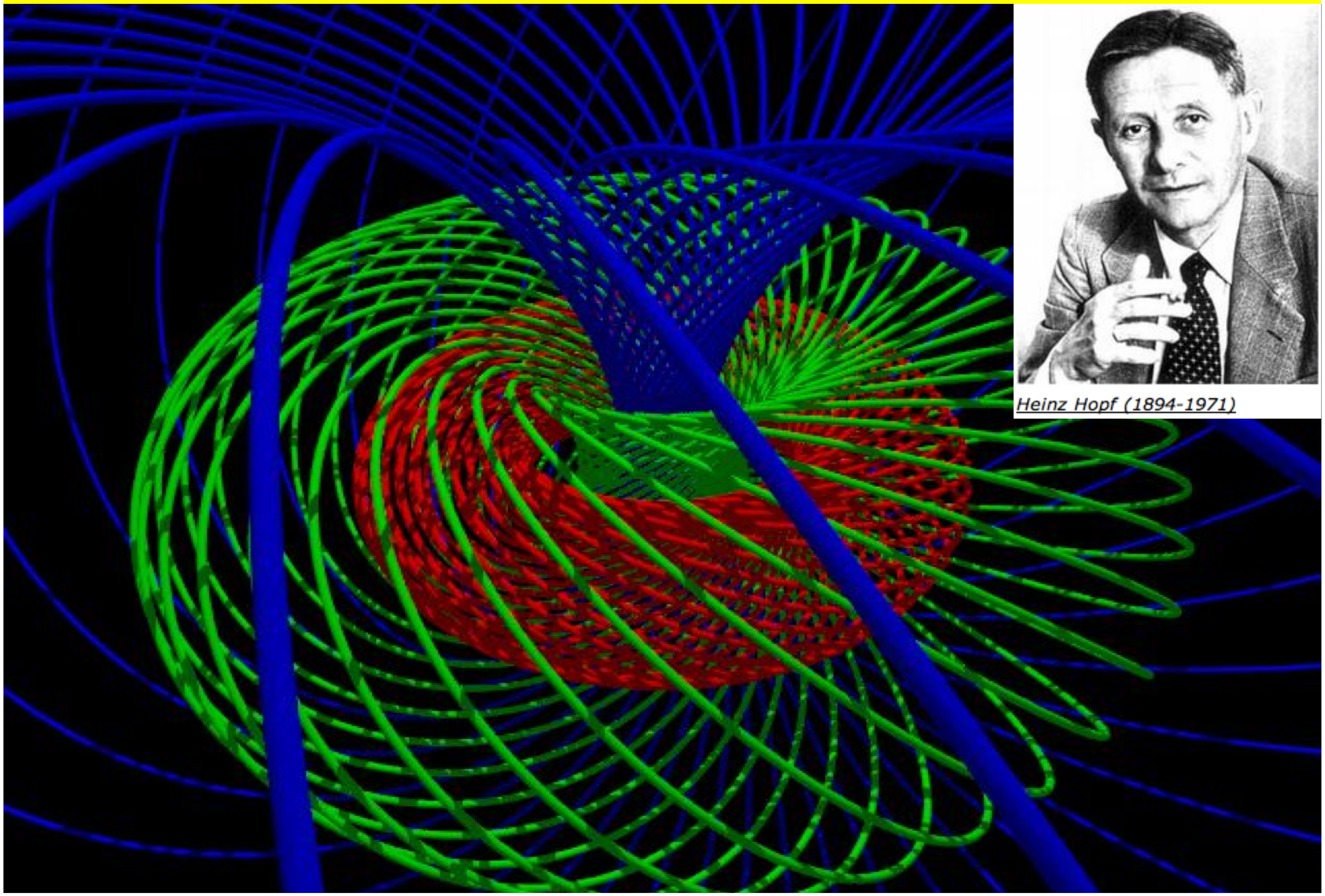
S^3 in terms of S^1 circles projected to 3D space & points on S^2





Topologically nontrivial!

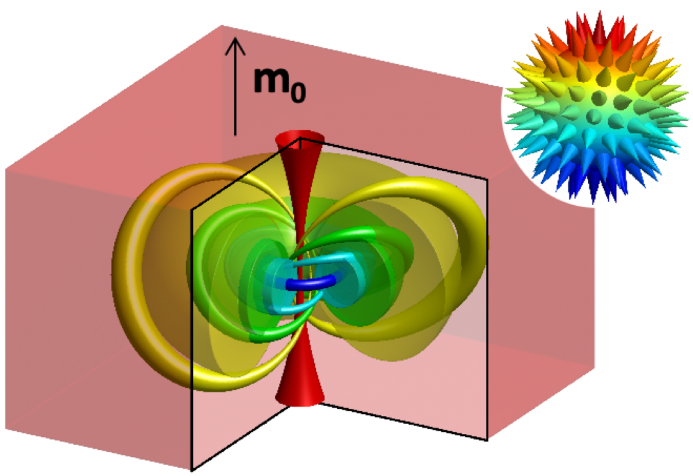
Hopf fibration



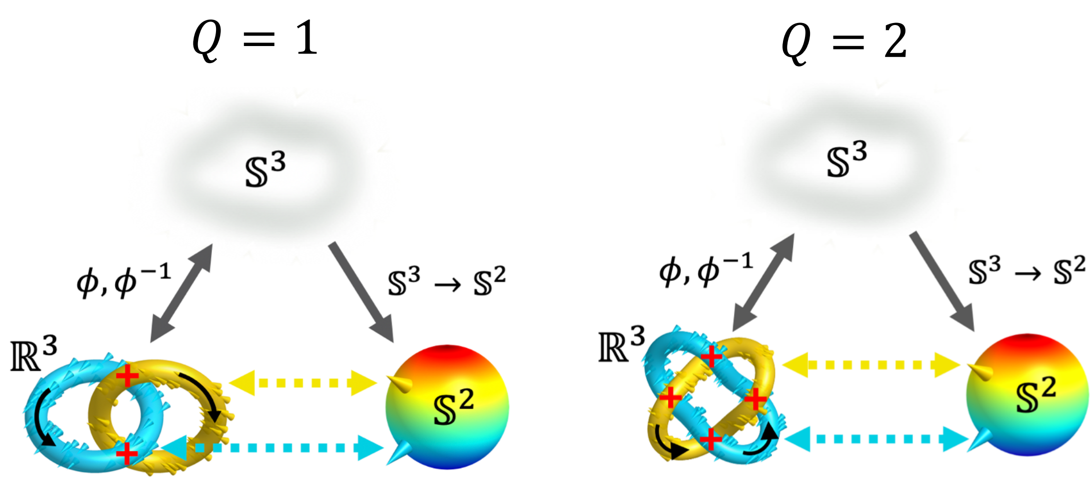
Heinz Hopf (1894-1971)

3D Hopf solitons

- $\pi_3(\mathbb{S}^2) = \mathbb{Z}$
- $\mathbb{R}^3 \cong \mathbb{S}^3$ when far-field uniform
- Hopf index $Q \in \mathbb{Z}$
- Ansatz form Hopf fibration

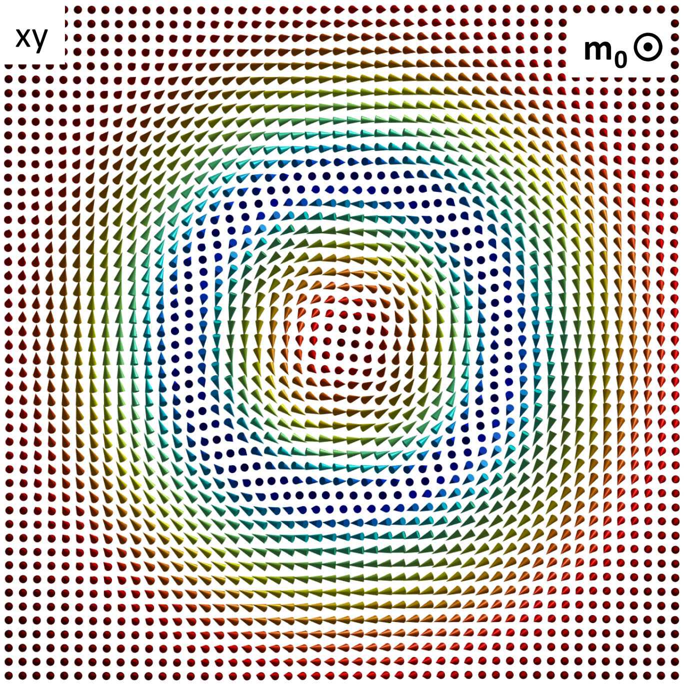


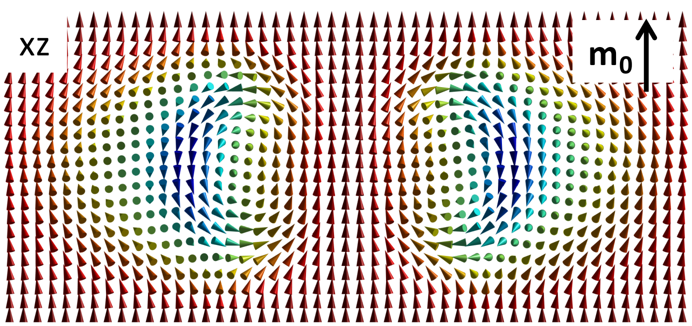
- Topological charge = Linking number of preimages $Q = \Sigma C/2$

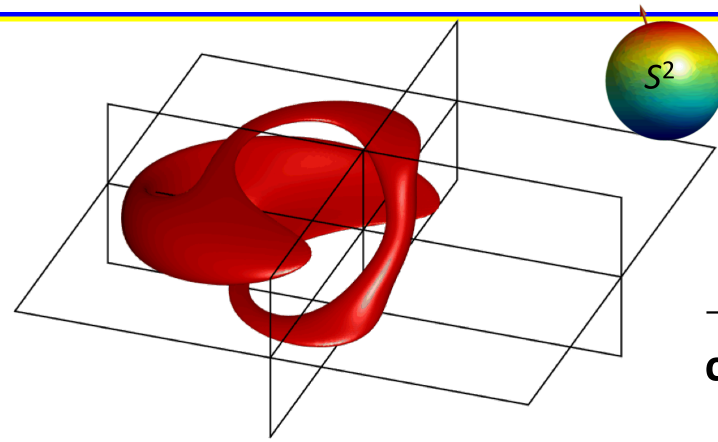


PJ Ackerman and II Smalyukh. *Nat. Mat.* **16**, 426 (2017)
PJ Ackerman and II Smalyukh. *Phys. Rev. X* (2017)
JSB Tai, PJ Ackerman, II Smalyukh, *PNAS U.S.A.* **115**, 921 (2018).

3D solitons, Numerical Modeling & Analysis

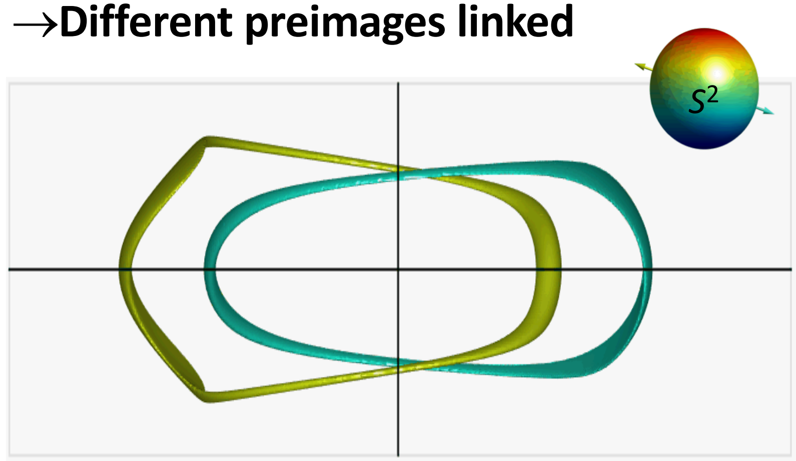




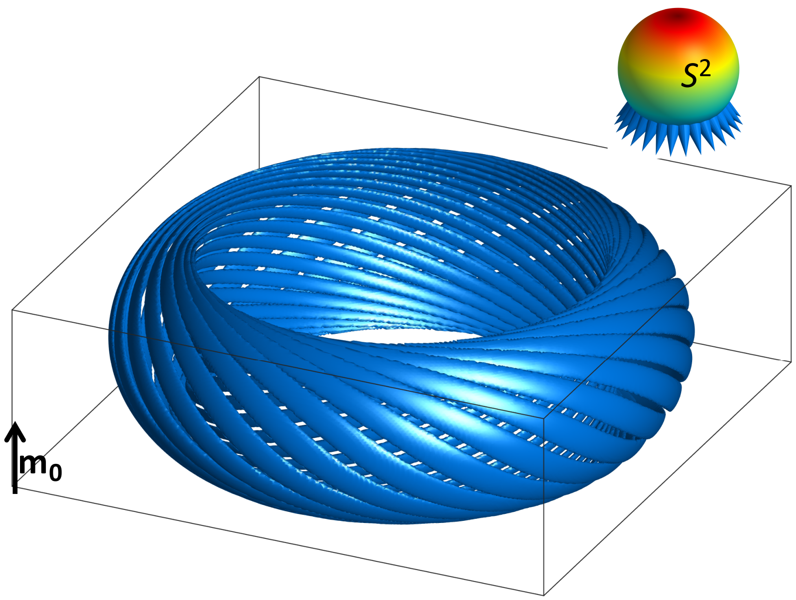


→All preimages - closed loops

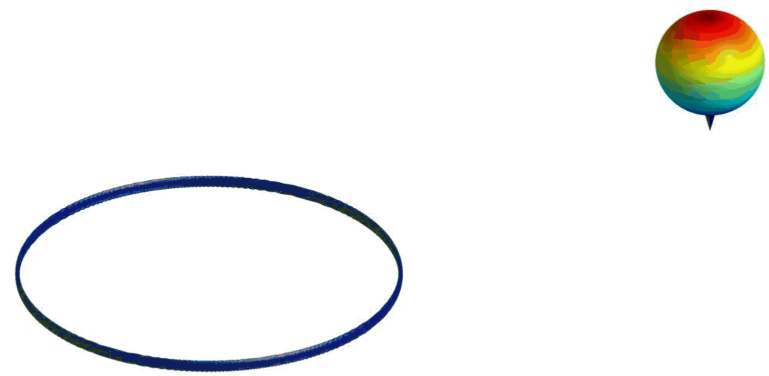
→Different preimages linked



Topological solitons with Hopf index Q=1



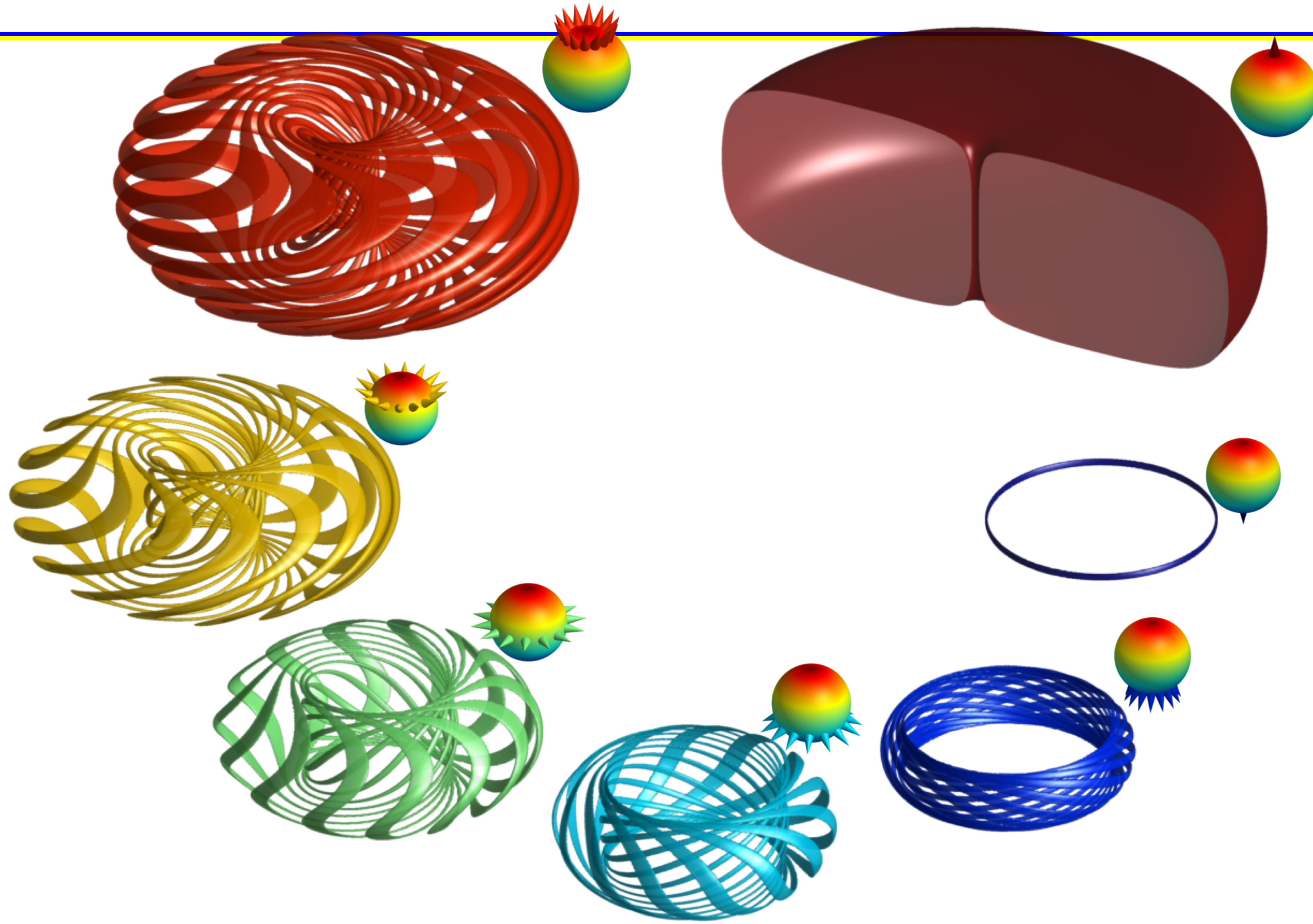
→Filling localized space with all preimages:



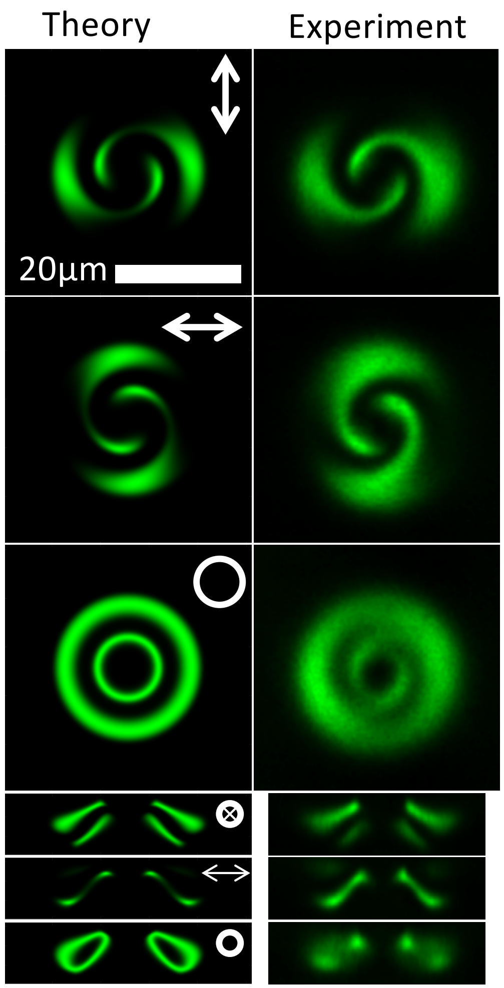
→Preimages of points with the same polar angle tile into tori
→Nested tori fill the 3D space

P.J. Ackerman & I.I. Smalyukh. *Nature Mater.* **16**, 426-432 (2017)

Nested tori of linked preimages

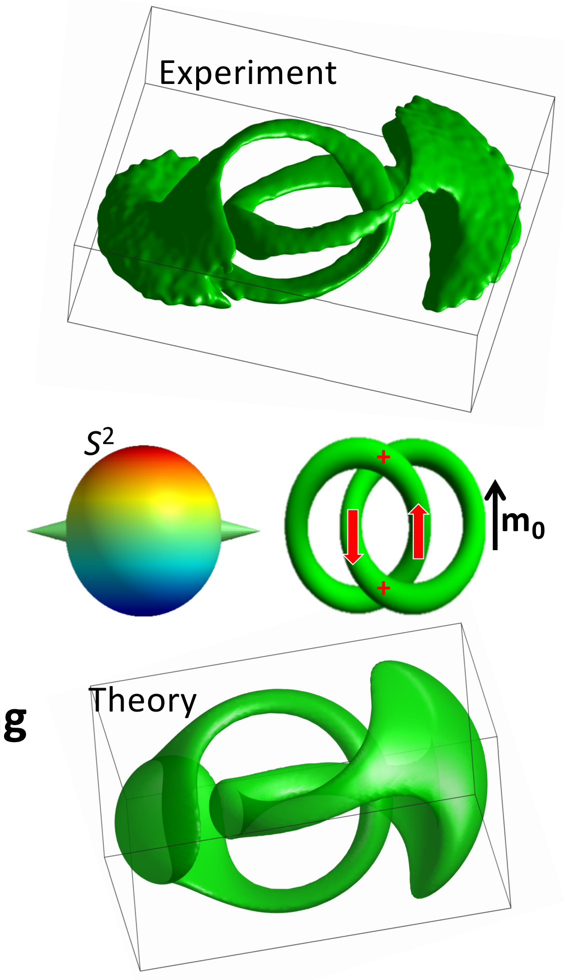


Numerical modeling versus experiment



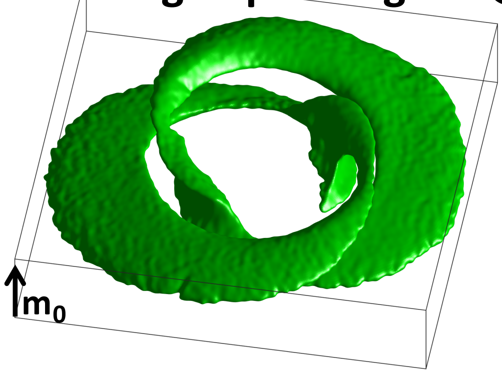
Computer-simulated & experimental images agree!!!

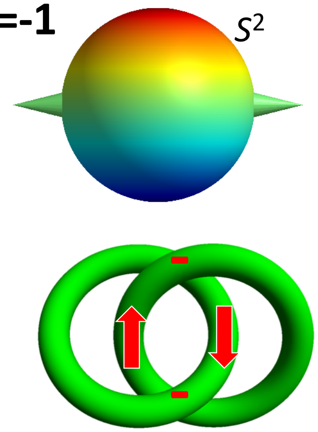
3-photon absorption based polarized 3D luminescence imaging

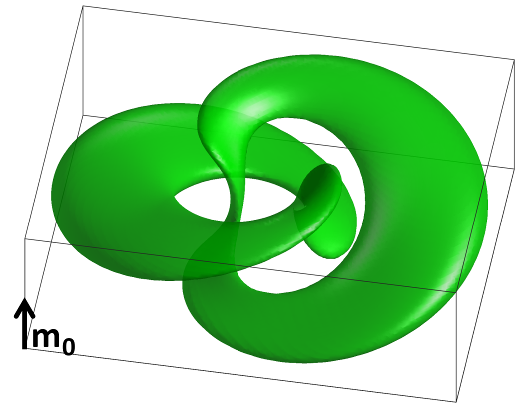


Q=-1 solitons & magnetic control of 2D crystals

→**Linking of preimages: Q=-1**



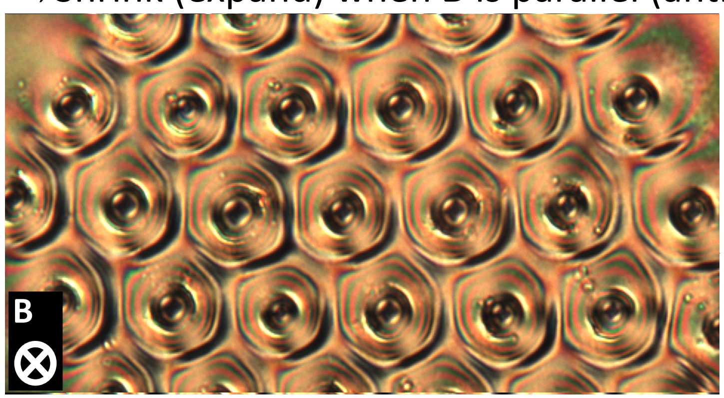


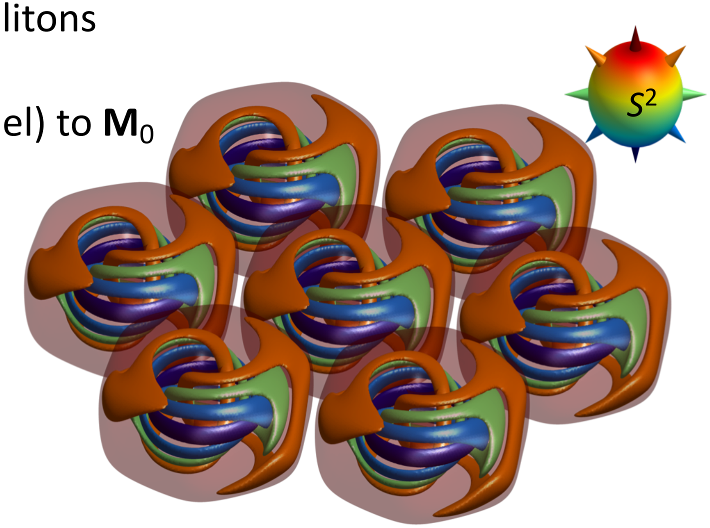


→Experimental & simulated 2D arrays of 3D solitons

→Control by applying **B**

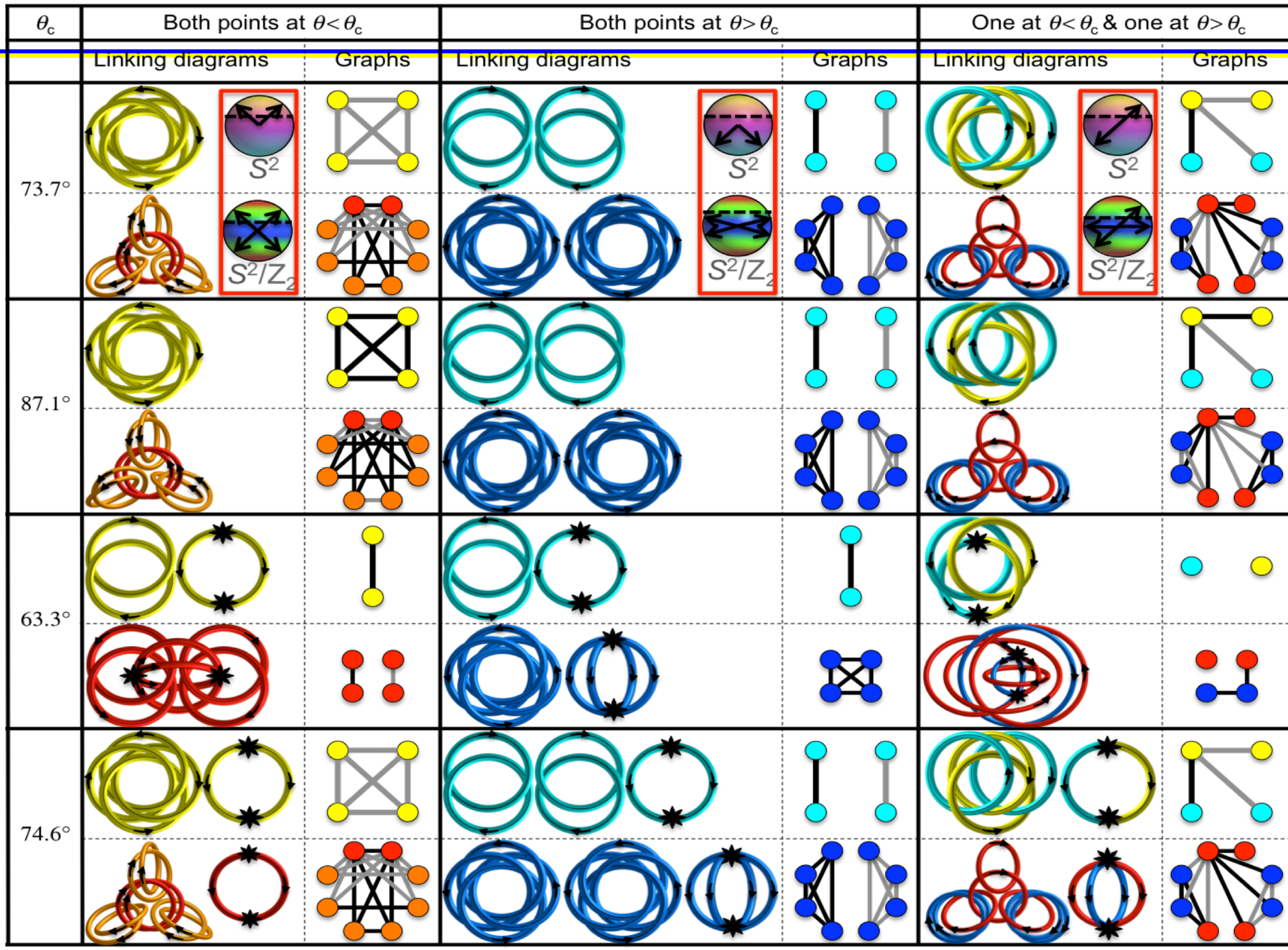
→Shrink (expand) when **B** is parallel (antiparallel) to $\mathbf{M}_0$





P. J. Ackerman and I. I. Smalyukh. *Phys Rev X* **7**, 011006 (2017)

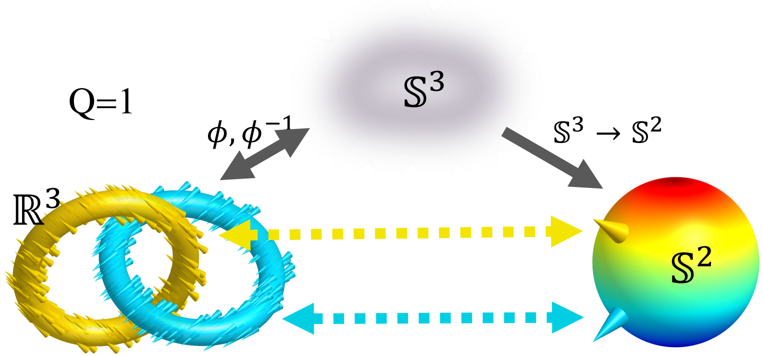
Experiments: beyond elementary Hopf solitons



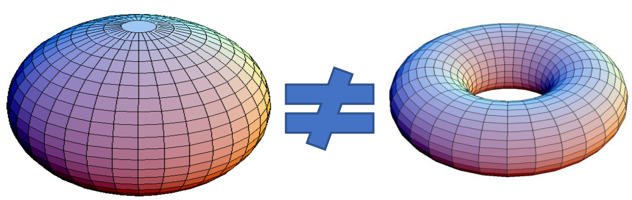
Sphere-sphere maps, homotopy theory

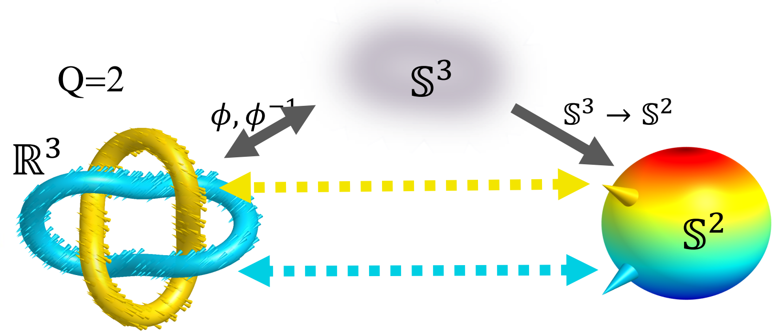
	π_1	π_2	π_3	π_4	π_5
$\mathbb{S}^0$	0	0	0	0	0
$\mathbb{S}^1$	$\mathbb{Z}$	0	0	0	0
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{S}^4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$\mathbb{S}^5$	0	0	0	0	0

homotopy group $\pi_i(S^n)$

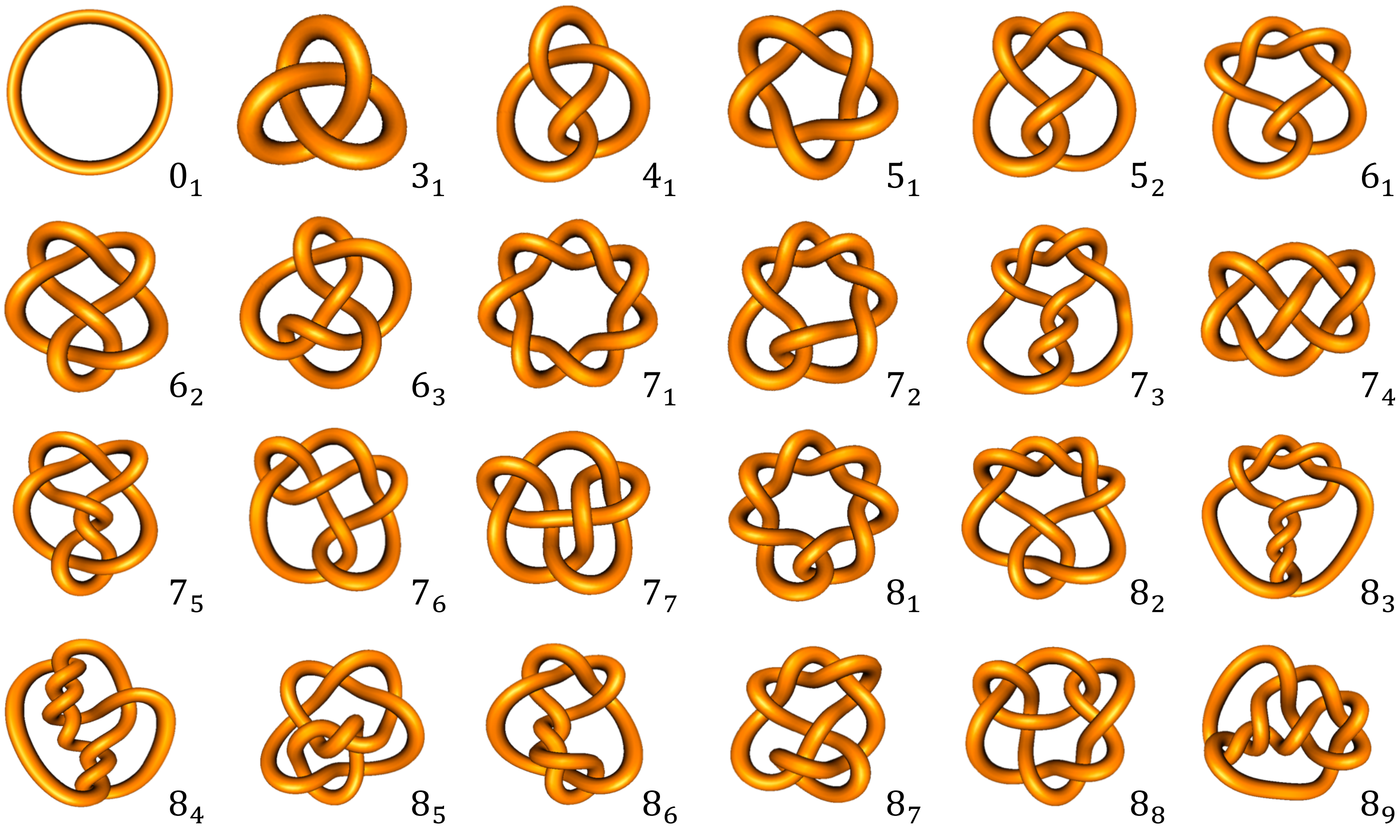


$\pi_3(S^2)$=Z solitons

Like 

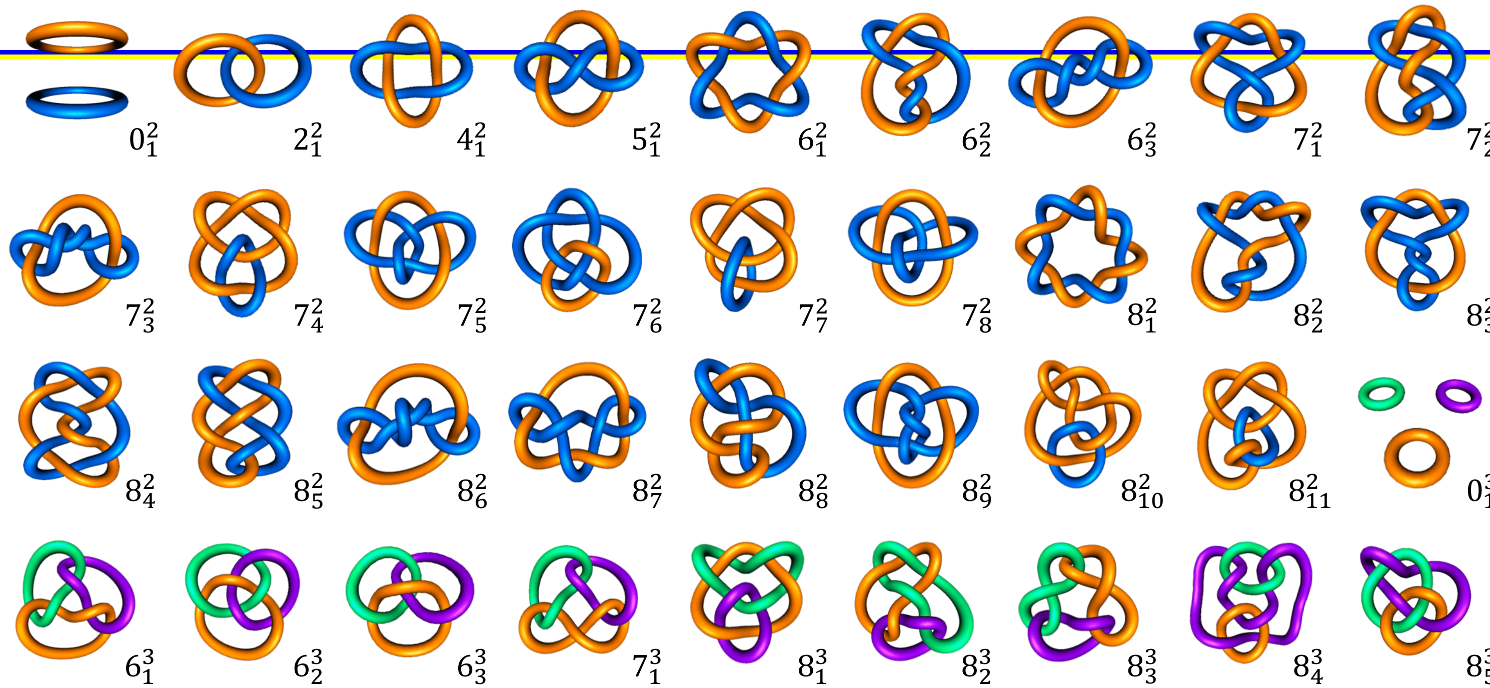


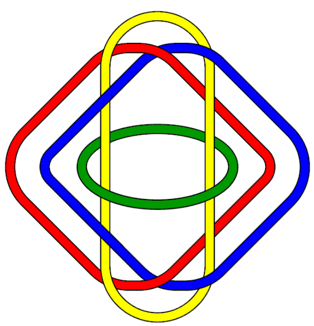
Diversity of knot topologies

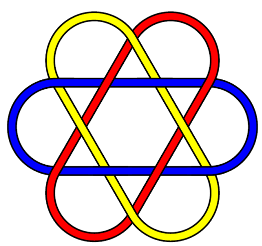


- All knots & links are topologically distinct and can lead to different realizations of solitons & vortices in physics fields

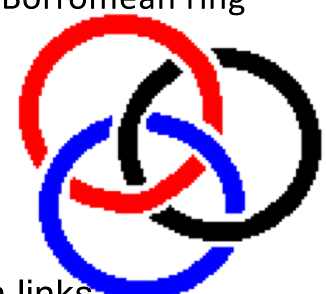
Diversity of multi-component link topologies







Borromean ring



12-crossing link Brunnian links

Topological transformation of hopfions ($\boldsymbol{E}$ field)

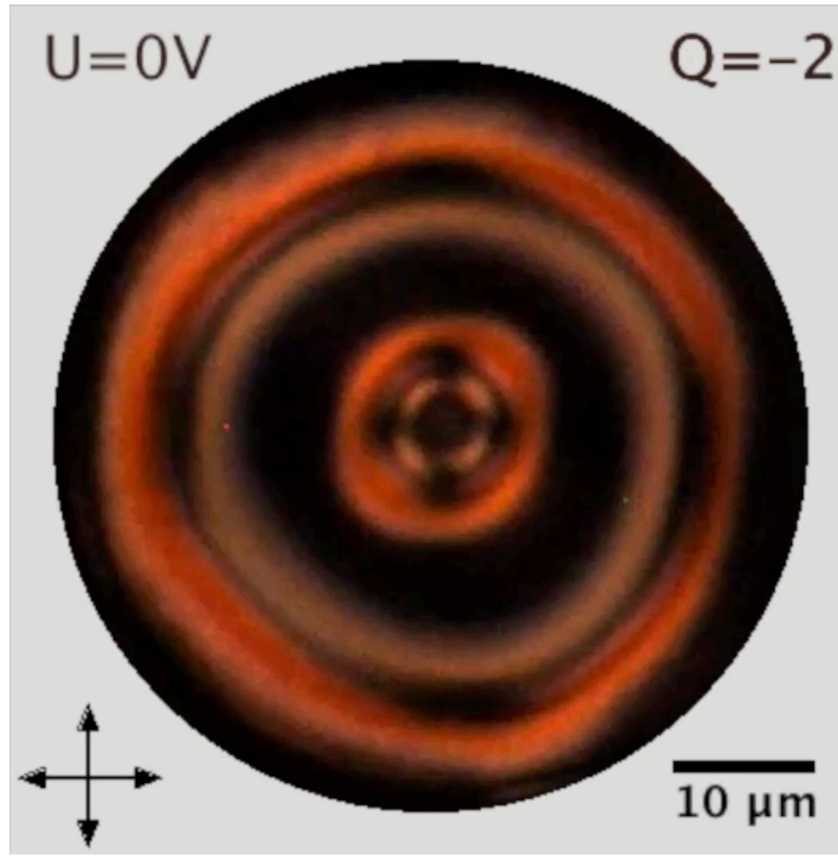
- Initial state:

$Q = -2$

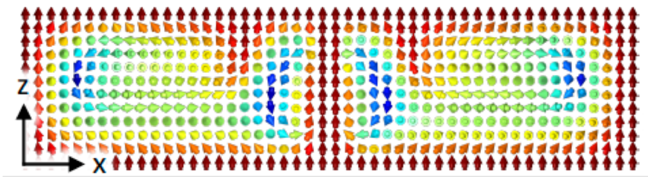
$$F_{\text{electric}} = -\frac{\varepsilon_0 \Delta\varepsilon}{2} \int d^3\boldsymbol{r}(\boldsymbol{E} \cdot \boldsymbol{m})^2 \, , \Delta\varepsilon < 0$$

- What happens to the intermediate state?
- Does topology switch at higher voltage?

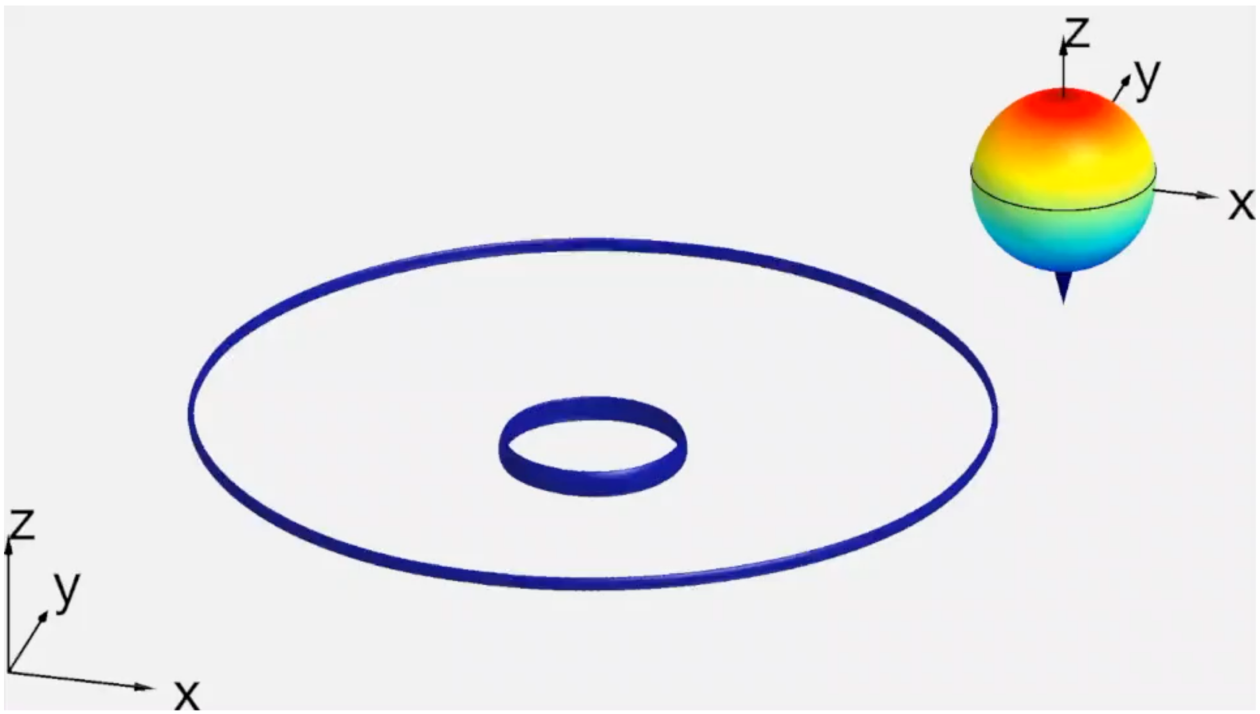
Video speed 2X

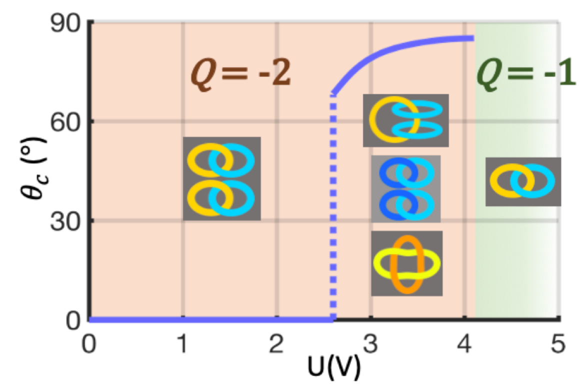


Visualization of the Q= -2 complex hopfion



Intermediate U



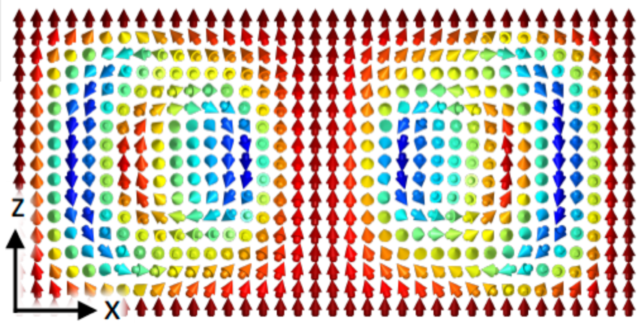


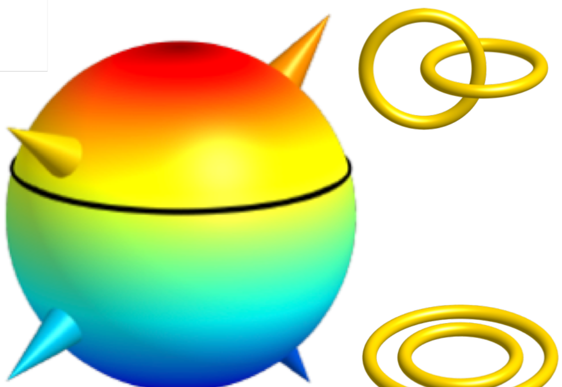
JSB Tai, PJ Ackerman, II Smalyukh, *PNAS U.S.A.* **115**, 921 (2018).

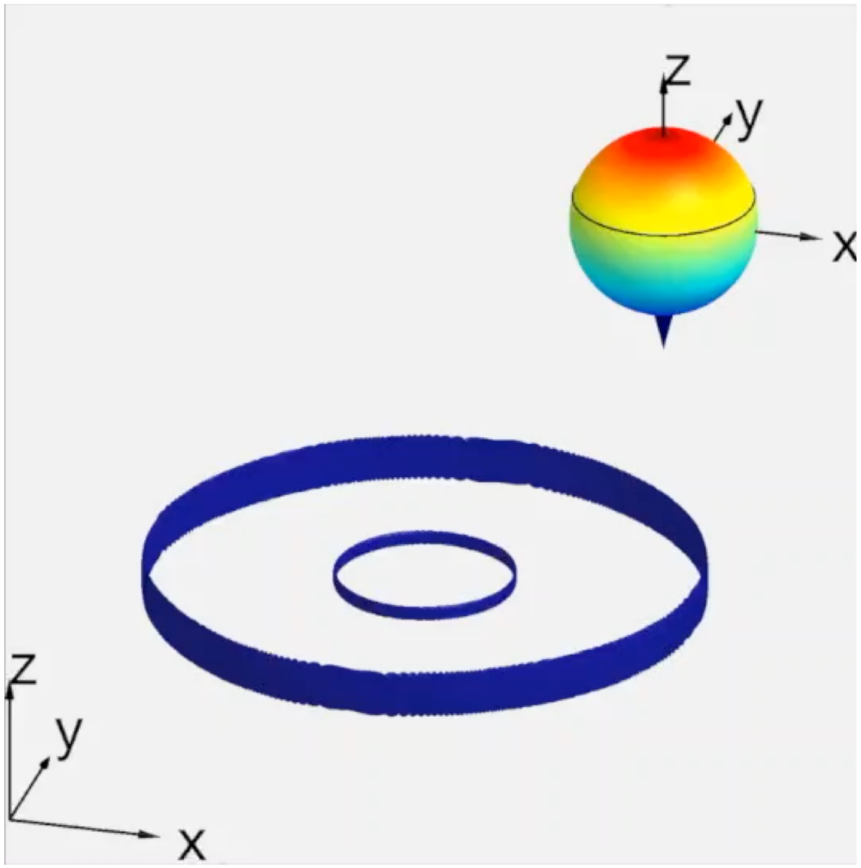
$\boldsymbol{H}$ - induced transformations

- Initial state $Q = 0$

$$F_{\text{magnetic}} = -\mu_0 M \int d^3\boldsymbol{r}\,(\boldsymbol{H}\cdot\boldsymbol{m})$$



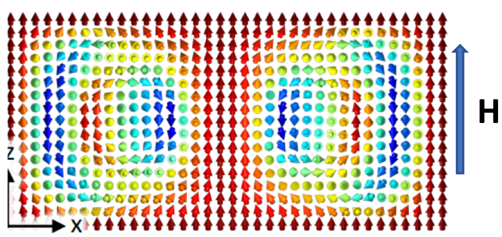


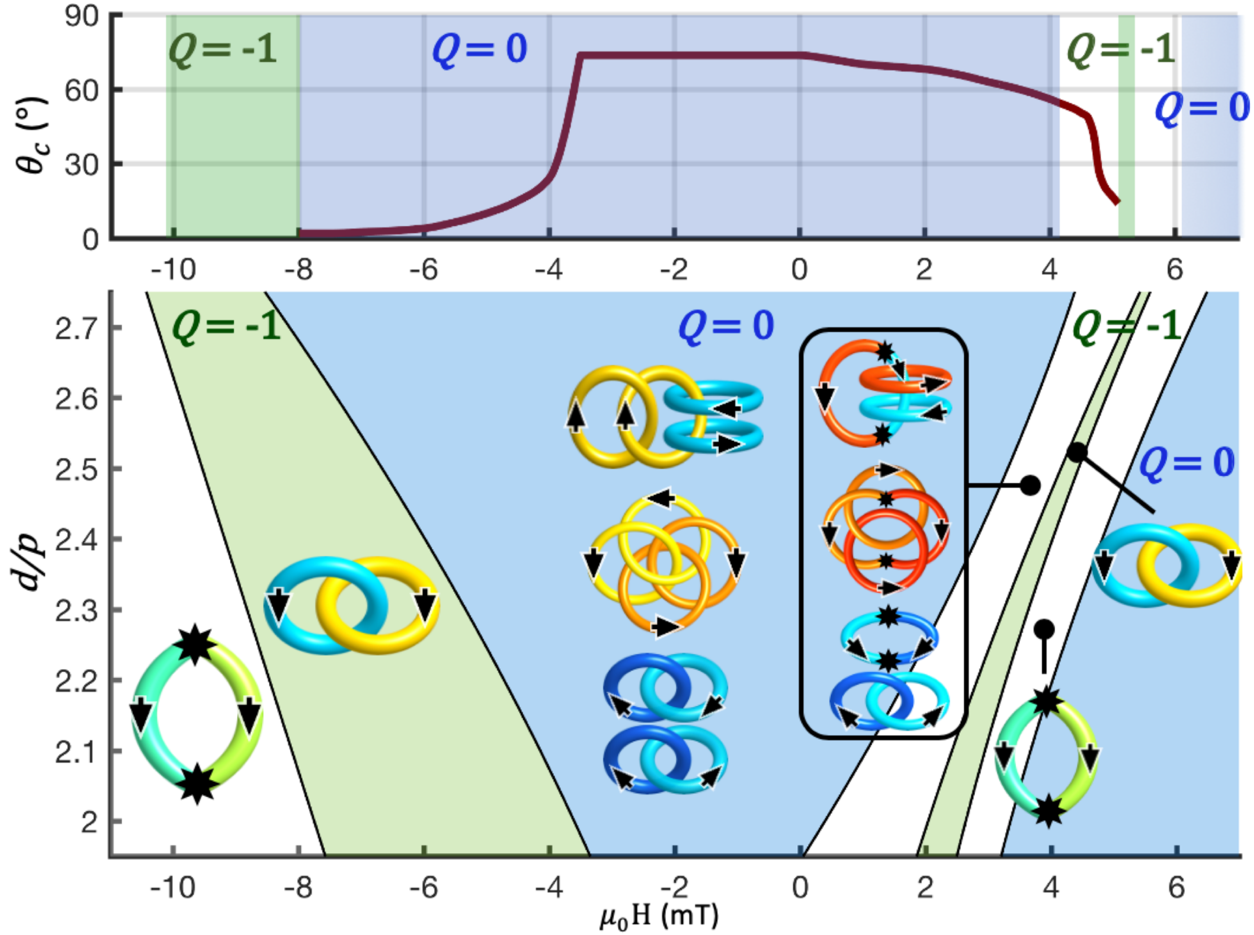


JSB Tai, PJ Ackerman, II Smalyukh, *PNAS U.S.A.* **115**, 921 (2018).

Topological & structural stability dependence on d/p & $\boldsymbol{H}$

- $\boldsymbol{H}$ along $\hat{z}$

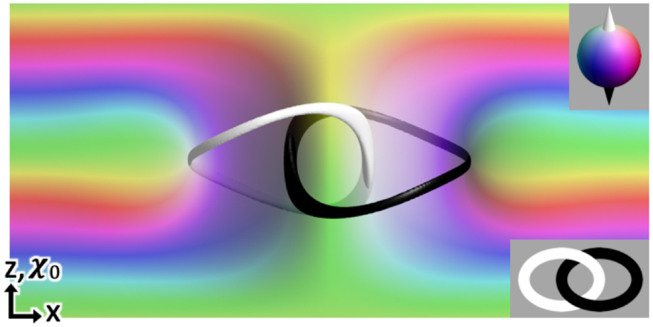


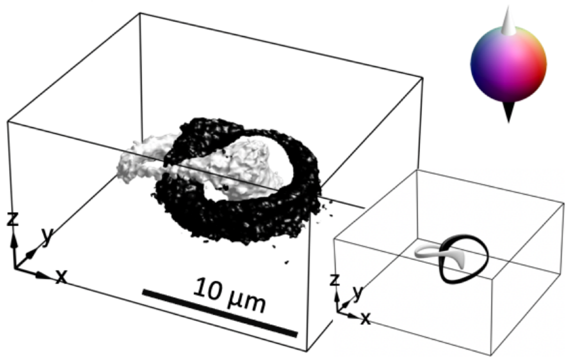


JSB Tai, PJ Ackerman, II Smalyukh, *PNAS U.S.A.* **115**, 921 (2018).

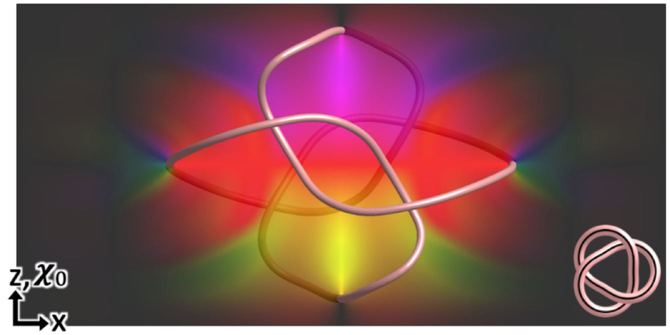
Heliknoton – solitons in a helical background

- Linking of preimage in $\boldsymbol{n}(\boldsymbol{r})$

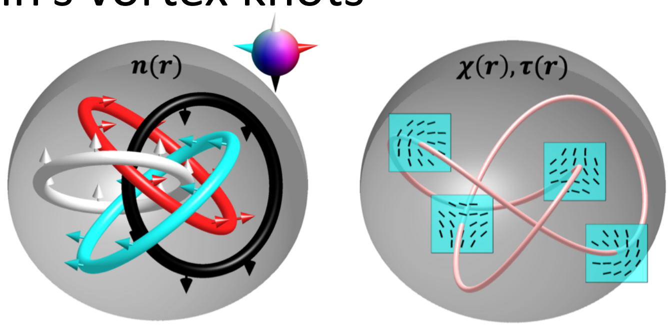




- Knotted Singular vortex lines in $\boldsymbol{\chi}(\boldsymbol{r})$ & $\boldsymbol{\tau}(\boldsymbol{r})$



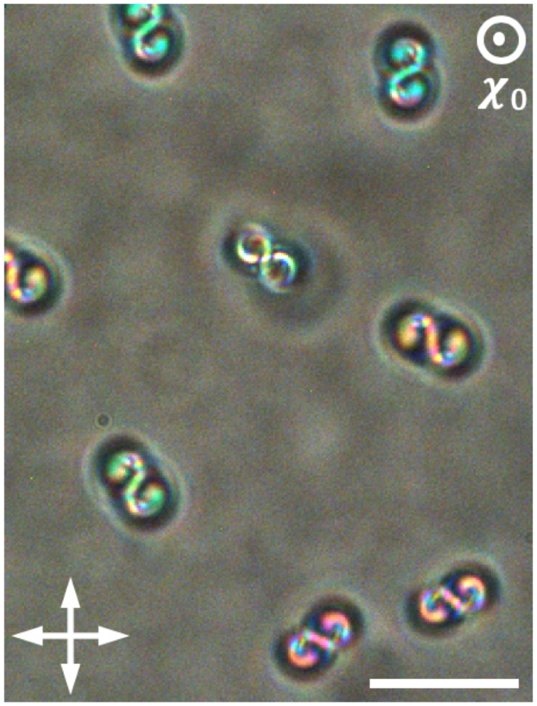
- Dual nature – Skyrme's knot soliton & Kelvin's vortex knots



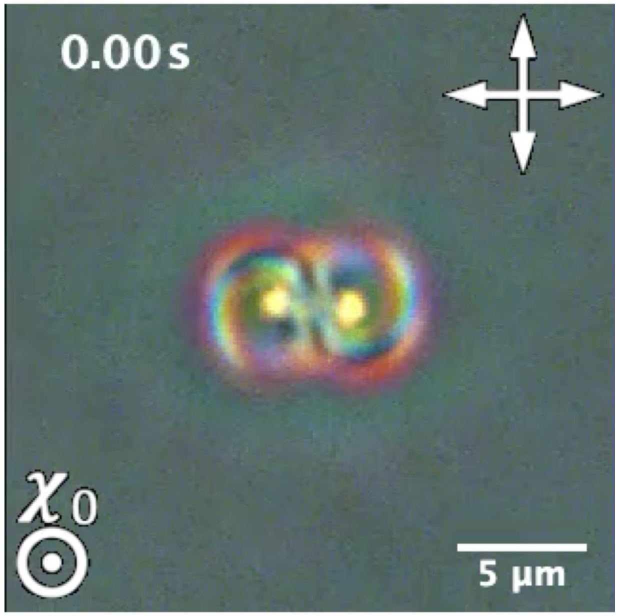
JSB Tai, II Smalyukh, *Science* **365**, 1449 (2019).

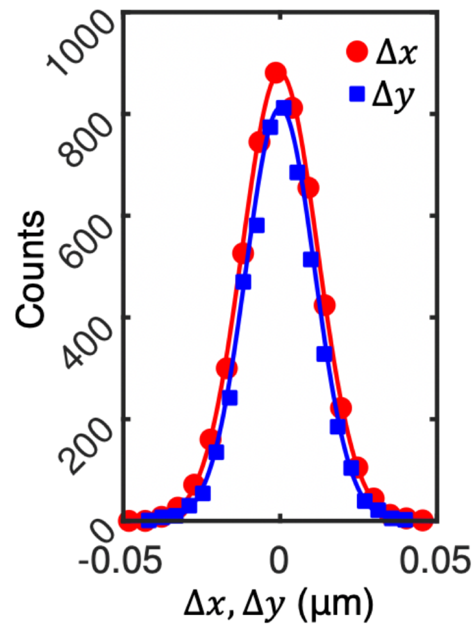
Heliknotons – particle-like

- Emerge with an applied ***E*** field along χ_0



- Brownian motion

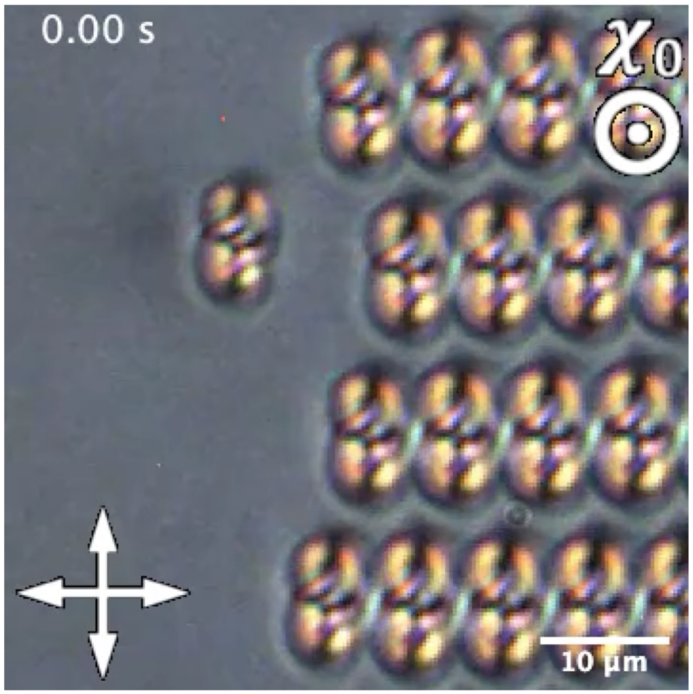




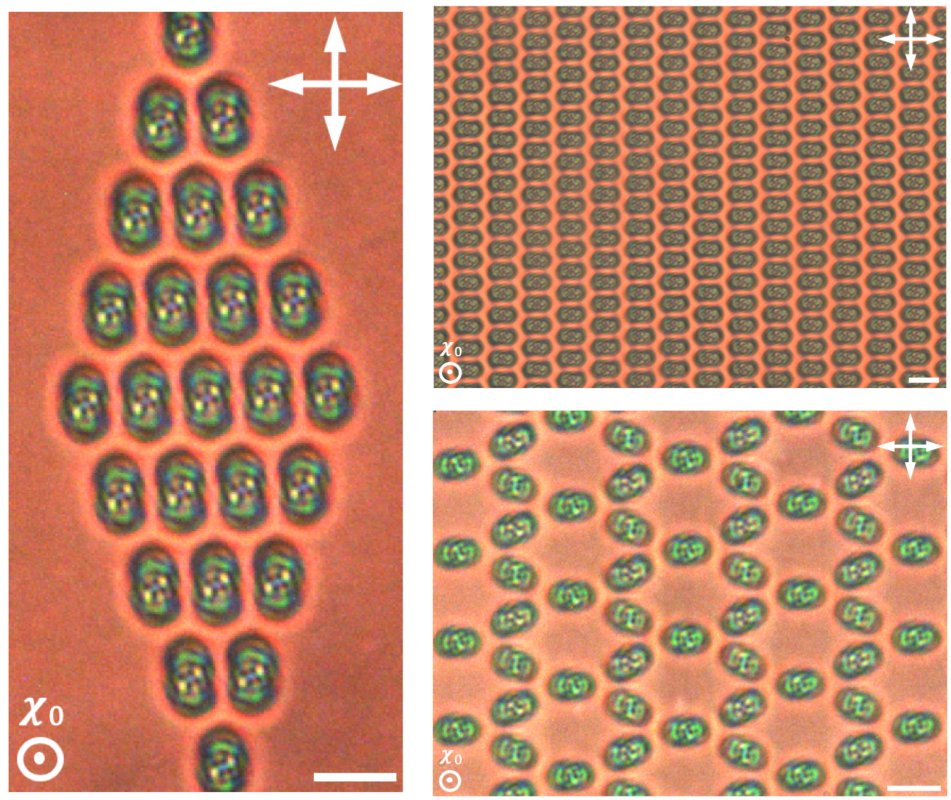
JSB Tai, II Smalyukh, *Science* **365**, 1449 (2019).

Heliknotons – self-assembled crystals

- Self-assemble into crystals



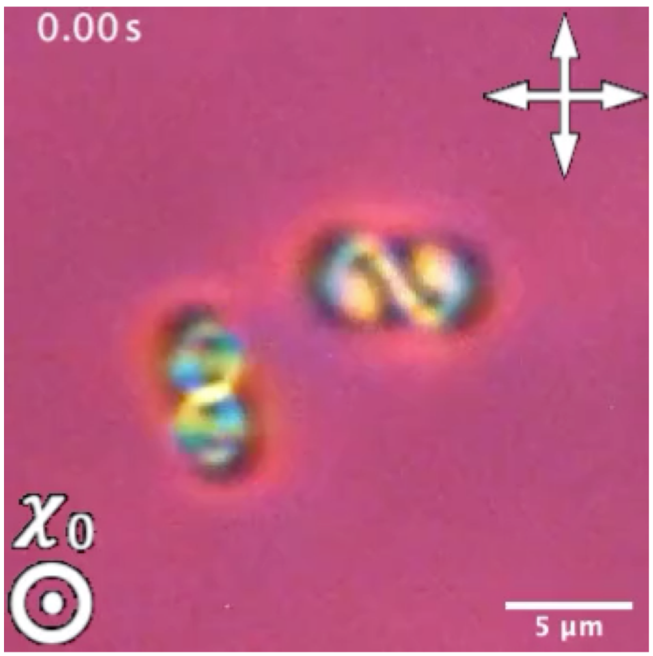
- Various assembled crystals

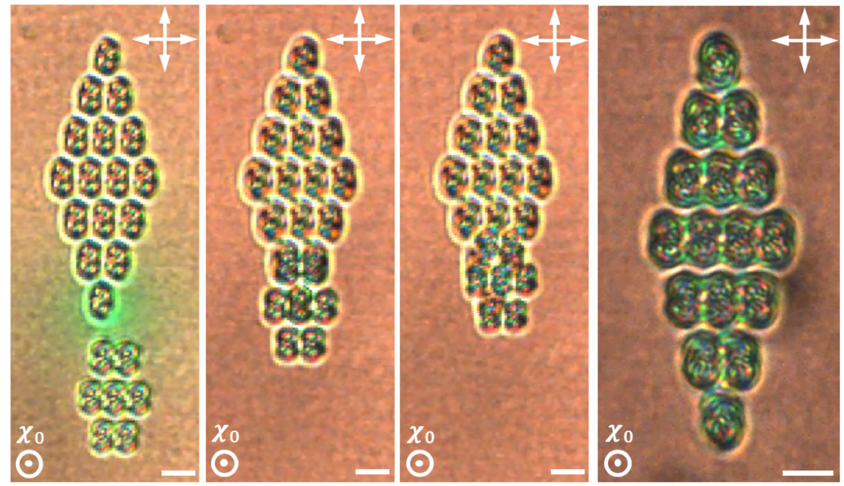


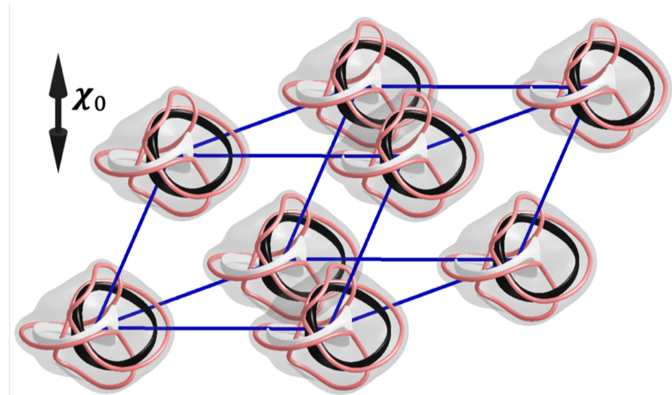
JSB Tai, II Smalyukh, *Science* **365**, 1449 (2019).

Heliknoton – 3D crystals

- 3D localized and 3D interaction



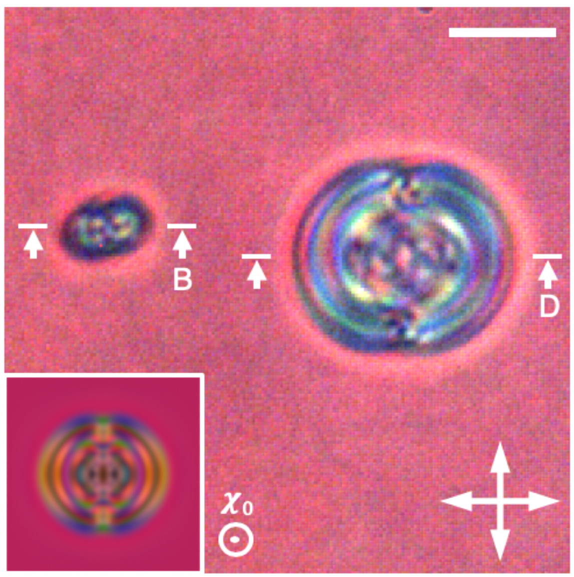




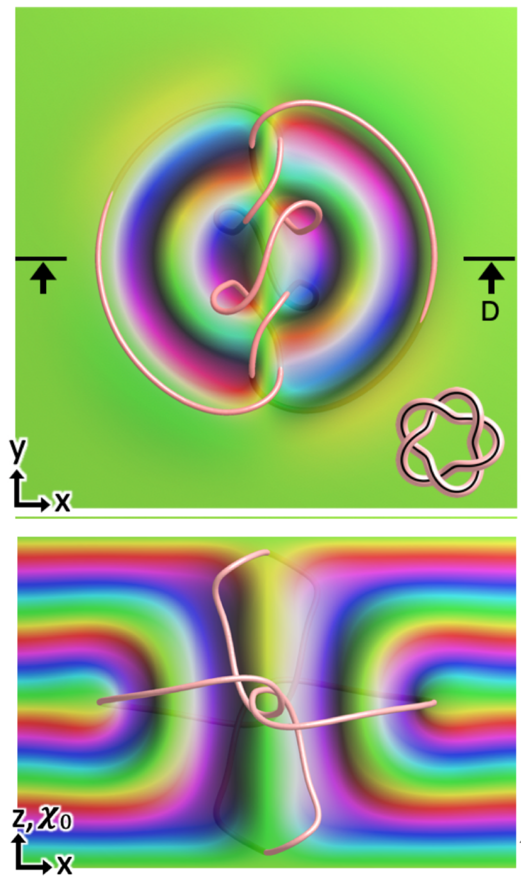
JSB Tai, II Smalyukh, *Science* **365**, 1449 (2019).

Heliknoton – high degree

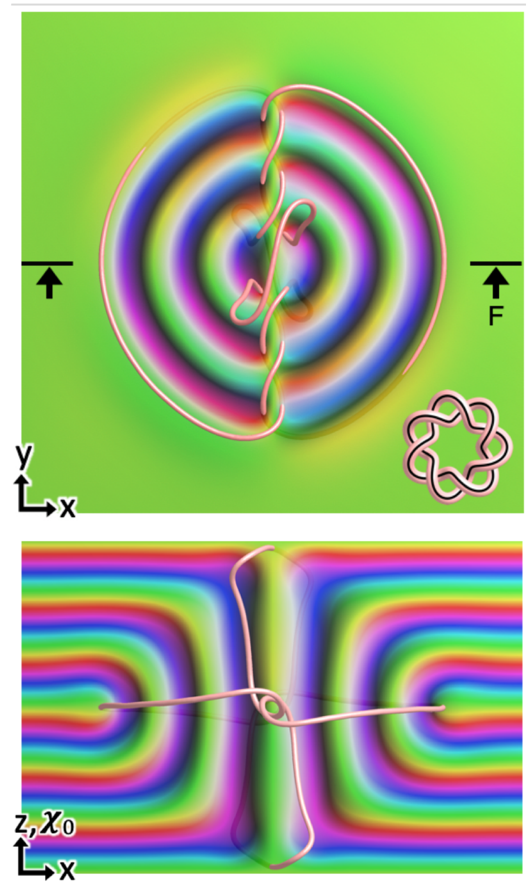
- Larger solitons observed



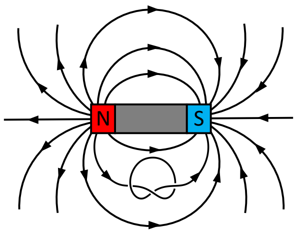
$Q = 2$

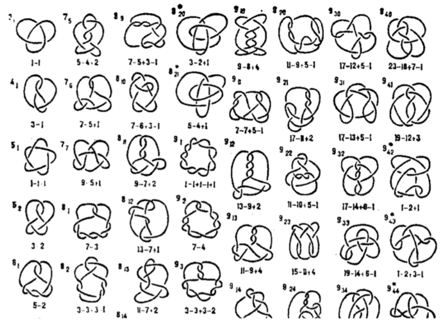


$Q = 3$

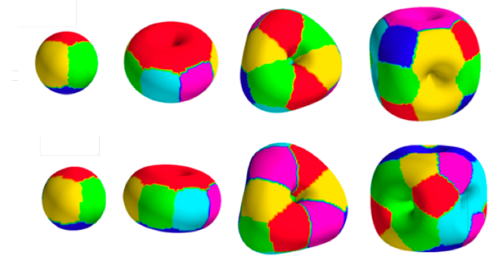


Knots and solitons – history



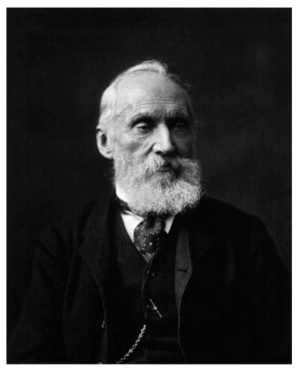


Naya & Sutcliffe, *Phys. Rev. Lett.* **121**, 232002 (2018)



Gauss

- Knots in fields as particles!!!



Lord Kelvin

- Models of atoms as knots
- knot theory

T. Skyrme

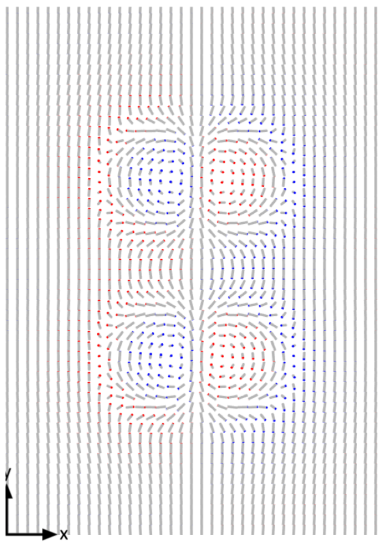
- Solitons stabilized by high-order terms (Skyrme model)
- Topological solitons as atomic nuclei

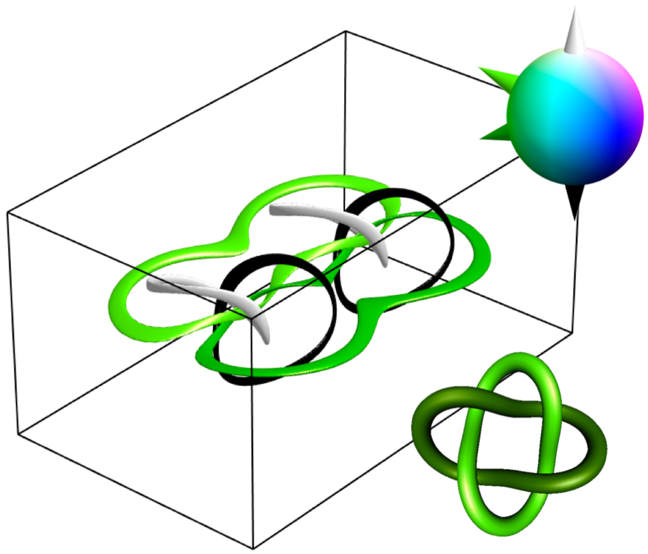
Ed Witten

- Skyrme model as effective model of QCD

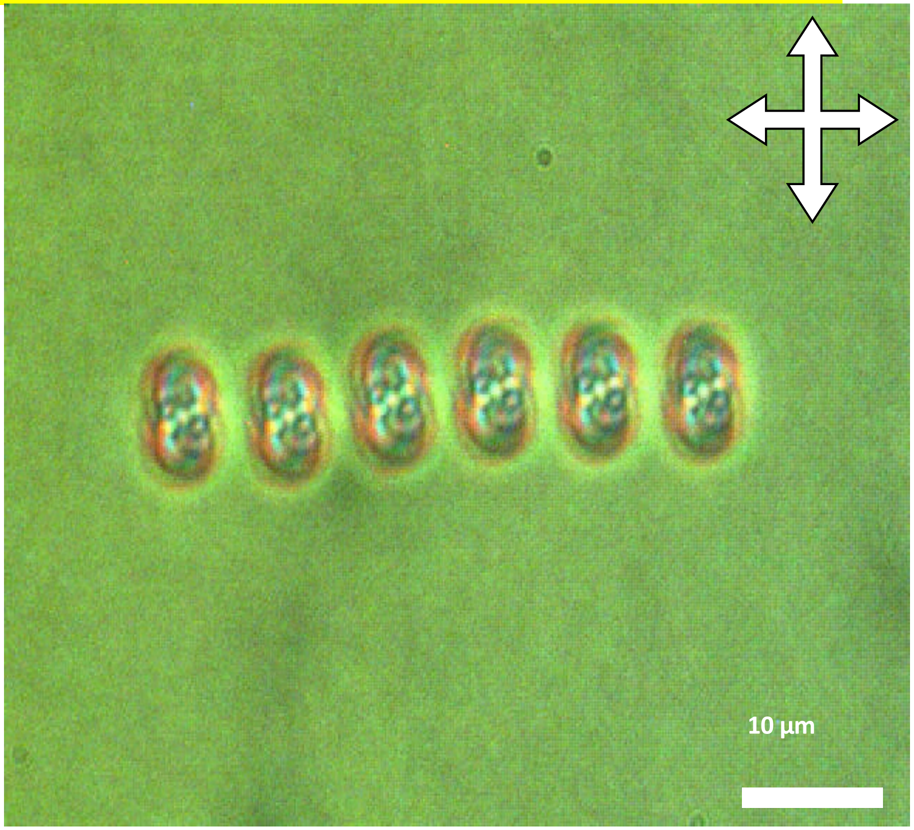
Heliknoton – high degree solitons

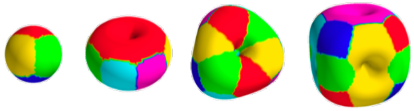
- Clustering





JSB Tai, II Smalyukh (2020).

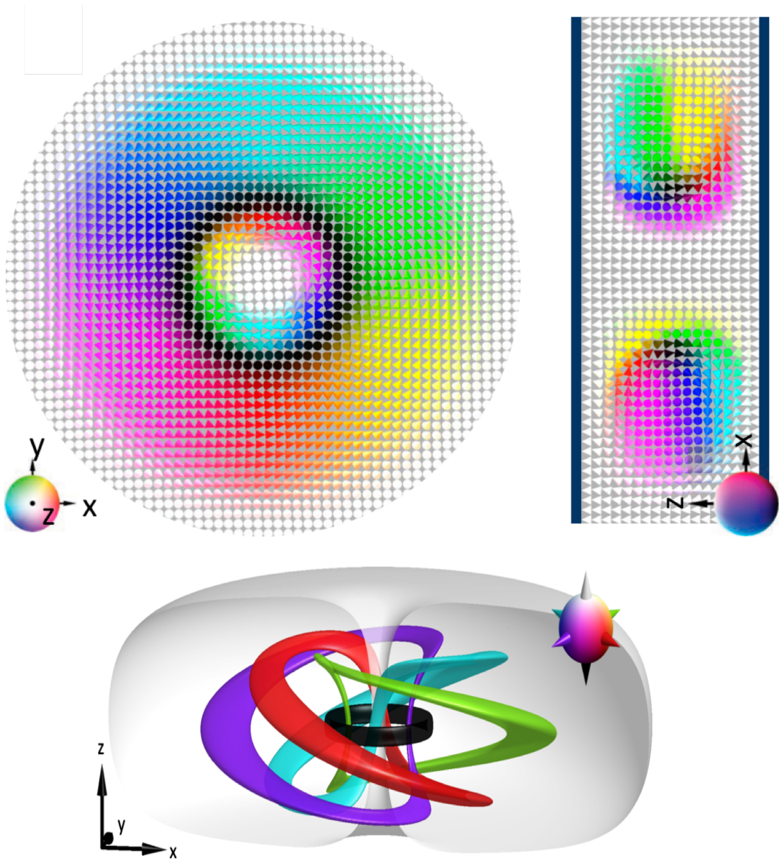




Naya & Sutcliffe, *PRL* **121**, 232002 (2018)

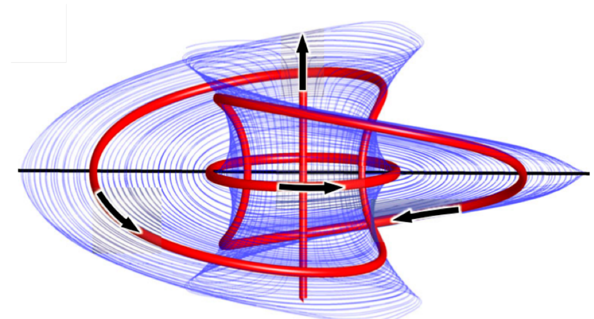
3D Hopf solitons in chiral magnets

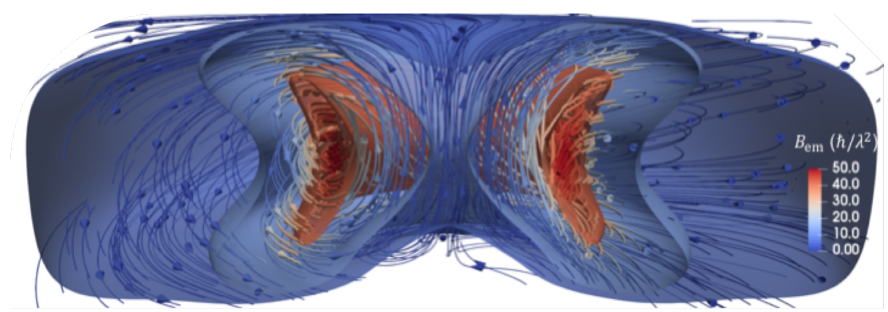
- Magnetic hopfion in a nanodisk



- Streamlines of emergent field form linked closed loops -> Hopf fibration

$$(\boldsymbol{B}_{\rm em})_i \equiv \hbar\varepsilon^{ijk}\boldsymbol{m}\cdot(\partial_j\boldsymbol{m}\times\partial_k\boldsymbol{m})/2$$

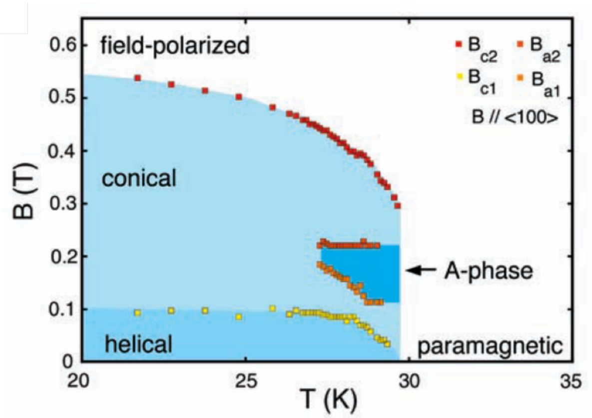




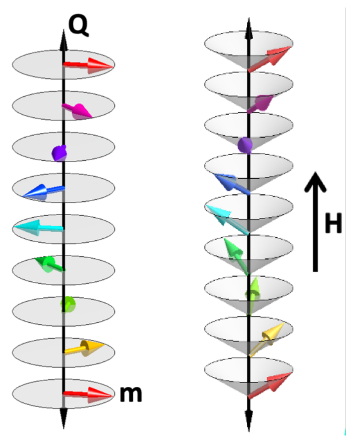
JSB Tai, II Smalyukh, *Phys. Rev. Lett.* **121**, 187201 (2018).

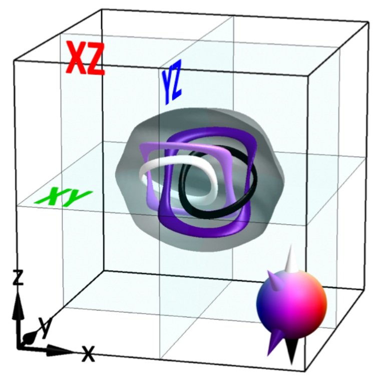
Magnetic heliknoton

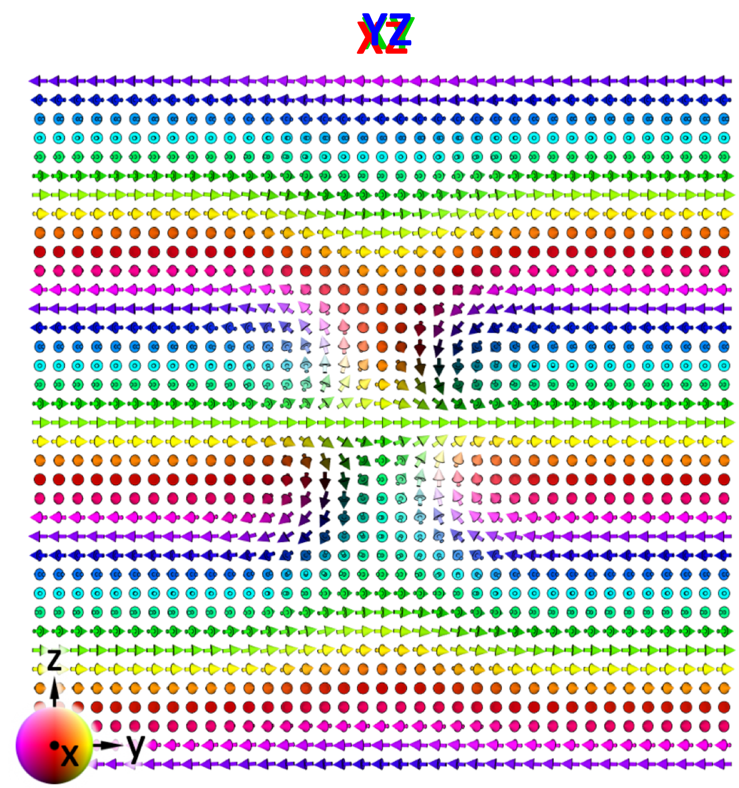
- Chiral magnet phase diagram



Mühlbauer et al. *Science* **323**, 915 (2009)

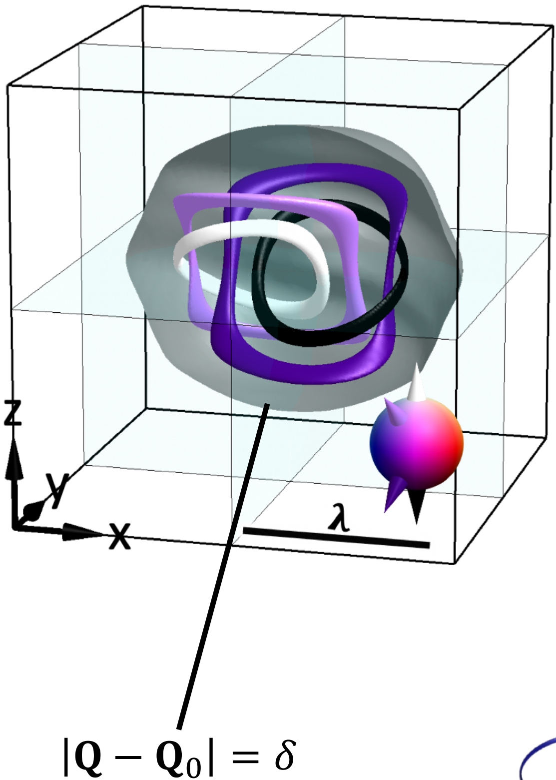


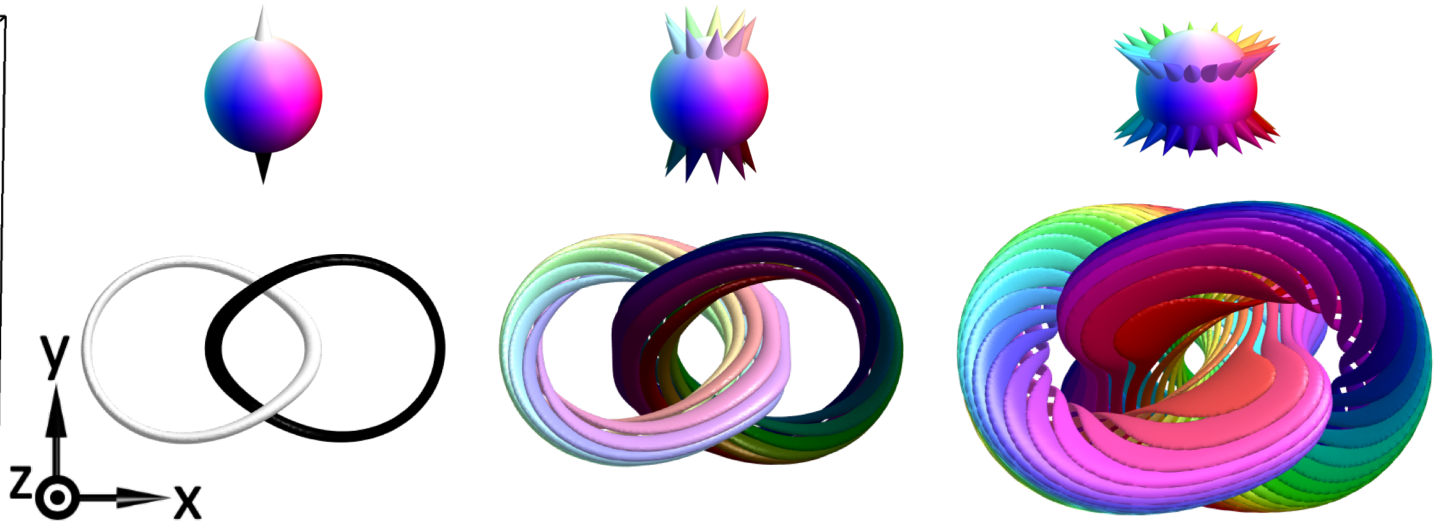


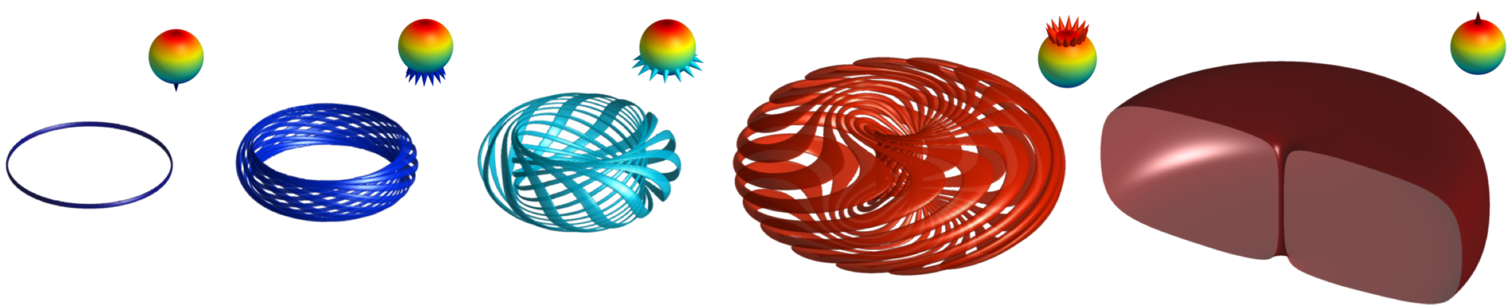


R Voinescu, JSB Tai, II Smalyukh, (2020).

Magnetic heliknoton – preimages





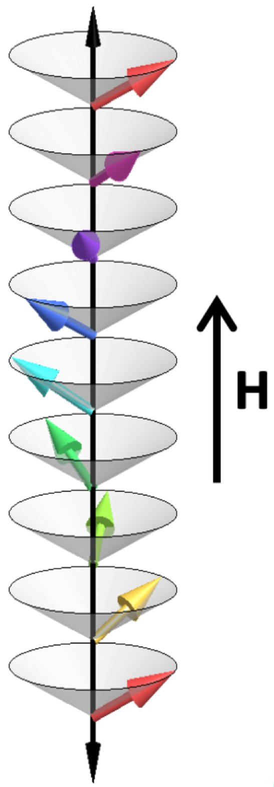


R. Voinescu, JSB Tai, II Smalyukh, (2020).

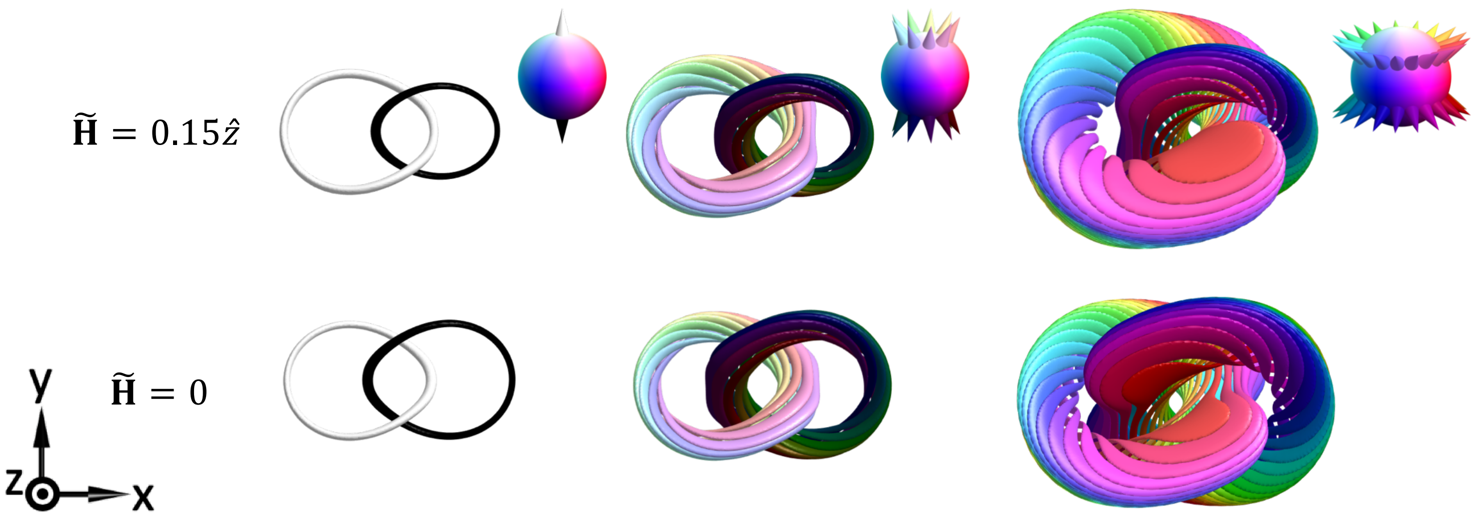
PJ Ackerman, II Smalyukh, *Nat. Mater.* ***16***, 436 (2017).

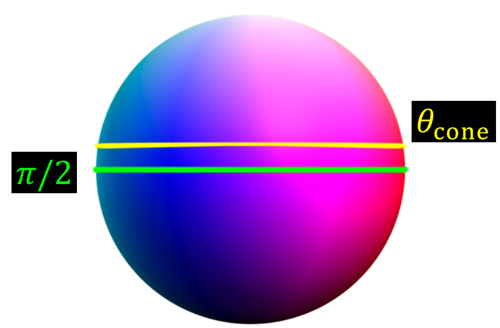
Heliknoton – conical background

- Broken inversion symmetry



$$\theta_{\mathrm{cone}} = \cos^{-1}\tilde{\mathrm{H}}$$

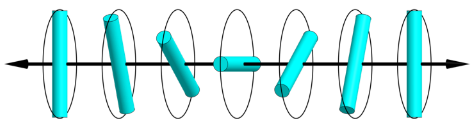


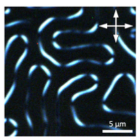


R Voinescu, JSB Tai, II Smalyukh (2020).

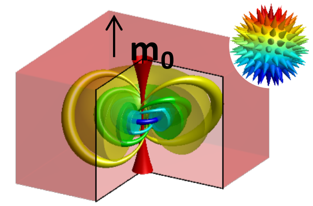
Topological solitons in condensed matters

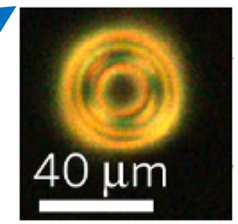
Twisted walls

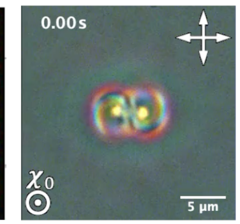




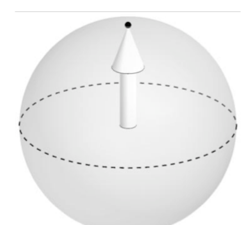
	π_1	π_2	π_3
$\mathbb{S}^0$	0	0	0
$\mathbb{S}^1$	$\mathbb{Z}$	0	0
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{S}^3$	0	0	$\mathbb{Z}$

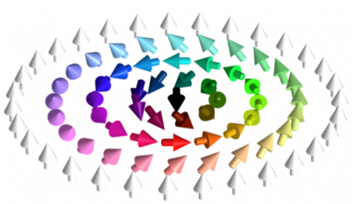






Hopfions & heliknotons





Skyrmions