KITP Program: Holographic Duality and Condensed Matter Physics

Recent Updates of Holographic Entanglement Entropy

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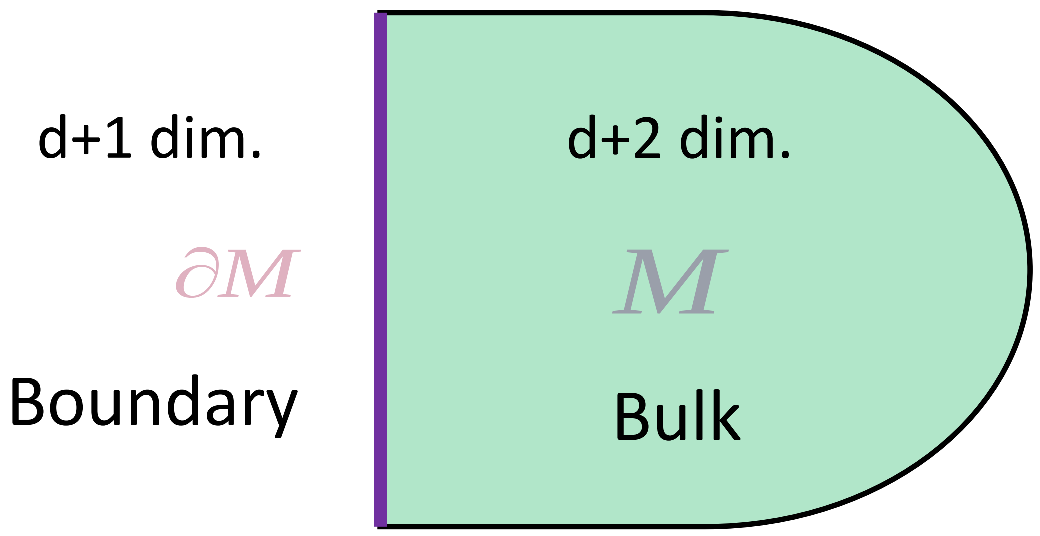
Contents

① Introduction

Holography (e.g. AdS/CFT [Maldacena 97])

⇒ Non-perturbative Definition of Quantum Gravity



$$Z_{QM}(\partial M) = Z_{Gravity}(M)$$

To understand the holography better and to explore the holography in general setups, we need suitable universal physical quantities.

Stationary BH ⇒ Mass M, Charge Q, Spin J.

(Thermodynamics)

Generic spacetime ⇒ We need much more quantities !

(Non-equilibrium, Topological, etc.)

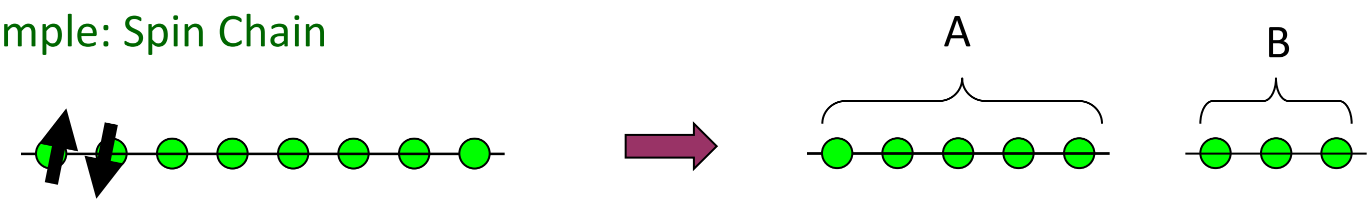
For this purpose, the entanglement entropy (EE) is a very useful quantity. [⇔ In cond-mat. , EE is useful to characterize quantum structures of ground states.]

Definition of Entanglement Entropy

Divide a quantum system into two subsystems A and B.

$$H_{tot} = H_A \otimes H_B \quad .$$

Example: Spin Chain



We define the reduced density matrix ρ_A for A by

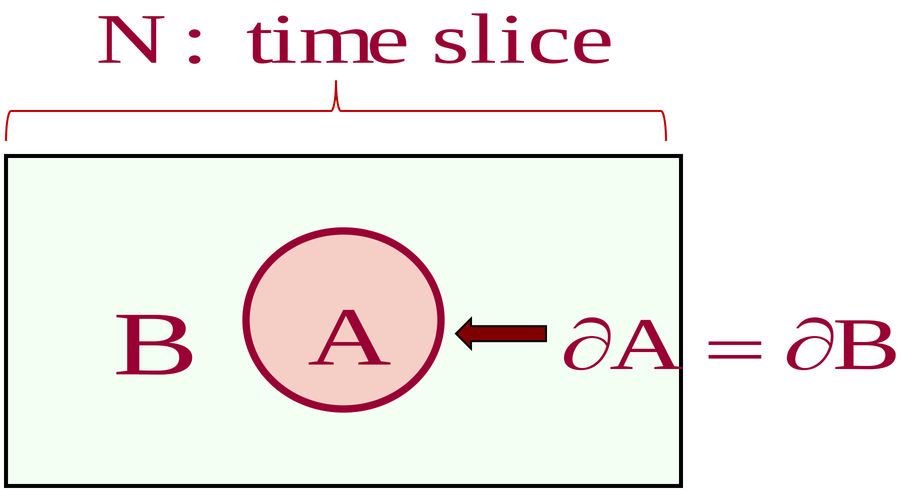
$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

taking trace over the Hilbert space of B .

Now the entanglement entropy S_A is defined by the von-Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .$$

In QFTs, it is defined geometrically:



Various Applications in other subjects

- **Quantum Information and Quantum Computing**

EE = the amount of quantum information

[see e.g. Nielsen-Chuang's text book 00]

- **Condensed Matter Physics**

EE = Efficiency of a computer simulation (DMRG) [Gaite 03,…]

➡ Divergent at quantum critical points !

[G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, 02,…]

➡ A new quantum order parameter !

[Topological entanglement entropy: Kitaev-Preskill 06, Levin-Wen 06]

Basic property: Area law

EE in d+1 dim. QFTs (in ground states) includes UV divergence

[Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

where ε is a UV cutoff (i.e. lattice spacing).

A comment: The holographic EE predicts that the area law is always true for any QFT if there is a UV fixed point.

Similar to the Bekenstein-Hawking formula of black hole entropy

$$S_{BH} = \frac{\mathrm{Area(horizon)}}{4G_N}.$$

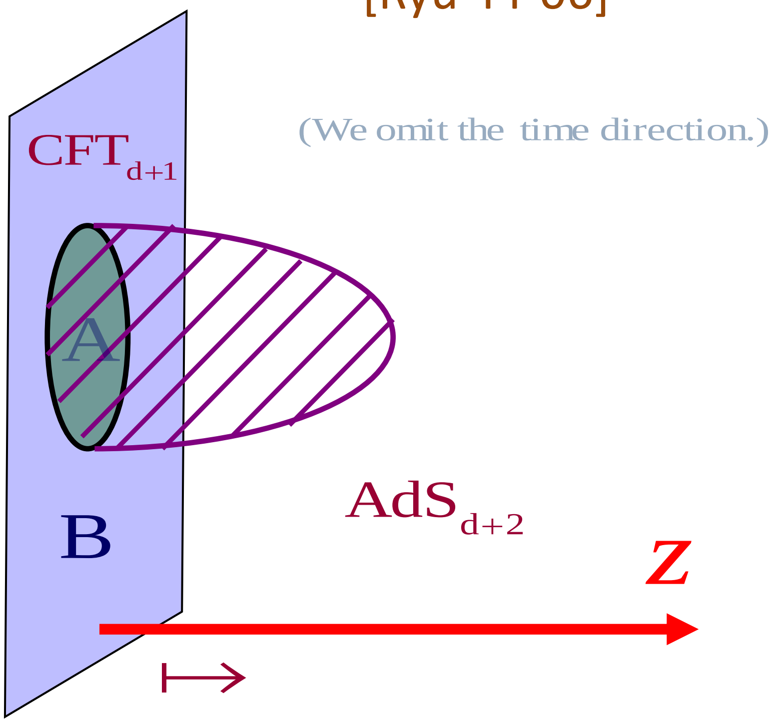
[EE = loop corrections to BH entropy, Susskind-Uglum 94,…]

② Holographic Entanglement Entropy

(2-1) Holographic Entanglement Entropy Formula

[Ryu-TT 06]

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$



γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \ \text{and} \ A \sim \gamma_A \ .$$

homologous

$$z > \varepsilon \quad \text{(UV cut off)}$$

$$ds^2_{AdS} = R^2_{AdS} \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2} .$$

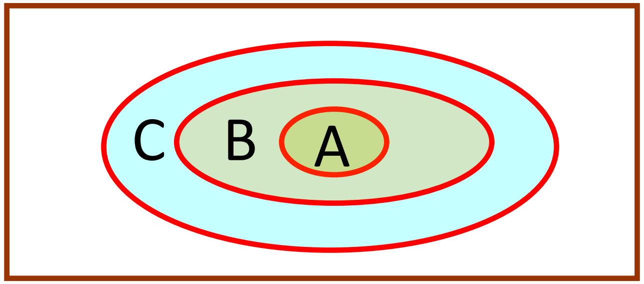
- In spite of a heuristic argument [Fursaev, 06] , there has been no complete proof. But, so many evidences and no counter examples.

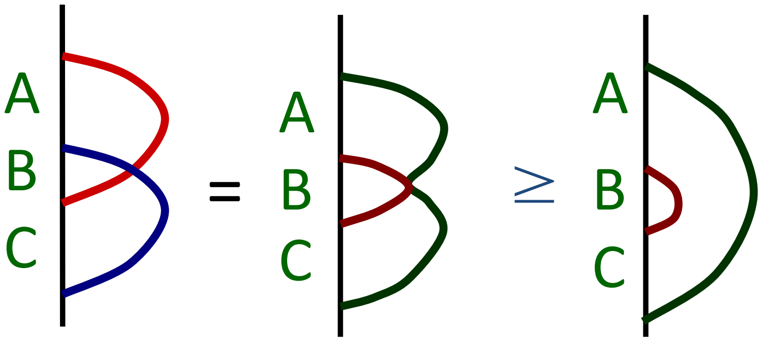
[A Partial List of Evidences]

- Area law follows straightforwardly [Ryu-TT 06]
- Agreements with analytical 2d CFT results for AdS3 [Ryu-TT 06]
- Holographic proof of strong subadditivity [Headrick-TT 07]
- Consistency of 2d CFT results for disconnected subsystems [Calabrese-Cardy-Tonni 09] with our holographic formula [Headrick 10]
- Agreement on the coefficient of log term in 4d CFT (~a+c) [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10]
- A direct proof when A = round ball [Casini-Hueta-Myers 11]
- Holographic proof of Cadney-Linden-Winter inequality [Hayden-Headrick-Maloney 11]

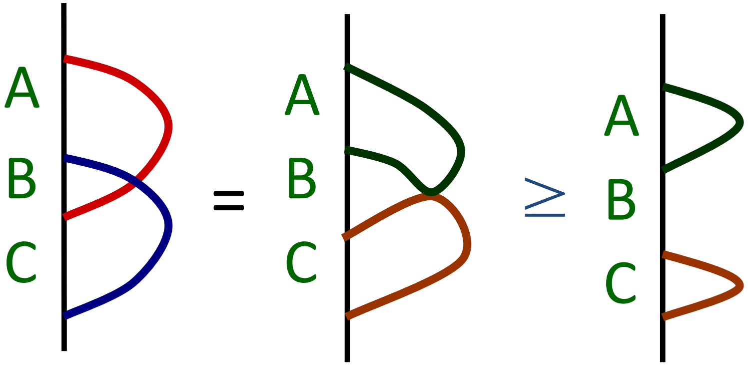
Holographic Proof of Strong Subadditivity [Headrick-TT 07]

We can easily derive the ***strong subadditivity***, which is known as the most important inequality satisfied by EE. [Lieb-Ruskai 73]





$$\Longrightarrow \quad S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_{B}$$

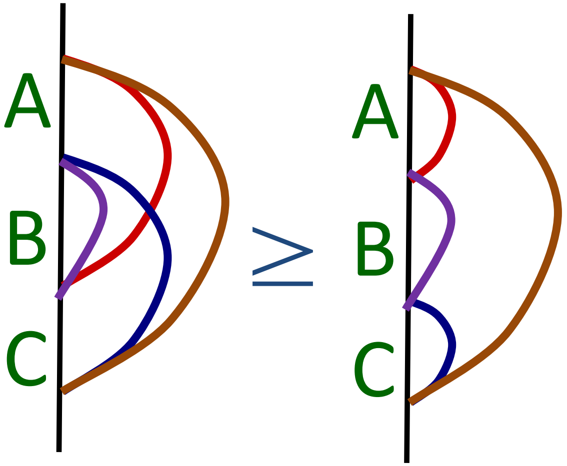


$$\Longrightarrow \quad S_{A+B} + S_{B+C} \geq S_{A} + S_{C}$$

Tripartite Information [Hayden-Headrick-Maloney 11]

Recently, the holographic entanglement entropy is shown to have a special property called ***monogamy***.

$$S_{AB}+S_{BC}+S_{AC}\geq S_A+S_B+S_C+S_{ABC}$$
$$\Leftrightarrow I(A:B)+I(A:C)\leq I(A:BC)$$



Comments:

(i) HEE argues that this is true for large N gauge theories.

(ii) This property is not always true for QFTs.

(iii) In 2+1 dim. mass gapped theories, this argues that the topological entanglement entropy is non-negative.

(iv) This property is also confirmed in time-dependent examples. [Balasubramanian-Bernamonti-Copland-Craps-Galli 11, Allais-Tonni 11]

- The area formula of HEE assumes the supergravity approximation (i.e. strongly coupled limit and large N limit).

⇒ The holographic formula is modified by higher derivatives. (deviations from strongly coupled limit, but still large N)

⇒ A precise formula was found for Lovelock gravities.

[Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11]

Ex. Gauss-Bonnet Gravity

$$S_{GBG} = -\frac{1}{16 G_N}\int dx^{d+2}\sqrt{g}\,[R - 2\Lambda + \lambda R_{AdS}^2 L_{GB}]$$

$$L_{GB} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 .$$

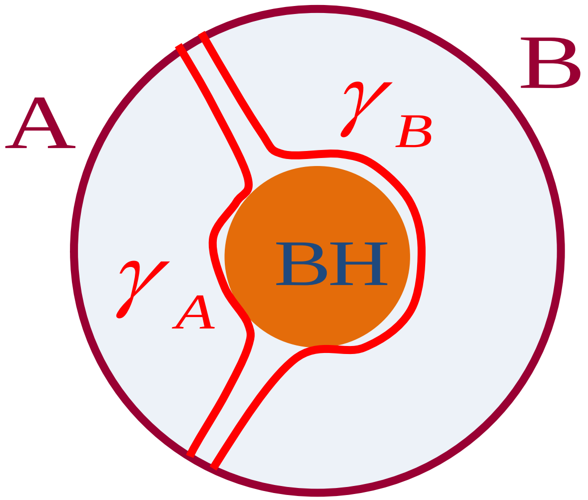
$$\Rightarrow \quad S_A = \underset{\gamma_A}{\mathrm{Min}}\left[\frac{1}{4G_N}\int_{\gamma_A} dx^d\sqrt{h}\,(1 + 2\lambda R_{AdS}^2 R)\right].$$

[But for general higher derivative theories, this is hard !]

Comments

- In the presence of a black hole horizon, the minimal surfaces typically wrap the horizon.

$\Rightarrow$ Reduced to the Bekenstein-Hawking entropy, consistently.



AdS BH $\Leftrightarrow$ Finite temp. CFT

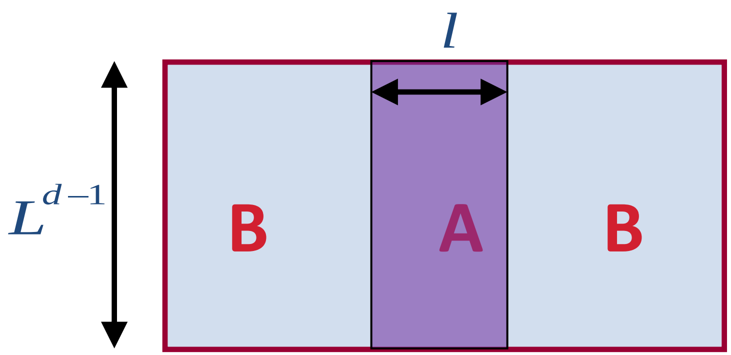
ρ_{tot} is not pure $\Leftrightarrow$ $S_A \neq S_B$.

- We need to replace minimal surfaces with extremal surfaces in the time-dependent spacetime. [Hubeny-Rangamani-TT 07]

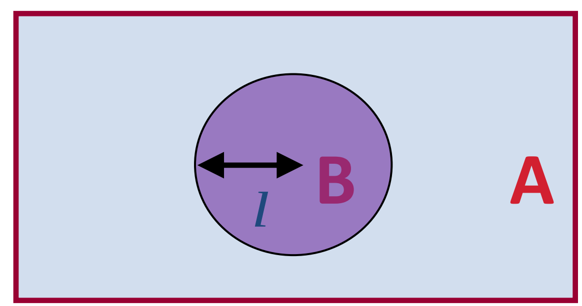
(2-2) Explicit Calculations of HEE

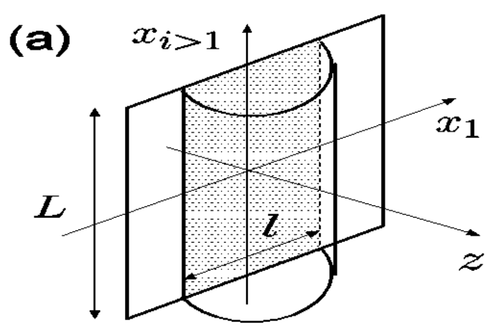
Two analytical examples of the subsystem A:

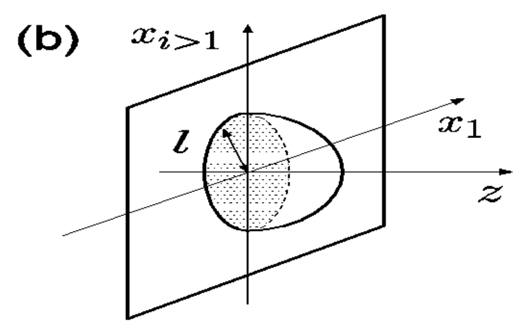
(a) Infinite strip



(b) Circular disk







Entanglement Entropy for (a) Infinite Strip from AdS

$$S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}}\left[\left(\frac{L}{\varepsilon}\right)^{d-1} - C\cdot\left(\frac{L}{l}\right)^{d-1}\right],$$

where $C = 2^{d-1}\pi^{d/2}\left(\Gamma\left(\frac{d+1}{2d}\right)\Big/\Gamma\left(\frac{1}{2d}\right)\right)^d.$

Area law divergence

This term is finite and does not depend on the UV cutoff.

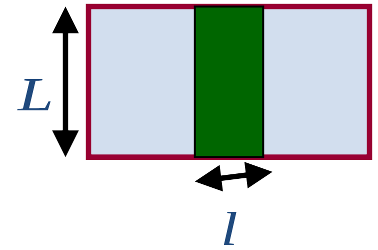
d=1 (i.e. AdS3) case:

$$S_A = \frac{R}{2G_N^{(3)}}\log\frac{l}{\varepsilon} = \frac{c}{3}\log\frac{l}{\varepsilon}.$$

Agrees with 2d CFT results
[Holzhey-Larsen-Wilczek 94 ;
Calabrese-Cardy 04]

Basic Example of AdS5/CFT4

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \Leftrightarrow N=4\ \mathrm{SU(N)}\ \ \mathrm{SYM}$$



$$\mathrm{CFT}: \quad S_A^{freeCFT} = K \cdot \frac{N^2 L^2}{\varepsilon^2} - 0.087 \cdot \frac{N^2 L^2}{l^2}.$$

$$\mathrm{Gravity}: \quad S_A^{AdS} = K' \cdot \frac{N^2 L^2}{\varepsilon^2} - 0.051 \cdot \frac{N^2 L^2}{l^2}.$$

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills.

[cf. 4/3 in thermal entropy, Gubser-Klebanov-Peet 96]

Entanglement Entropy for (b) Circular Disk from AdS

[Ryu-TT 06]

$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[p_1\left(\frac{l}{\varepsilon}\right)^{d-1} + p_3\left(\frac{l}{\varepsilon}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{\varepsilon}\right) + p_d & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{\varepsilon}\right)^2 + q\log\left(\frac{l}{\varepsilon}\right) & (\text{if } d = \text{odd}) \end{cases} \right],$$

where $p_1 = (d-1)^{-1}, p_3 = -(d-2)/[2(d-3)], \ldots.$

$\ldots\ldots \quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$

Area law divergence

A universal quantity in odd dimensional CFT ?
(Note: in mass gapped theories this is the topological EE.)

Conformal Anomaly (~central charge)

2d CFT   c/3・log(l/ε)

4d CFT   -4a・log(l/ε)

Comments: EE in Odd dim. CFT (e.g. 2+1 dim.)

Recently, [Casini-Hueta-Myers 11] proved for any odd dim. CFTs:

$$S_A = \log Z(S^{d+1})$$

if the subsystem A = d dim. round ball.

Also, in this setup, [Myers-Sinha 10] proved that the holography tells us that the finite part of S_A monotonically decreases under the RG flow.

⇒ A `c'-theorem for odd dim. QFTs

EE = `c-function' .

(2-3) HEE and Thermalization

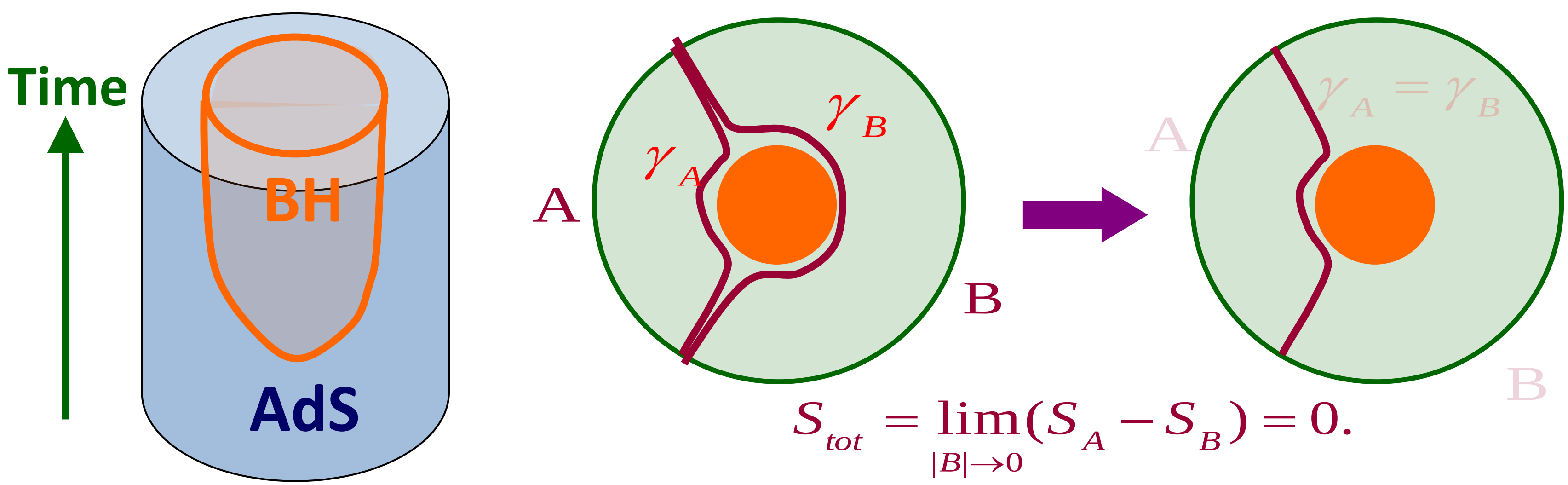
BH formation $\Leftrightarrow$ Thermalization [e.g. Chesler-Yaffe 08', Bhattacharyya-Minwalla 09', …]

ρ_{tot} is pure (unitary evolution)

$\Rightarrow S_{tot} = 0$, even if $\exists$ dynamical horizon. [Arrastia-Aparicio-Lopez 10, Ugajin-TT 10]

On the other hand, $S_A^{finite} \propto$ Size of BH $\neq 0$,

$\rightarrow$ EE = Coarse-grained entropy

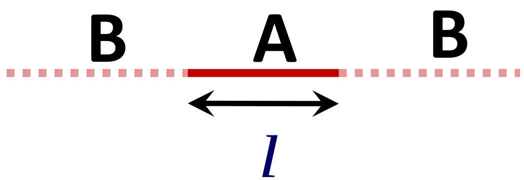


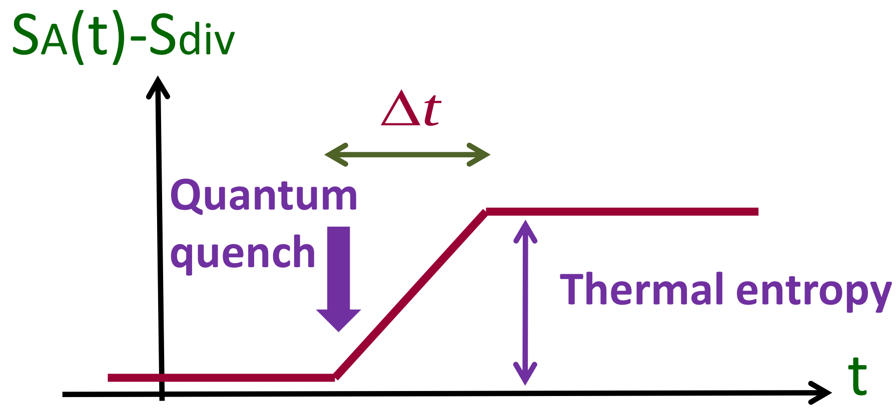
$$S_{tot} = \lim_{|B|\to 0}(S_A - S_B) = 0.$$

Time Evolutions of HEE under Quantum Quenches

In 1+1 dim. CFTs, we expect a linear growth of EE after a quantum quench.

[Calabrese-Cardy 05]





Causality → $\Delta t \approx \frac{l}{2}$

HEE reproduced the same result.

[Arrastia-Aparicio-Lopez 10]

In higher dim., the thermalization time depends on the shape of A.

HEE predicts: A = strip → $\Delta t > \frac{l}{2}$, A = round disk → $\Delta t \approx \frac{l}{2}$

[Albash-Johnson 10, Balasubramanian-Bernamonti-de Boer-Copland-Craps- Keski-Vakkuri-Müller-Schäfer-Shigemori-Staessens 10, 11]

③ Can HEE probe Fermi Surfaces ?

[Ogawa-Ugajin-TT, in preparation]

(3-1) Logarithmic Violation of Area Law

In d dim. lattice models that the area law of EE is violated logarithmically in free fermion theories. [Wolf 05, Gioev-Klich 05]

$$S_A \sim L^{d-1} \log L, \qquad (L = size\ of\ A).$$

Comments:

(i) This property can be understood from the logarithmic EE in 2D CFT, which approximates the radial excitations of fermi surface.

(ii) It is natural to expect that this property is true for non-Fermi liquids. [Swingle 09,10, Zhang-Grover-Vishwanath 11 etc.]

Note in this lattice calculation assumes

$$\varepsilon^{-1} \ \text{(UV cut off)} \sim k_F \ .$$

Instead, in our holographic context which corresponds to a continuous limit, we are interested in the case $\varepsilon^{-1} >> k_F$.

In this case, we expect

$$S_A = (div.) + \eta \cdot (L \cdot k_F)^d \log L k_F + \cdots.$$

Below we would like to see if we can realize this behavior in HEE.

We assume that all physical quantities can be calculable in the classical gravity limit ($\Leftrightarrow \exists O(N^2)$ ***Fermi surfaces).***

(3-2) Holographic Construction

The metric ansatz: $ds^2 = \frac{R_{AdS}^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + dx^2 + dy^2\right).$

$$\text{Asymp. AdS} \Rightarrow f(0) = g(0) = 1.$$

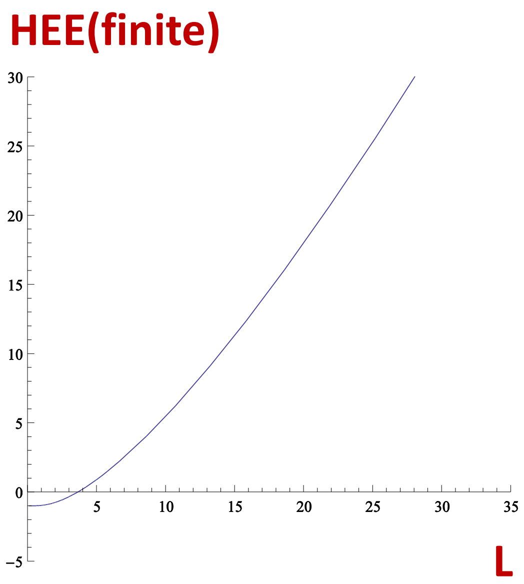
(Below we work d=2 i.e. AdS4/CFT3 setup.)

The logarithmical behavior of EE occurs iff

$$g(z) \rightarrow \left(\frac{z}{z_F}\right)^2 \qquad (z \rightarrow \infty).$$

Note: f(z) does not affect the HEE.

z_F^{-1} is dual to the fermi energy.



(3-3) Null Energy Condition

To have a sensible holographic dual, a necessary condition is known as the null energy condition:

$$T_{\mu\nu}N^{\mu}N^{\nu} \geq 0 \qquad \text{for any null vector } N^{\mu}.$$

In the IR region, the null energy condition argues

$$g(z) \propto z^{2}, \qquad f(z) \propto z^{-2m} \quad \Rightarrow \quad m \geq 2.$$

The specific heat behaves like

$$C \propto T^{\alpha} \quad \text{with} \quad \alpha \leq \frac{2}{3}.$$

Notice that this excludes standard Landau fermi liquids.

In summary, we find that ***classical gravity duals only allow non-fermi liquids***.

Comments:

(i) Our definition of classical gravity duals is so restrictive that it does not include either the emergent AdS2 geometry [Faulkner-Liu-McGreevy-Vegh 09, Cubrovic-Zaanen-Schalm 09] nor the electron stars (or Lifshitz) [Hartnoll-Polchinski-Silverstein-Tong 09, Hartnoll-Tavanfar 10] .

(ii) More generally, the background with $g(z) \propto z^{2n}$ leads to

$$S_A^{finite} \propto L^{\frac{2n}{n+1}}.$$

(iii) We can embed this background in an effective gravity theory:

$$S_{EMS} = \frac{1}{16 G_N} \int dx^{d+2} \sqrt{-g} \left[R - 2\Lambda - W(\phi) F_{\mu\nu} F^{\mu\nu} - \partial_\mu \phi \partial^\mu \phi - V(\phi) \right].$$

if W and V behave in the large ϕ limit as follows

$$V(\phi) + 2\Lambda \approx -\frac{(p^2 + 12p + 32)}{4 R_{AdS}^2} \cdot e^{-\sqrt{\frac{2}{p-2}}\phi},$$

$$W(\phi) \approx \frac{8 A^2}{z_F^2\, p(8+p) R^2} e^{3\sqrt{\frac{2}{(p-2)}}\phi},$$

$$\Rightarrow \ f(z) \propto z^{-p}, \quad g(z) \propto z^2, \qquad (p > 2).$$

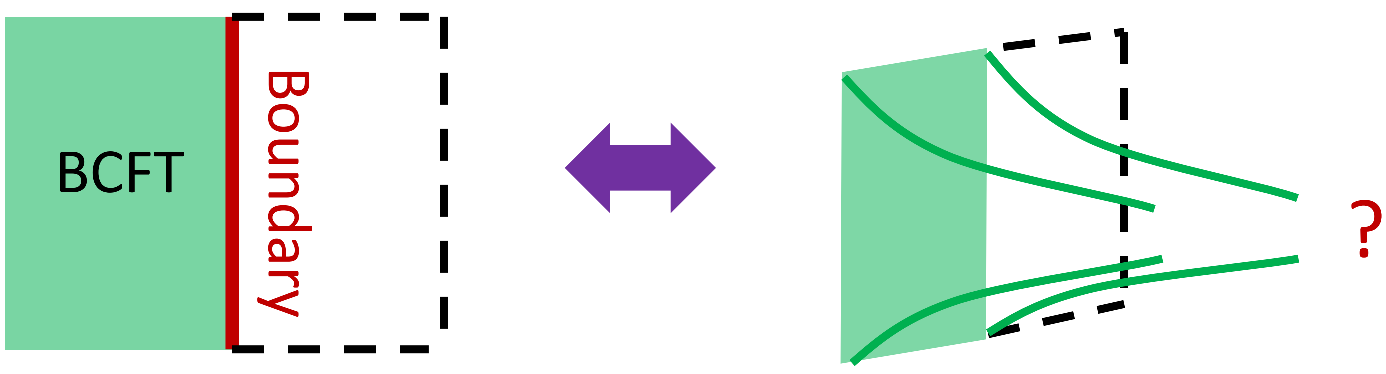
④ HEE and Holographic Dual of BCFT

(4-1) AdS/BCFT

What is a holographic dual of CFT on a manifold with Boundary (BCFT) ?

$$\mathrm{CFT}_d:\ SO(d,2) \quad \Leftrightarrow \quad \mathrm{AdS}_{d+1}$$

$$\mathrm{BCFT}_d:\ SO(d-1,2) \quad \Leftrightarrow \quad \mathrm{AdS}_d$$



[Earlier studies: Karch-Randall 00 (BCFT,DCFT),…
Bak-Gutperle-Hirano 03, Clark-Freedman-Karch-Schnabl 04 (Janus CFT)]

AdS/BCFT Proposal [Fujita-Tonni-TT 11]

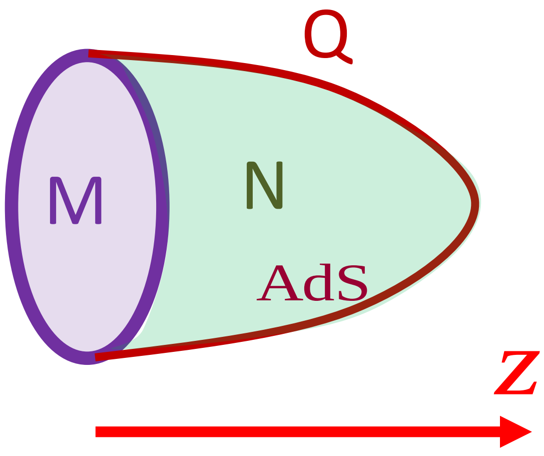
In addition to the standard AdS boundary M, we include an extra boundary Q, such that ∂Q=∂M.

$$I_E = -\frac{1}{16\pi G_N}\int_N \sqrt{g}(R-2\Lambda-L_{matter}) - \frac{1}{8\pi G_N}\int_Q \sqrt{h}(K-L^Q_{matter}).$$

EOM at boundary leads to the Neumann b.c. on Q :

$$K_{ab} - Kh_{ab} = 8\pi G_N T^Q_{ab} \quad .$$

Conformal inv. ⇒ $T^Q_{ab} = -Th_{ab}.$



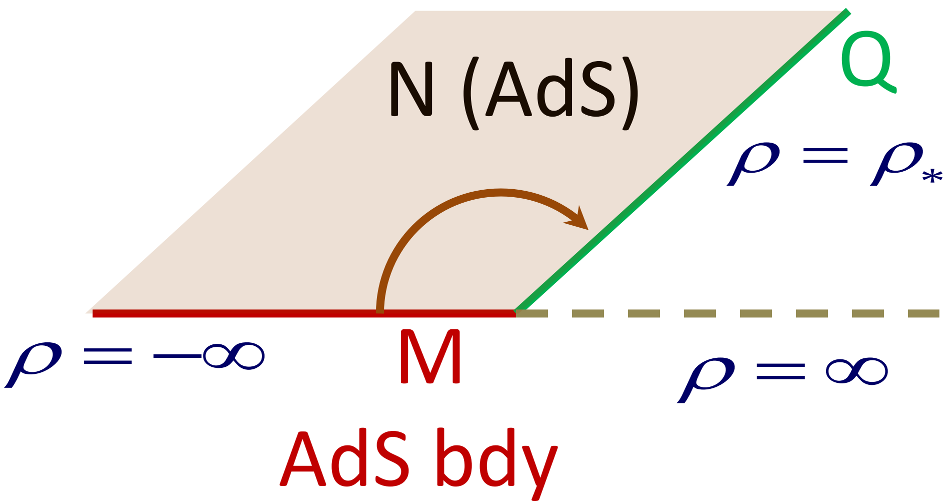
(4-2) Simplest Example

Consider the AdS slice metric:

$$ds^2_{AdS(d+1)} = d\rho^2 + \cosh^2(\rho/R)ds^2_{AdS(d)} \quad .$$

Restricting the values of ρ to $-\infty < \rho < \rho_*$ solves the boundary condition with

$$T = \frac{d-1}{R}\tanh\frac{\rho_*}{R} \quad .$$



(4-3) Holographic Boundary Entropy

The boundary entropy [Affleck-Ludwig 91]

Sbdy measures the degrees of freedom at the boundary.

The g-theorem:

Sbdy monotonically decreases under the RG flow in CFT.

[proved by Friedan -Konechny 04]

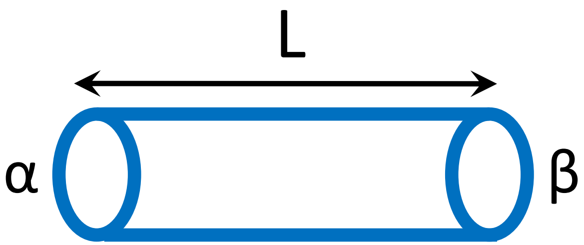
Definition 1 (Disk Amplitude)

It is simply defined from the disk amplitude

$$S_{bdy(\alpha)} = \log g_\alpha \,, \qquad g_\alpha \equiv \langle 0 | B_\alpha \rangle.$$

Definition 2 (Cylinder Amplitude)

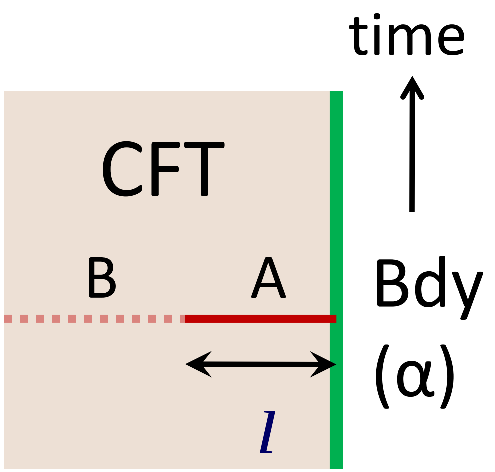
$$Z^{cylinder}_{(\alpha,\beta)} = \langle B_\alpha | e^{-HL} | B_\beta \rangle \underset{L\to\infty}{\approx} \underbrace{g_\alpha g_\beta}_{\text{Boundary Part}} \underbrace{e^{-E_0 L}}_{\text{Bulk Part}}.$$



Definition 3 (Entanglement Entropy)

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A],$$

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad .$$



In 2D BCFT, the EE generally behaves like

$$S_A = \underbrace{\frac{c}{6}\log\frac{l}{\varepsilon}}_{\text{Bulk Part}} + \underbrace{\log g_\alpha}_{\text{Boundary Part}} \quad .$$

[Calabrese-Cardy 2004]

In our setup, HEE can be found as follows

$$S_A = \frac{\text{Length}}{4G_N} = \frac{1}{4G_N}\int_{-\infty}^{\rho_*} d\rho = \frac{c}{6}\log\frac{l}{\varepsilon} + \underline{\frac{\rho_*}{4G_N}} \quad .$$

Boundary Entropy

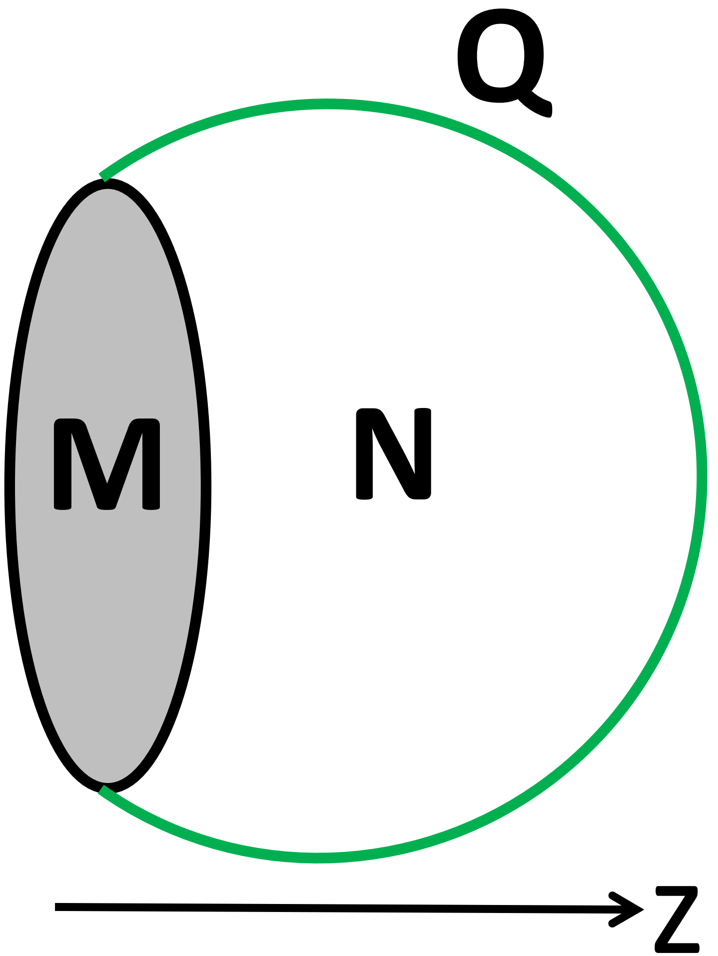
[Earlier calculations: Azeyanagi-Karch-Thompson-TT 07 (Non-SUSY Janus),
Chiodaroli-Gutperle-Hung, 10 (SUSY Janus)]

Also $S_{bdy} = \rho_* / 4G_N$ can be confirmed in other two definitions.

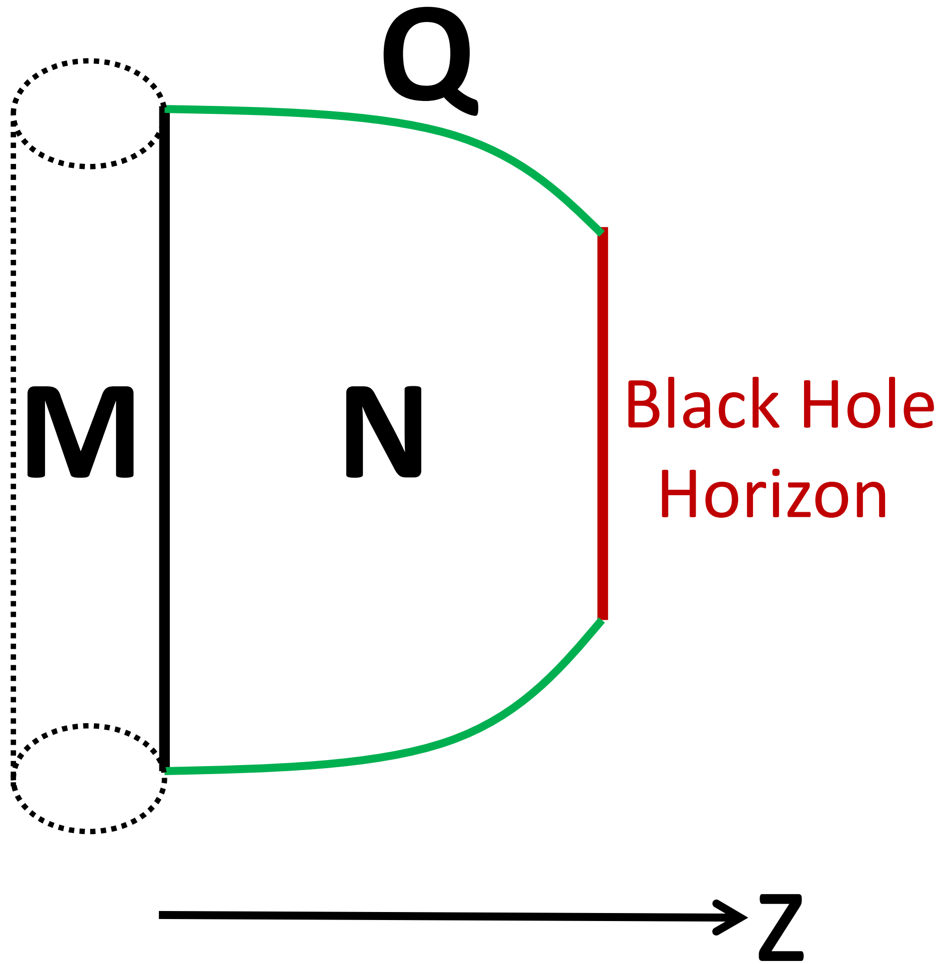
$$I_{Disk} = \frac{R}{4G_N}\left(\frac{r^2}{2\varepsilon^2} + \frac{r\sinh(\rho_*/R)}{\varepsilon} + \log\frac{\varepsilon}{r} - \underline{\frac{\rho_*}{R}} - \frac{1}{2}\right).$$

$$I_{Cylinder} = \frac{\pi}{3} c \cdot l \cdot T_{BH} + \underline{\frac{\rho_*}{2G_N}}.$$

Holographic Dual of Disk



Holographic Dual of Cylinder



(4-4) Holographic g-Theorem

Consider the surface Q defined by $x = x(z)$ in the Poincare metric

$$ds^2 = R^2\left(\frac{dz^2 - dt^2 + dx^2 + (d\vec{w})^2}{z^2}\right).$$

We impose the null energy condition for the boundary matter

i.e. $T^Q_{ab} N^a N^b \geq 0$ for any null vector N^a.

[cf. Hol. C-theorem: Freedman-Gubser-Pilch-Warner 1999, Myers-Sinha 2010]

For the null vector, $N^t = 1, \quad N^z = 1/\sqrt{1+(x')^2}\ , \quad N^x = x'/\sqrt{1+(x')^2}\ ,$

we find the constraint

$$(K_{ab} - Kh_{ab})N^a N^b = -\frac{R \cdot x''}{z(1+(x')^2)^{3/2}} \geq 0.$$

Thus we simply get $x''(z) \leq 0$ from the null energy condition.

Define the holographic g-function:

$$\log g(z) = \frac{R^{d-1}}{4G_N} \cdot \mathrm{Arcsinh}\left(\frac{x(z)}{z}\right) = \frac{R^{d-2}}{4G_N} \cdot \rho_*(z).$$

Then we find $$\frac{\partial \log g(z)}{\partial z} = \frac{x'(z)z - x(z)}{\sqrt{z^2 + x(z)^2}} \leq 0 \quad ,$$

because $(x'z - x)' = x''z \leq 0$.

For d=2, at fixed points $\log g(z)$ agrees with the boundary entropy.

For any d, $\rho_*(z)$ is a monotonically decreasing function w.r.t. z.

This is our holographic g-theorem !

Example: AdS4/BCFT3

In this case, we obtain

$$I_E = \frac{R^2}{2G_N}\left[\frac{\pi}{2} + \arctan\left(\sinh\frac{\rho_*}{R}\right) - \frac{1}{24}\sinh\frac{3\rho_*}{R}\right.$$
$$\left. - \underbrace{\left(\sinh\frac{\rho_*}{R}\right)\log r_B} + \left(\log\cosh\frac{\rho_*}{R} - \frac{33}{24} - \log 2\right)\sinh\frac{\rho_*}{R}\right].$$

Conformal anomaly
in odd dim. CFT ?

➡ This should come from the 2 dim. boundary !

Boundary central charge

As the usual central charge in 2 dim. CFT, we can define a boundary central charge in BCFT3 as follows:

$$r_B \frac{\partial \log Z_{Ball}}{\partial r_B} = -\frac{1}{2\pi} \left\langle \int_\Sigma dx^2 \sqrt{g_b} T_\mu^\mu \right\rangle = \frac{c_{bdy}}{6} \chi(\Sigma).$$

In our holographic calculation, we obtain

$$c_{bdy} = \frac{3R^2}{2G_N} \sinh \frac{\rho_*}{R}.$$

Our holographic g-theorem leads to a c-theorem for c_{bdy} .

➡ Our conjecture: this is true for all BCFT3.

⑤ Conclusions

- The entangled entropy (EE) is a useful bridge between gravity (string theory) and cond-mat physics.

Gravity	⇔	Entanglement	⇔	Cond-mat. systems $\lvert\Psi\rangle$
$g_{\mu\nu}$		$S_A \approx \mathrm{Area}$		

- Classical gravity duals $\Rightarrow$ Non-fermi liquids

- AdS/BCFT can be useful to probe the boundary physics.
  The boundary central charge etc.

Future Problems

- Proof of HEE ?
- Complete Higher derivative corrections to HEE ?
- 1/N corrections to HEE ?
- More on HEE and Fermi Liquids ?
- HEE for non-AdS spacetimes ?
- What is an analogue of the Einstein eq. for HEE ?
- .
- .
- Superconductors in AdS/BCFT ?
- Topological Insulators and AdS/BCFT ?
- .