Micro-physics of stellar interiors

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How does micro-physics enter?

Forward problem

- Nobody would do it in this way (!), **but …**

$$\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\frac{1}{\rho}\nabla p - \nabla\phi$$

$$\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] = 0$$

$$\left[\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s\right] = -\frac{1}{T}\epsilon_{\mathrm{nucl}} - \frac{1}{\rho T}\,\mathrm{div}\mathbf{F}$$

$$\Delta\phi = 4\pi G\rho$$

… let’s illustrate stellar evolution and pulsation by using the same equations, both for the equilibrium and the oscillation problem

First: the equilibrium solution

- **Put all red stuff to 0!**
- **Obtain the usual equilibrium equations (for simplicity assume (i) spherical symmetry; (ii) no convection)**

$$\frac{dp}{dr} = -\frac{GM_r\rho}{r^2}$$

$$L_r = 4\pi r^2 F$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon_{\mathrm{nuc}}$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{64\pi\sigma r^2 T^3}$$

Material properties are mandatory!

- **So far all equations the same for all stars**
- **Richness of variety of stars only enters with the constituent equations**

$$\rho = \rho(T, p, \mathbf{X})$$

$$\kappa = \kappa(T, \rho, \mathbf{X})$$

$$\epsilon = \epsilon_{\mathrm{nuc}}(T, \rho, \mathbf{X})$$

$$s = s(T, \rho, \mathbf{X})$$

$$\mathbf{F} = \mathbf{K}(\nabla T; T, \rho, \mathbf{X})$$

(Diffusive radiation : $F = -\frac{16\sigma T^4}{3\kappa\rho}\nabla T$)

Microphysics is not only for the variety of stars:

Material properties are also essential for stellar (parametric) evolution!

This is a seismology conference, therefore go to next step:

Second: oscillation equations

- Once equilibrium model found, solve red part by putting black part = 0

$$\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\frac{1}{\rho}\nabla p - \nabla\phi = 0$$

$$\left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{v})\right] = 0$$

$$\left[\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s\right] = -\frac{1}{T}\epsilon_{nuc} - \frac{1}{\rho T}\mathrm{div}\mathbf{F} = 0$$

- If appropriate, use linearization, spherical harmonics, adiabatic $\Rightarrow \nu_{n\ell}$

Result: model frequencies

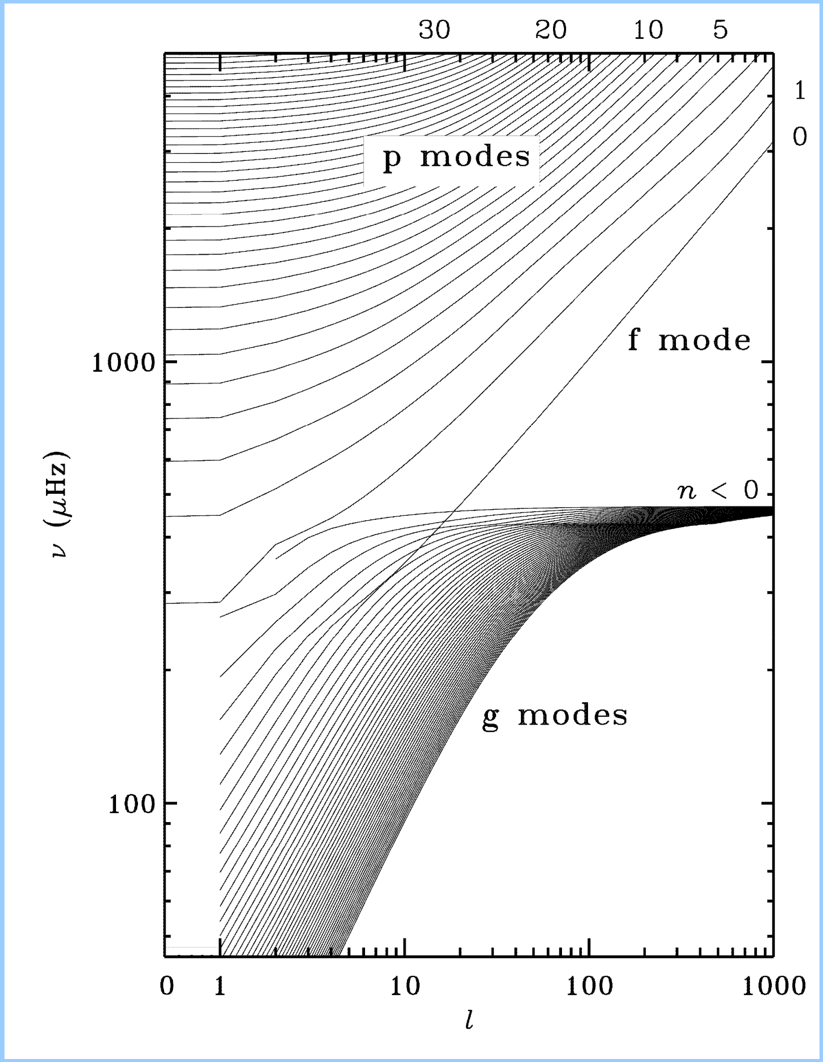
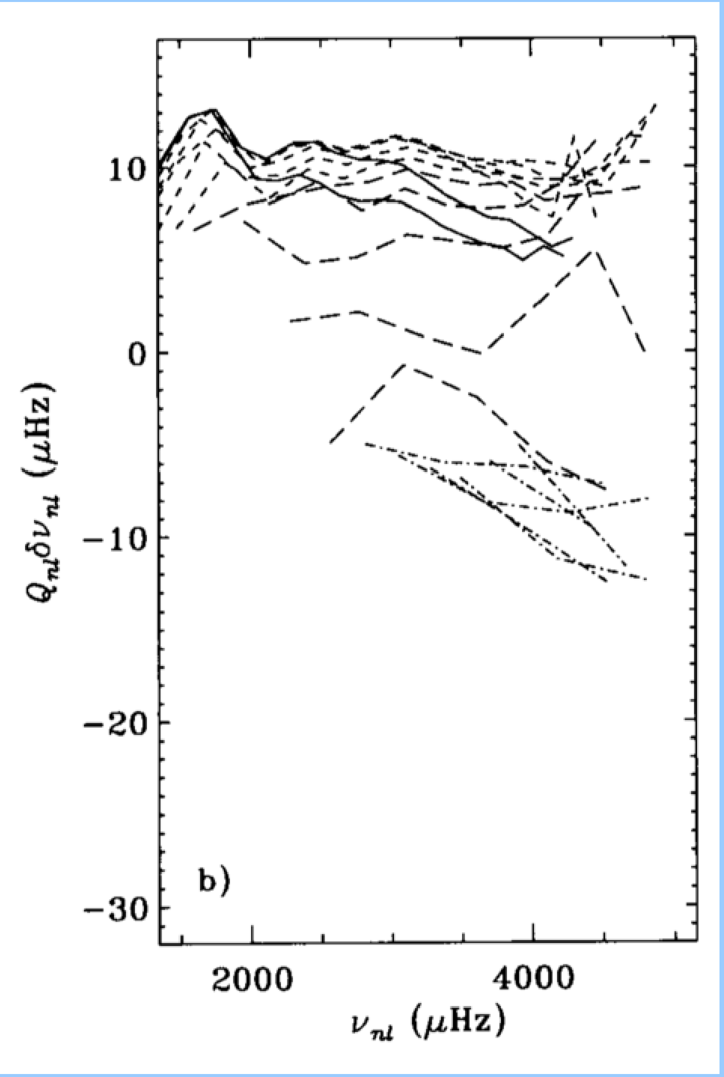


Figure:

J. Christensen-Dalsgaard

Allow “O-C Diagrams”

- O = “Observation”
- C = “Computation”



(Q_{nl} is a scale factor. Details see: Christensen-Dalsgaard & Däppen 1992, A&A Rev. **4**, 267)

Tools to test physics

Since the equation of state is the most basic ingredient, let’s go into more detail!

In the Sun and solar-like stars: fortunate situation

- In the convection zone, the stratification is (largely) adiabatic, its structure is mainly determined by thermodynamics.
- Little “contamination” from opacity
- Helioseismology can probe locally

  (to some degree also asteroseismology, e.g., in abundance determination)

Elementary stellar thermodynamics

- In stellar physics, before 1975, normal (non-degenerate) stars were successfully modeled by

$$pV = (\Sigma_i N_i)kT$$

- Provided: N_i from Saha equation:

  Good to 90% accuracy!

In early helioseismology

- From 1975-1985, more refined equations of state, mainly

- Detailed chemical composition
- Fermi-Dirac electrons
- Debye-Hückel screening

good to 95-99% accuracy!

The **dominant** non-ideal effect turned out to be...**DH!**

- Screened Coulomb potential

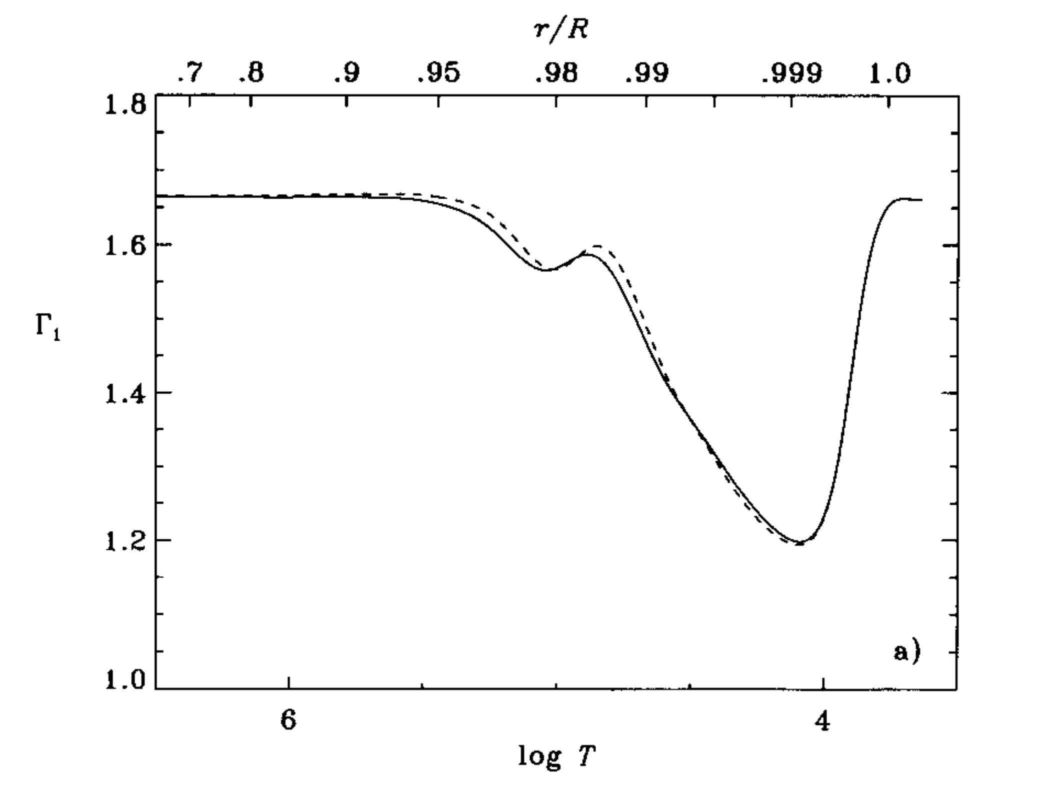
$$V_D = \frac{Q_1 Q_2}{r} \exp^{-r/r_D}$$

(=the [static] Debye-Hückel approximation)

Impact: two similar solar models...

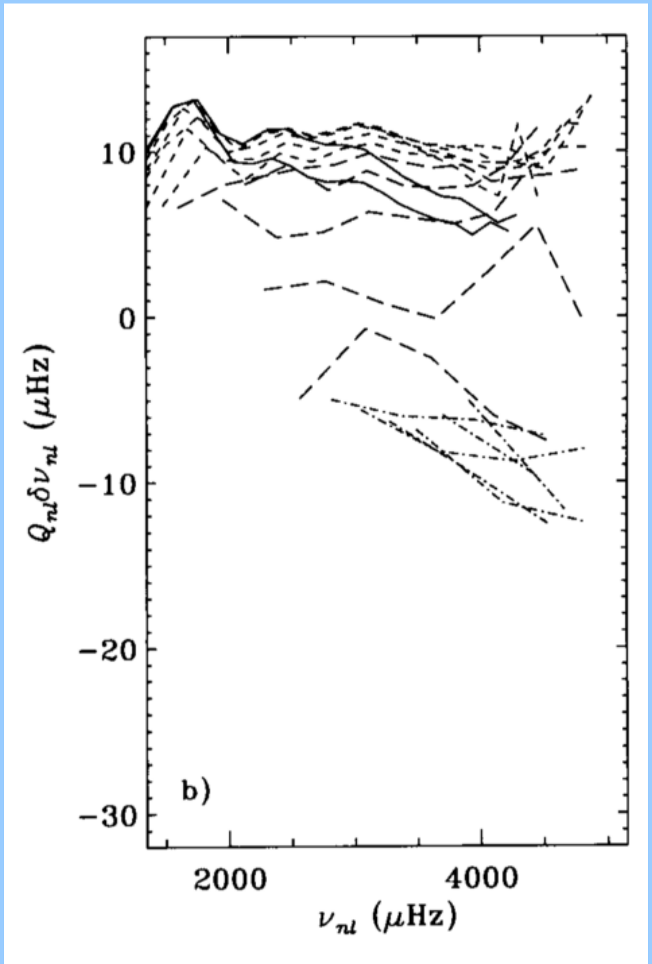
- Identical models, except for their equations of state. One is with Debye-Hückel screening, one without. Their adiabatic exponents are:

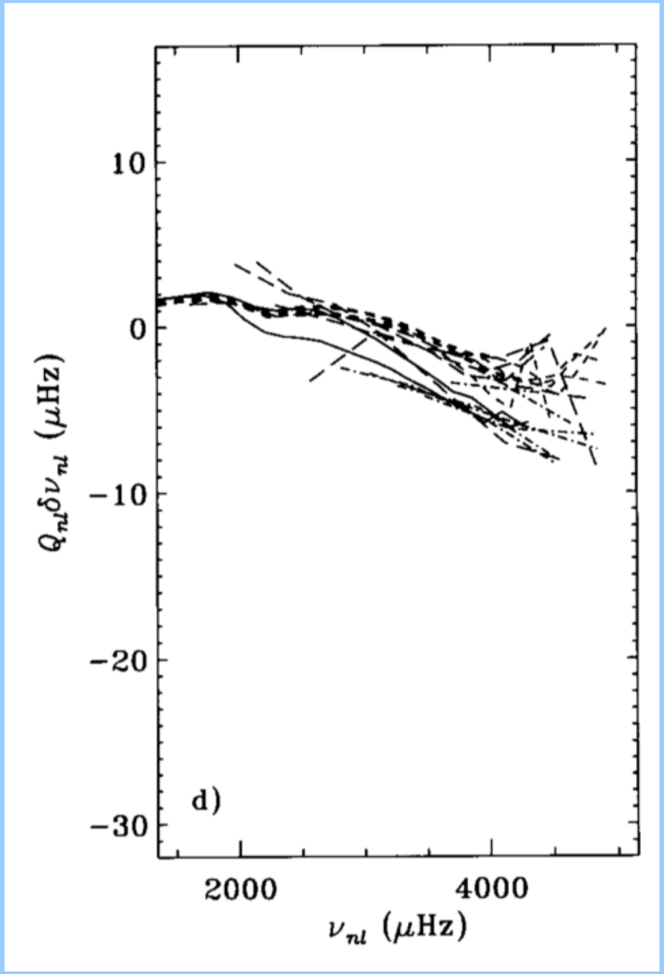
Dashed: with screening
Solid: without screening



From: Christensen-Dalsgaard & Däppen 1992, A&A Rev. **4**, 267

...and their O-C diagrams





without screening **with screening**

from: Christensen-Dalsgaard & Däppen 1992, A&A Rev. **4**, 267

To my chagrin, the physicists were not overwhelmed!

A tangent -
Seismic abundance determination

Acoustic modes (largely) governed by the adiabatic sound speed

$$c_{\mathsf{ad}}^2 = \gamma_1 \frac{p}{\rho} \propto \frac{T}{\mu}$$

$$\gamma_1 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

(often denoted $\mathit{\Gamma}_1$ to Douglas Gough's chagrin!)

Intuitive reasoning

$$\gamma_1 \approx \gamma = \frac{c_p}{c_V}$$

$$c_p - c_V \approx 1$$ (1st Law; molar units)

c_p , c_V **each grow in ionization zones (analogous to latent heat!)**

$$\gamma = \frac{c_p}{c_V}$$

drops in ionization zones: down from 5/3 to about 1.2 for a hydrogen plasma

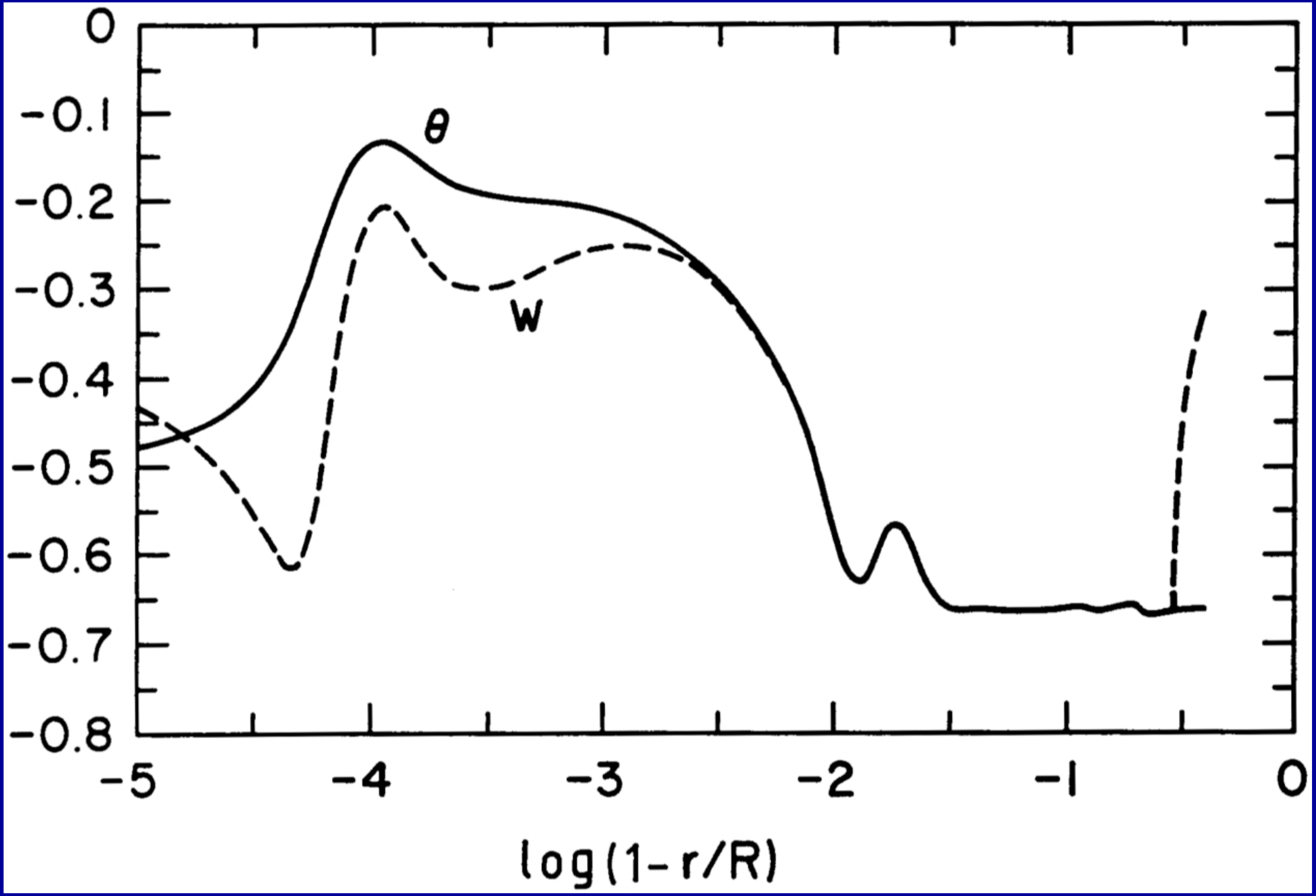
Supertool for abundances

Douglas Gough (Catania 1983) applies adiabatic constraint to the hydrostatic equation:

$$W = \frac{\mathrm{d}c^2}{\mathrm{d}r}\frac{1}{g(r)} = \frac{1-\gamma_1-\gamma_{1,\rho}}{1-\gamma_{1,c^2}} = \Theta$$

What would that look like inside the Sun?

The helium bump!



Gough and WD (Liège 1984)

Result of the helium hump method (and similar techniques, e.g., by Dziembowski, Thompson, Vorontsov, Antia, Basu…)

$Y = 0.24 \ldots 0.25$

inside the solar convection zone

Whatever the value of Y thus obtained, the result is entangled with the uncertainty in the equation of state!

Practical tools for modelers I

Phenomenological options

- *e.g.* Eggleton, Faulkner & Flannery (EFF) (A&A 1973)

- On the plus side
  Useful computational tool, remarkably accurate
  Broad range of applicability
  Even includes relativistic electrons

- On the minus side
  Misses dominant non-ideal term (later fixed, see next)
  Ad-hoc pressure ionization term
  Can lead to artificial phase transitions

There are improved descendants of EFF, great tools for modelers!

- ... CEFF

... by J. Christensen-Dalsgaard and WD (1991), who added a Coulomb (C) term, following progress in helioseismology

- ... SIREFF

... by Swenson, Irwin and Rogers (1996) who added excited-states terms, later extended into:

- ... FreeEOS

... by Irwin (excellent tool, freely downloadable from [http://freeeos.sourceforge.net/], newest version: 2008)

WARNING:

BORING STUFF!

(even more of it in the Appendix, if interested…)

Two main approaches: introduction

- Free-energy minimization
  chemical picture
  intuitive, but highly practical

- Grand-canonical expansions
  Physical picture
  systematic method for non-ideal corrections

Chemical picture

- Treat compounds as fundamental entities
- Reactions $(\mathrm{H} \leftrightarrow \mathrm{H}^+ + \mathrm{e}^- \ , \ \text{etc.})$
- Constraints $(N_\mathrm{H} + N_\mathrm{p} = \text{const.} \ , \ \text{etc.})$
- Minimize $F(T, V, N_\mathrm{H}, N_\mathrm{p}, N_{\mathrm{e}^-}, \ldots)$

  !!!subject to constraints!!!
- In practice, cook a free energy (intuition!)

$$F_\mathrm{tot} = F_\mathrm{nuc} + F_\mathrm{e} + F_\mathrm{interactions} + \ldots$$

- Consistent (formally!) $p = -(\frac{\partial F}{\partial V})_T \ , \ \text{etc.}$

Example: MHD

- **Fairly conventional realization (chemical picture)**
- **Key ingredient: occupation probabilities**

Hummer, D.G. & Mihalas, D.M. 1988, *ApJ* **331**, 794;

Mihalas, D.M., Däppen, W. & Hummer, D.G. 1988, *ApJ* **331**, 815

Däppen,W., Mihalas, D.M., Hummer, D.G. & Mihalas, B.W. 1988, *ApJ* **332**, 261

$$Z_{jk}^{\text{int}} = \sum_i w_{ijk}\ g_{ijk} \text{exp}\left(-\beta E_{ijk}\right)$$

$$(w_{ijk})_{\text{neutral}} = \text{exp}\left[-(4\pi/3V)\sum_{j',k'} N_{j'k'}(r_{ijk} + r_{1j'k'})^3\right]$$

$$(w_{ijk})_{\text{charged}} = \text{exp}\left\{-\left(\frac{4\pi}{3V}\right) 16\left[\frac{(Z_{jk}+1)^{1/2}e^2}{K_{ijk}^{1/2}\chi_{ijk}}\right]^3 \sum_{j',k'} N_{j'k'} Z_{j'k'}^{3/2}\right\}$$

Physical picture

- Only electrons and nuclei are fundamental
- No reactions
- Quantum mechanics and statistical mechanics dealt with simultaneously
- Nothing to minimize
- Consistent (not just formally!)

Example: OPAL/ACTEX

- **First successful stellar modeling with an equation of state in the physical picture (LLNL)**

  Rogers, F.J. 1986, *ApJ* **310**, 723;
  Rogers, F.J., Swenson, F.J. & Iglesias, C.A. 1996, *ApJ* **456**, 902
  Rogers, F.J. & Nayfonov, A. 2002, *ApJ* **576**, 1064

- **Key points: systematic expansions**
  (z = activity)

$$\frac{p}{k_B T} = z + z^2 b_2 + z^3 b_3 + \ldots \; ; \quad \rho = \frac{z}{k_B T}\left(\frac{\partial p}{\partial z}\right)$$

Planck-Larkin Partition Function

$$\mathrm{PLPF} = \sum_{nl} (2l+1) \left[\exp\left(-\frac{E_{nl}}{kT}\right) - \; 1 \; + \frac{E_{nl}}{kT} \right]$$

Domain of validity of OPAL/ACTEX equation of state for stellar models

> 0.1 solar masses

MHD vs. OPAL in the Sun

c^2 Inversions (numerical; Sun-Model)

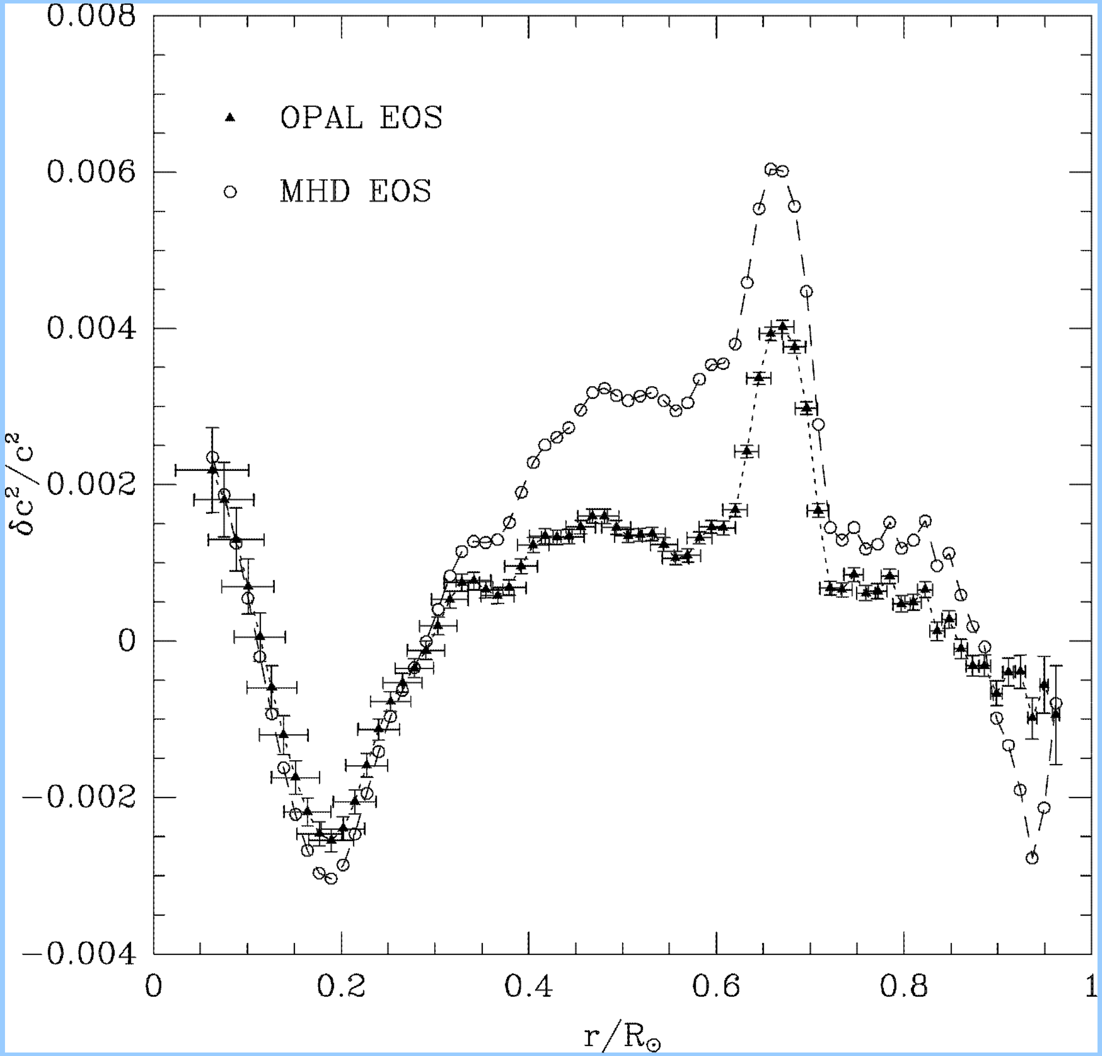
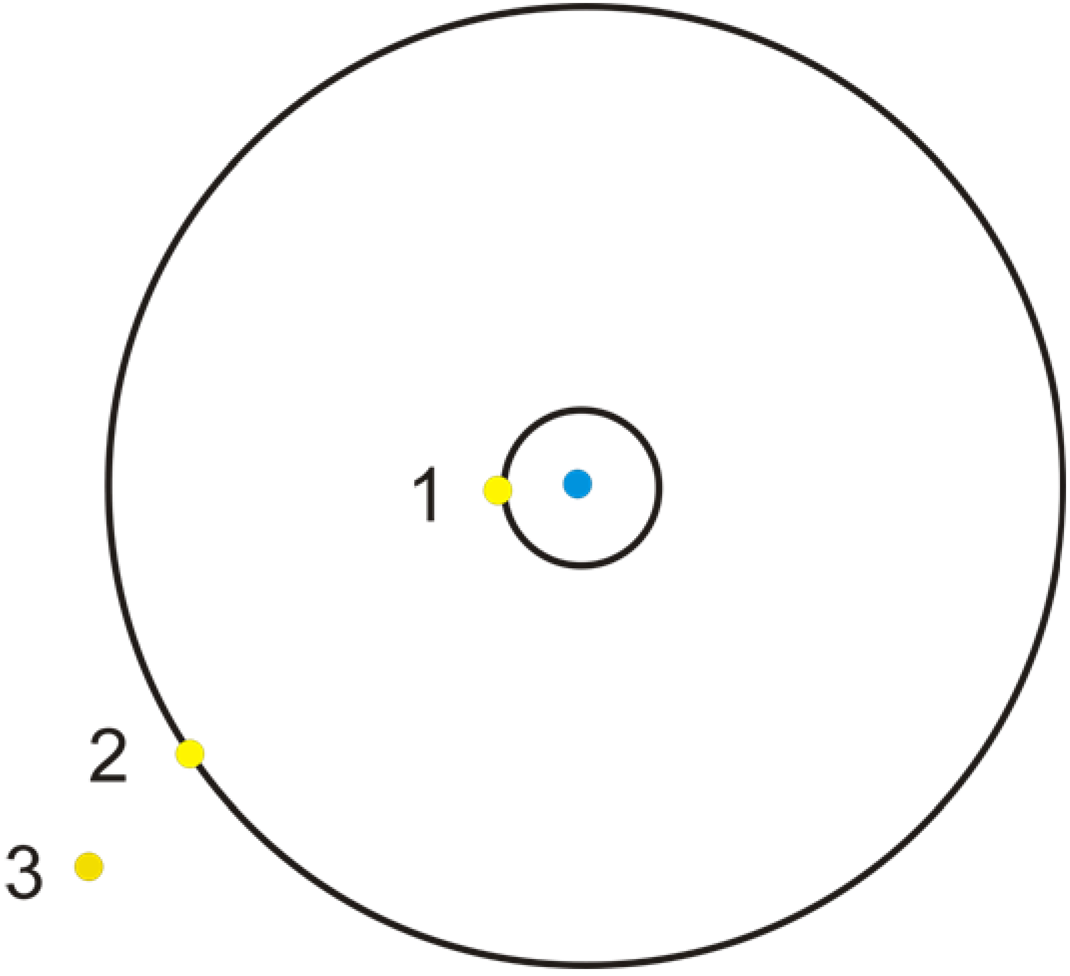


Figure from:
S. **Basu**

OPAL fares better than MHD...

- Why? Likely answer:
- There is no PLPF in MHD
- There are no scattering states in MHD
- Open question: is it fundamentally impossible to find PLPF entirely from within the chemical picture?

More precisely:
consider 3 states of a H atom…



1: ground state
2: weakly bound
3: continuum

force law for electrons (C=Coulomb, F=free)

	CEFF	MHD	OPAL
1: ground state	C	C	C
2: weakly bound	F	C	C
3: continuum	F	F	C

Treatment of excited states:

OPAL good, CEFF at least consistent, MHD inconsistent
(as are all chemical picture formalisms with excited states but without PLPF)

FYI: more of it in an appendix!

Further possibilities towards very high precision…

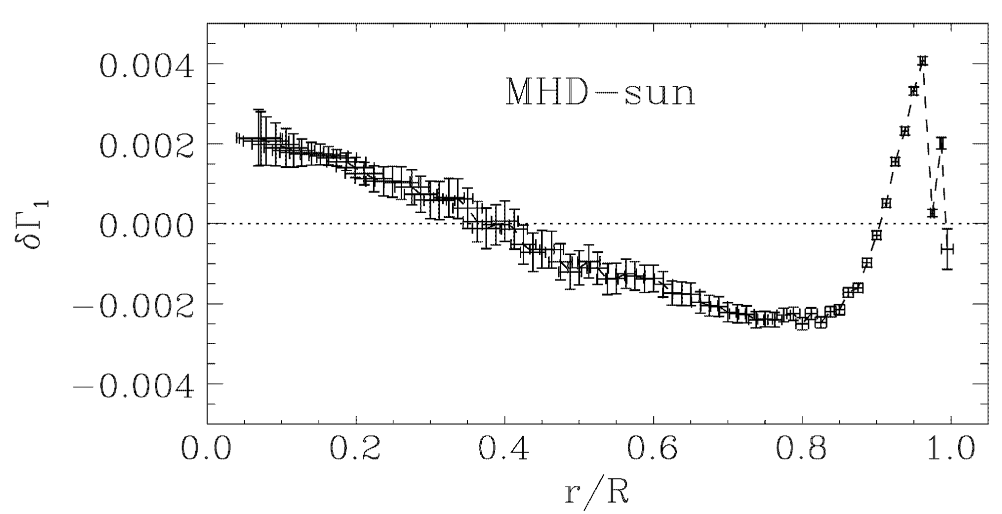
To illustrate: a small effect - relativistic electrons in the Sun

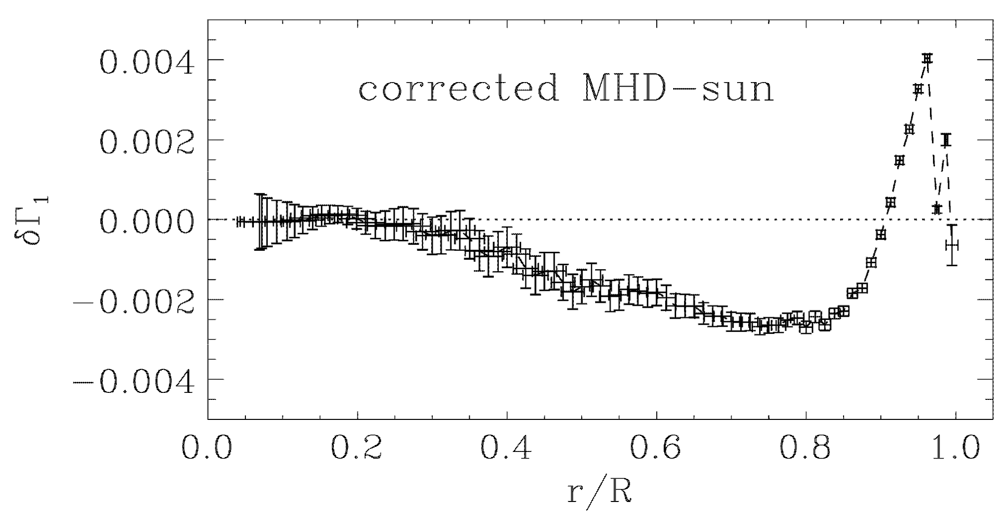
- Relativistic corrections are expected to be small, central temperature

$$kT \approx 1 \text{ keV} \ll 511 \text{ keV}$$

- And yet: the effect can be observed!!
  (Elliot & Kosovichev 1998, ApJ, **500** L199)

Models with and without relativistic electrons





Figures from:

Elliot, J.R. & Kosovichev, A.G.

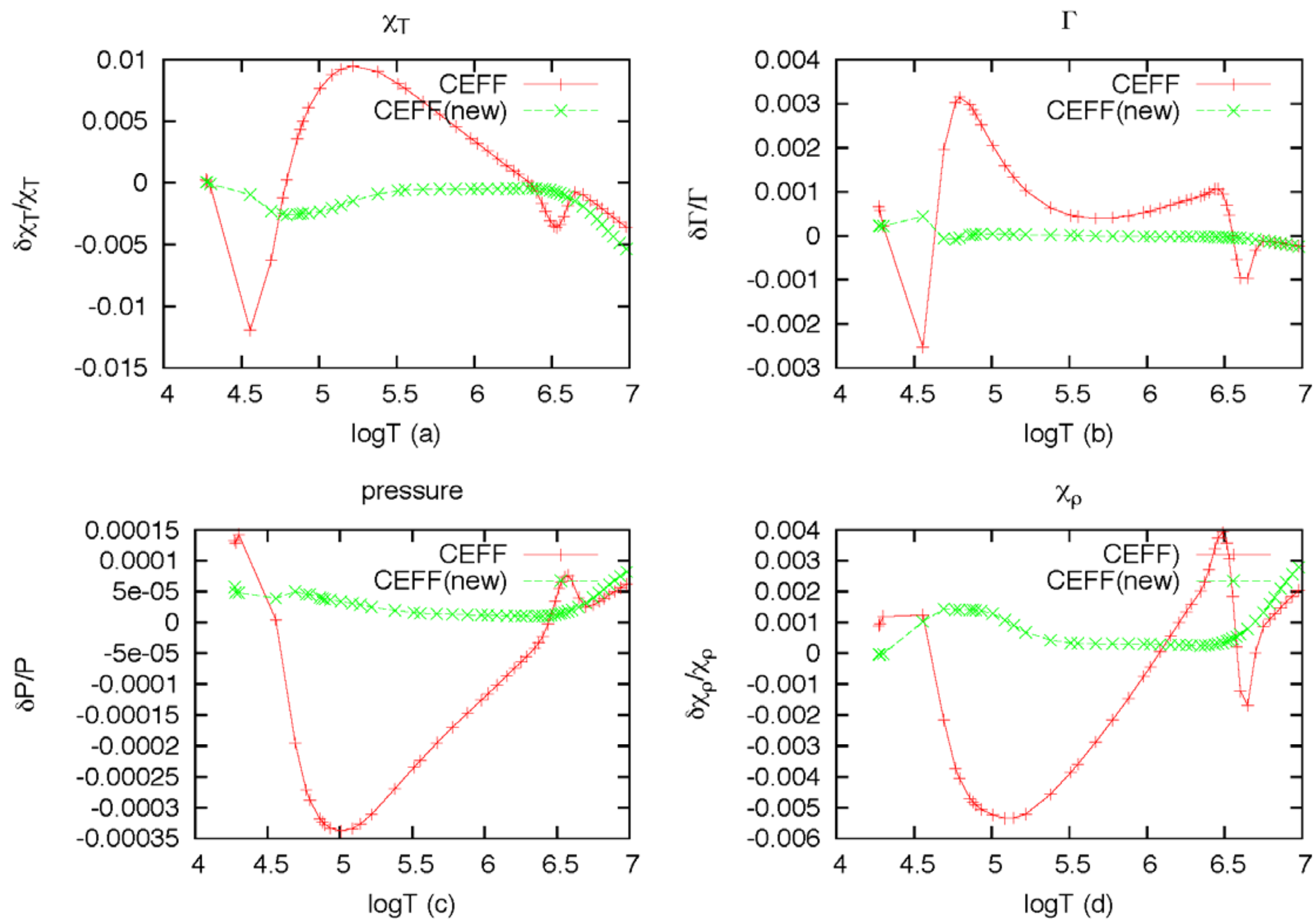
1998, *ApJ*, **500** L199

Practical tools for modelers II

Tool for the community

- OPAL emulator as an in-line formalism, starting out from CEFF (Hsiao-Hsuan Lin)
- Higher precision than FreeEOS but only for Sun, while FreeEOS has larger stellar domain
- Work in progress, in several steps, current state on following slide

Results (Hsiao-Hsuan Lin, 2011)



A further option is SAHA-S

SAHA-S is a serious alternative to OPAL. It is based on a systematic development, and it contains a modified Planck-Larkin partition function.

See:

Starostin, A.N. & Roerich, V.C., Phys. A: Math. Gen. 39(2006)4431

Truly Exact Options

- There are exact density expansions, for instance the Feynman-Kac [FK] path-integral computations [1] or Green-function calculations [2].
- The coefficients in these expansions are indeed exact!
- Problem is the domain of applicability!
- Why? See Appendix, if you insist!

[1] Alastuey, A. & Perez, A. 1992, Europhys.Lett., 20, 19

[2] Kraeft, W.-D., Kremp, D., Ebeling,W., Röpke, G., 1986, Quantum Statistics of Charged Particle Systems (Plenum)

$$
\begin{aligned}
\beta P \;=\; & \sum_{\alpha} \rho_{\alpha} - \frac{\kappa_D^3}{24\pi} \\
& + \frac{\pi}{6}(\ln 2 - 1) \sum_{\alpha,\beta} \beta^3 e_{\alpha}^3 e_{\beta}^3 \rho_{\alpha}\rho_{\beta} \\
& - \frac{\pi}{\sqrt{2}} \sum_{\alpha,\beta} \rho_{\alpha}\rho_{\beta}\lambda_{\alpha\beta}^3 Q(x_{\alpha\beta}) - \frac{\pi}{3}\beta^3 \sum_{\alpha,\beta} \rho_{\alpha}\rho_{\beta} e_{\alpha}^3 e_{\beta}^3 \ln(\kappa_D \lambda_{\alpha\beta}) \\
& + \frac{\pi}{\sqrt{2}} \sum_{\alpha} \frac{(-1)^{2\sigma_{\alpha}+1}}{(2\sigma_{\alpha}+1)} \lambda^3{}_{\alpha\alpha} \rho_{\alpha}^2 E(x_{\alpha\alpha}) \\
& - \frac{3\pi}{2\sqrt{2}} \beta \sum_{\alpha,\beta} e_{\alpha} e_{\beta} \kappa_D \rho_{\alpha}\rho_{\beta} {\lambda_{\alpha\beta}}^3 Q(x_{\alpha\beta}) \\
& - \frac{\pi}{2}\beta^4 \sum_{\alpha,\beta} \rho_{\alpha}\rho_{\beta} e_{\alpha}^4 e_{\beta}^4 \kappa_D \ln(\kappa_D \lambda_{\alpha\beta}) \\
& + \frac{3\pi}{2\sqrt{2}} \beta \sum_{\alpha} \frac{(-1)^{2\sigma_{\alpha}+1}}{(2\sigma_{\alpha}+1)} \lambda_{\alpha\alpha}^3 \rho_{\alpha}^2 e_{\alpha}^2 \kappa_D E(x_{\alpha\alpha}) \\
& + \frac{1}{16} \sum_{\alpha} \frac{\beta^2 \hbar^2 e_{\alpha}^2}{m_{\alpha}} \kappa_D^3 \rho_{\alpha} + \pi\left(\frac{1}{3} - \frac{3}{4}\ln 2 + \frac{1}{2}\ln 3\right) \times \\
& \sum_{\alpha,\beta} \beta^4 e_{\alpha}^4 e_{\beta}^4 \kappa_D \rho_{\alpha}\rho_{\beta} \\
& + C_1 \sum_{\alpha,\beta,\gamma} \beta^5 e_{\alpha}^3 e_{\beta}^4 e_{\gamma}^3 \kappa_D^{-1} \rho_{\alpha}\rho_{\beta}\rho_{\gamma} \\
& + C_2 \sum_{\alpha,\beta,\gamma,\delta} \beta^6 e_{\alpha}^3 e_{\beta}^3 e_{\gamma}^3 e_{\delta}^3 \kappa_D^{-3} \rho_{\alpha}\rho_{\beta}\rho_{\gamma}\rho_{\delta}
\end{aligned}
$$

More in the Appendix!

Finally, let’s not forget the **physical issues** in the equation of state

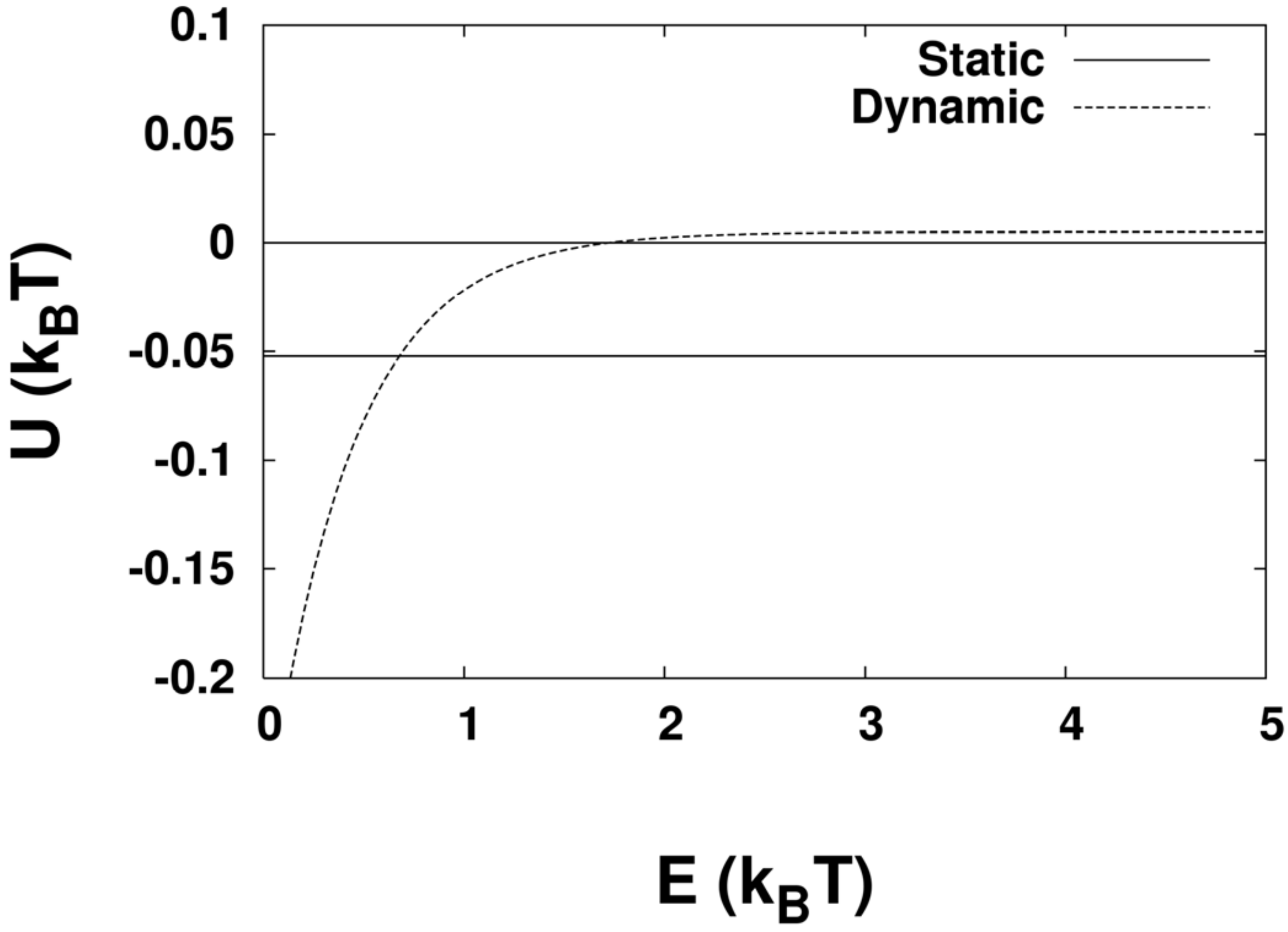
- Various smaller competing - but relevant - effects, among other:

- Existence and population of excited states
- Diffraction and exchange terms
- Parametric “size” in hard-spheres
- Relativistic correction for electrons

One current issue in nuclear reactions

Dynamic screening in solar and stellar nuclear reactions

Dynamic screening energy at the turning point for pairs of protons with a given relative kinetic energy [Mao et al. ApJ 701(2009)1204]



Screening energies and the ratio of screened to unscreened nuclear reaction rates for solar p-p reactions [Mussack & Däppen, ApJ, 729(2011)96]

Case	Screening energy U	Reaction-rate correction
Unscreened	0	1
Statically screened	$U_0 = -Z_1 Z_2 e^2 / R_D$	1.042
Dynamically screened	$U_0(E) = k_B T \left(0.005 - 0.281 \exp\left(-2.35 E / k_B T\right)\right)$	0.996

Our simulations suggest that dynamic effects obliterate the static-screening enhancement. In other words, they imply that at the the most probable collision energy (of 3-4 kT) the screening cloud is virtually absent and the naked Coulomb potential has to be tunneled through.

But,

Anderegg et al. [Phys. Plasmas 17(2010)055702] have used an apparent "duality" to infer the solar screening results from a system under totally different physical conditions, in their case magnetization experiments on a laser-cooled ionic system. They think that their experiments confirm Salpeter's view of static screening enhancement.

Not so fast!

However, since that work is based on the assumption of a faithful mapping between two entirely different physical systems, and furthermore, since the present state of those experiments corresponds to a stronger coupled plasma than in the center of the Sun, this is unlikely the last word on the issue of dynamic screening.

Conclusions

- There are many stars
- They have different ages and chemical compositions
- Solar and stellar constraints will improve microphysics
- Thus better astrophysics and better physics from the Sun and the stars

APPENDIX –

Density and activity expansions

Look at the ideal Fermi-Dirac gas: the exact solution (but implicit!) ...

$$\frac{p}{kT} = \frac{1}{\lambda^3} f_{5/2}(\tilde{z})$$

$$\frac{N}{V} = \frac{1}{\lambda^3} f_{3/2}(\tilde{z})$$

$$\lambda = \frac{h}{\sqrt{2\pi mkT}}; \qquad \tilde{z} = e^{\frac{\mu}{kT}}$$ = fugacity

$$f_{5/2}(\tilde{z}) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\ x^2 \ln(1 + \tilde{z}e^{-x^2})$$

$$f_{3/2}(\tilde{z}) = \tilde{z} \frac{d}{d\tilde{z}} f_{5/2}(\tilde{z})$$

… and its high-temperature virial expansion

$$\frac{pV}{NkT} = 1 + \frac{1}{2^{5/2}} \frac{N\lambda^3}{V} + \ldots$$

[obtained from expansion of Fermi integrals and elimination of $\tilde{z}$]

This is a nice explicit classical limit (and it includes the relevant criterion)!

Expand without eliminating: fugacity / activity expansion

$$\frac{p}{kT} = \frac{1}{\lambda^3}\left[\tilde{z} - \frac{1}{2^{5/2}}\tilde{z}^2 + ...\right] = z - \frac{\lambda^3}{2^{5/2}}z^2 + ...$$

$$\frac{N}{V} = \frac{1}{\lambda^3}\left[\tilde{z} - \frac{1}{2^{3/2}}\tilde{z}^2 + ...\right] = z - \frac{\lambda^3}{2^{3/2}}z^2 + ...$$

Which is based on the natural definition of the activity

$$z = \frac{1}{\lambda^3}\tilde{z}$$

Any real system has an activity expansion (ACTEX) …

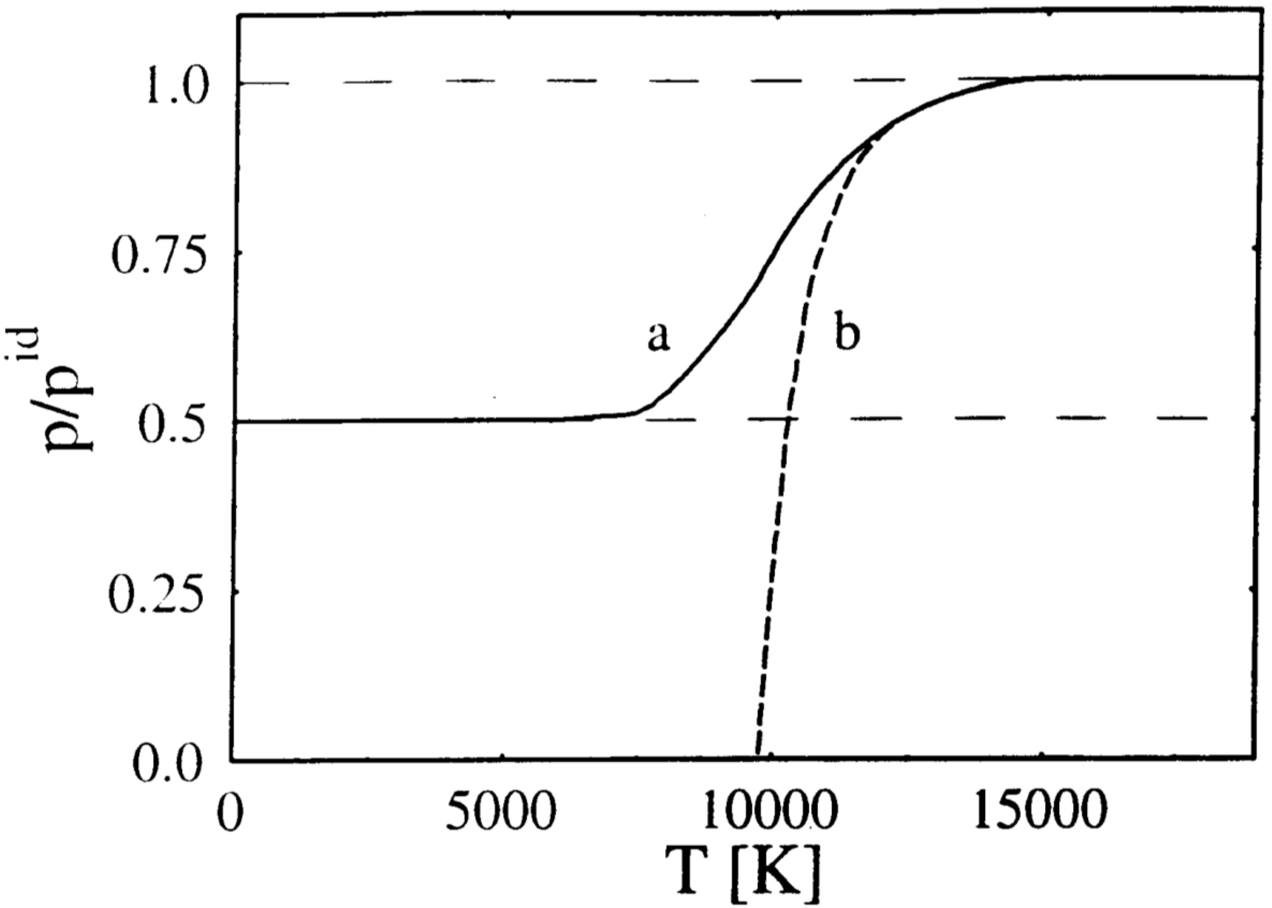
$$\frac{p}{kT} = z - b_2 z^2 + \ldots$$

$$\frac{N}{V} = z - 2b_2 z^2 + \ldots$$

… because the grand-canonical partition function delivers these coefficients b_i (of course infinitely many)

Radically different behavior already to first non-ideal order (here illustrated with H-H_2 system)

(a: activity   b: virial)



For reacting systems, activity expansions are much better suited than virial expansions!

See: Kremp D., Schlanges, M. & Kraeft, W.-D., Quantum Statistics of Nonideal Plasmas (Springer, Berlin, 2005)