Kavli Institute for Theoretical Physics, Santa Barbara, 9 September 2019

In Search of Universality

Collisionless Fun and Games in Kinetic Phase Space

Toby Adkins & Alex Schekochihin
(Merton College & U of Oxford)

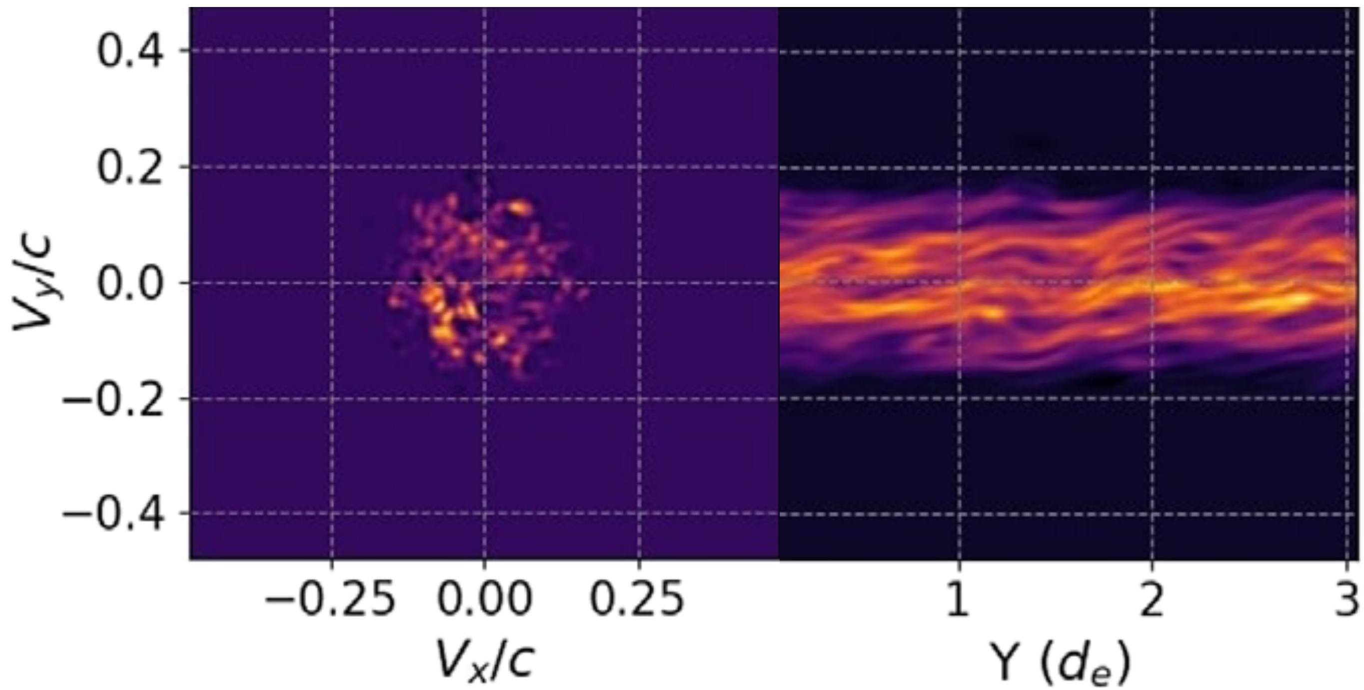
with thanks to
Joseph Parker (RAL), Paul Dellar (Oxford Maths), Edmund Highcock (Real World),
Anjor Kanekar, Michael Nastac, Bill Dorland (Maryland),
Greg Hammett (Princeton),
Romain Meyrand (Otago)

Trigger warning: EQUATIONS

Journal of Plasma Physics **84**, 905840107 (2018)
++new paper in preparation

Universal Equilibria & Kinetic Turbulence

On Wednesday, Ammar Hakim and/or Jimmy Juno will show you these plots:



from Skoutnev et al. ApJ 872, L28 (2019): **distribution resulting from relaxation of two beams**

$$\boxed{f = f_0(\mathbf{v}) + \delta f(\mathbf{r}, \mathbf{v})}$$

Is there a universal collisionless equilibrium?
(or classes of equilibria independent of precise initial conditions)

Fine structure in phase space.
What is the structure of this "phase-space turbulence"?

Universal Equilibria & Kinetic Turbulence

$$\partial_t f + \mathbf{v} \cdot \nabla f + \frac{e}{m} \nabla \varphi \cdot \partial_{\mathbf{v}} f = 0$$

collisionless plasma (one species: electrons)

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$$\partial_t f_0 = \partial_{\mathbf{v}}\cdot\sum_{\mathbf{k}} i\mathbf{k}\frac{e}{m}\langle\varphi^*_{\mathbf{k}} f_{\mathbf{k}}\rangle$$ slow equilibrium evolution

$$f = f_0(\mathbf{v}) + \delta f(\mathbf{r},\mathbf{v}) = f_0(\mathbf{v}) + \sum_{\mathbf{k}} f_{\mathbf{k}}(\mathbf{v}) e^{i\mathbf{k}\cdot\mathbf{r}}$$

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Fast, fine-scale fluctuations:

$$\varphi_{\mathbf{k}} = -\frac{4\pi e}{k^2}\int d\mathbf{v} f_{\mathbf{k}}$$

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Phase-space correlation function of δf determines evolution of f_0

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$$f_{\mathbf{k}\omega} = \frac{e}{m}\frac{\varphi_{\mathbf{k}\omega}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}\mathbf{k}\cdot\partial_{\mathbf{v}} f_0 + h_{\mathbf{k}\omega}$$

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linear response

turbulence

$$\partial_t f_{\mathbf{k}} + i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}} + i\frac{e}{m}\left(\varphi_{\mathbf{k}}\mathbf{k}\cdot\partial_{\mathbf{v}} f_0 + \sum_{\mathbf{p}}\varphi_{\mathbf{p}}\mathbf{p}\cdot\partial_{\mathbf{v}} f_{\mathbf{k}-\mathbf{p}}\right) = 0$$

initial condition

$$f_{\mathbf{k}\omega} = \frac{e}{m}\frac{\varphi_{\mathbf{k}\omega}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}\mathbf{k}\cdot\partial_{\mathbf{v}} f_0 + h_{\mathbf{k}\omega} \qquad h_{\mathbf{k}\omega} = \frac{ig_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v} + i0} + \mathit{TURBULENCE}$$

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A "collisionless collision integral"

$$\boxed{\begin{aligned}\partial_t f_0 &= \partial_{\mathbf{v}}\cdot\sum_{\mathbf{k}} i\mathbf{k}\frac{e}{m}\langle\varphi_{\mathbf{k}}^* f_{\mathbf{k}}\rangle = \partial_{\mathbf{v}}\cdot\int d\mathbf{v}''\left[\mathsf{D}(\mathbf{v},\mathbf{v}'')\cdot\partial_{\mathbf{v}} f_0(\mathbf{v}) - \mathsf{D}(\mathbf{v}'',\mathbf{v})\cdot\partial_{\mathbf{v}''} f_0(\mathbf{v}'')\right]\\ \mathsf{D}(\mathbf{v}'',\mathbf{v}) &= -\frac{16\pi^2 e^4}{m^2}\sum_{\mathbf{k}}\frac{\mathbf{k}\mathbf{k}}{k^4}\mathrm{Im}\iint\frac{d\omega' d\omega}{(2\pi)^2}\frac{e^{i(\omega'-\omega)t}}{\epsilon^*_{\mathbf{k}\omega'}\epsilon_{\mathbf{k}\omega}}\int d\mathbf{v}'\frac{\langle h^*_{\mathbf{k}\omega'}(\mathbf{v}')h_{\mathbf{k}\omega}(\mathbf{v})\rangle}{\omega - \mathbf{k}\cdot\mathbf{v}'' + i0}\end{aligned}}$$

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Properties of phase-space turbulence decide to what equilibrium f_0 relaxes and how quickly

$$\partial_t f_{\mathbf{k}} + i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}} + i\frac{e}{m}\Big(\underbrace{\varphi_{\mathbf{k}}\mathbf{k}\cdot\partial_{\mathbf{v}} f_0}_{\text{linear response}} + \underbrace{\sum_{\mathbf{p}}\varphi_{\mathbf{p}}\mathbf{p}\cdot\partial_{\mathbf{v}} f_{\mathbf{k}-\mathbf{p}}}_{\text{turbulence}}\Big) = 0$$

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Oh, and is there Landau damping in a turbulent plasma?

Kinetic Turbulence

OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f^*_{\mathbf{k}}(\mathbf{v}')\rangle$

$$\partial_t f_{\mathbf{k}} + \underbrace{i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}}}_{\text{phase mixing}} + i\frac{e}{m}\left(\underbrace{\varphi_{\mathbf{k}}\mathbf{k}\cdot\partial_{\mathbf{v}} f_0}_{\text{linear response}} + \underbrace{\sum_{\mathbf{p}}\varphi_{\mathbf{p}}\mathbf{p}\cdot\partial_{\mathbf{v}} f_{\mathbf{k}-\mathbf{p}}}_{\text{turbulence}}\right) = 0$$

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$$+(\text{same eqn})^*(\mathbf{v}') \times f_{\mathbf{k}}(\mathbf{v})$$

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break correlators using **Kraichnan model**:

$$\boxed{\frac{e^2}{m^2}\langle\varphi_{\mathbf{k}}(t)\varphi_{\mathbf{k}'}^*(t')\rangle = 2\varkappa_k\delta_{\mathbf{k},\mathbf{k}'}\delta(t-t')}$$

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$$\frac{e}{m}\langle\varphi_{\mathbf{p}} f_{\mathbf{k}-\mathbf{p}}(\mathbf{v}) f^*_{\mathbf{k}}(\mathbf{v}')\rangle = \varkappa_p i\Big(\mathbf{p}\cdot\partial_{\mathbf{v}} C_{\mathbf{k}} + \mathbf{p}\cdot\partial_{\mathbf{v}'} C_{\mathbf{k}-\mathbf{p}}\Big)$$

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$$\partial_t C_{\mathbf{k}} + i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')C_{\mathbf{k}} = S_{\mathbf{k}} + D\left(\partial_{\mathbf{v}} + \partial_{\mathbf{v}'}\right)^2 C_{\mathbf{k}} + 2\sum_{\mathbf{p}} \varkappa_p \mathbf{p}\cdot\partial_{\mathbf{v}}\mathbf{p}\cdot\partial_{\mathbf{v}'}\left(C_{\mathbf{k}-\mathbf{p}} - C_{\mathbf{k}}\right)$$

$$D = \frac{1}{d}\sum_{\mathbf{p}} p^2 \varkappa_p$$ QL diffusion

the interesting bit (to be dealt with in a moment)

$$S_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = 2\varkappa_k \mathbf{k}\cdot\partial_{\mathbf{v}} f_0(\mathbf{v})\mathbf{k}\cdot\partial_{\mathbf{v}'} f_0(\mathbf{v}')$$

linear source

Linear Phase Mixing

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$$\partial_t C_{\mathbf{k}} + i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')C_{\mathbf{k}} = S_{\mathbf{k}}$$

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linear source

Consider for a moment the case with no nonlinearities:

$$C_{\mathbf{k}} = S_{\mathbf{k}}\frac{1-e^{-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')t}}{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')} \to 2\varkappa_k\left[\mathbf{k}\cdot\partial_{\mathbf{v}} f_0(\mathbf{v})\right]^2 \pi\delta\left(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')\right)$$

Distribution function becomes very fine-scaled in phase space (and requires collisions to regularise it)

Kinetic Turbulence

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On these grounds, let $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') \to C_{\mathbf{k}}(\mathbf{u},\mathbf{w}),\ \mathbf{u} = \dfrac{\mathbf{v}+\mathbf{v}'}{2},\ \mathbf{w} = \mathbf{v}-\mathbf{v}',\ \partial_{\mathbf{u}} \ll \partial_{\mathbf{w}}$

Kinetic Turbulence

OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f^*_{\mathbf{k}}(\mathbf{v}')\rangle$

$$\partial_t C_{\mathbf{k}} + i\mathbf{k}\cdot\mathbf{w}C_{\mathbf{k}} \approx S_{\mathbf{k}}(\mathbf{u},\mathbf{u}) + 2\sum_{\mathbf{p}} \varkappa_p (\mathbf{p}\cdot\partial_{\mathbf{w}})^2 \left(C_{\mathbf{k}} - C_{\mathbf{k}-\mathbf{p}}\right)$$

$$S_{\mathbf{k}}(\mathbf{u},\mathbf{u}) = 2\varkappa_k [\mathbf{k}\cdot\partial_{\mathbf{u}} f_0(\mathbf{u})]^2$$

Consider for a moment the case with no nonlinearities:

$$C_{\mathbf{k}} = S_{\mathbf{k}} \frac{1 - e^{-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')t}}{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')} \to 2\varkappa_k \left[\mathbf{k}\cdot\partial_{\mathbf{v}} f_0(\mathbf{v})\right]^2 \pi\delta\big(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')\big)$$

Distribution function becomes very fine-scaled in phase space (and requires collisions to regularise it)

On these grounds, let $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') \to C_{\mathbf{k}}(\mathbf{u},\mathbf{w}),\ \mathbf{u} = \dfrac{\mathbf{v}+\mathbf{v}'}{2},\ \mathbf{w} = \mathbf{v}-\mathbf{v}',\ \partial_{\mathbf{u}} \ll \partial_{\mathbf{w}}$

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Finally, Fourier-transform in **w**:

$$C_{\mathbf{k}}(\mathbf{u},\mathbf{w}) = \int d\mathbf{s}\, C_{\mathbf{k},\mathbf{s}}(\mathbf{u}) e^{-i\mathbf{s}\cdot\mathbf{w}}$$

"Wigner function"

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Kinetic Cascade

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This equation describes a journey through phase space of a conserved "energy":

$$W = \iint d\mathbf{r}d\mathbf{v}\langle \delta f^2\rangle = V\sum_{\mathbf{k}}\int d\mathbf{v}\langle |f_{\mathbf{k}}(\mathbf{v})|^2\rangle = V\sum_{\mathbf{k}}\iint d\mathbf{u}d\mathbf{s}\, C_{\mathbf{k},\mathbf{s}}(\mathbf{u})$$

$$\frac{dW}{dt} = V\sum_{\mathbf{k}}\int d\mathbf{u}\, S_{\mathbf{k}}(\mathbf{u},\mathbf{u})$$ **– (collisional regularisation at large *s* and/or *k*)**

energy source at small *s* and *k*

Phase Mixing vs. Mode Coupling

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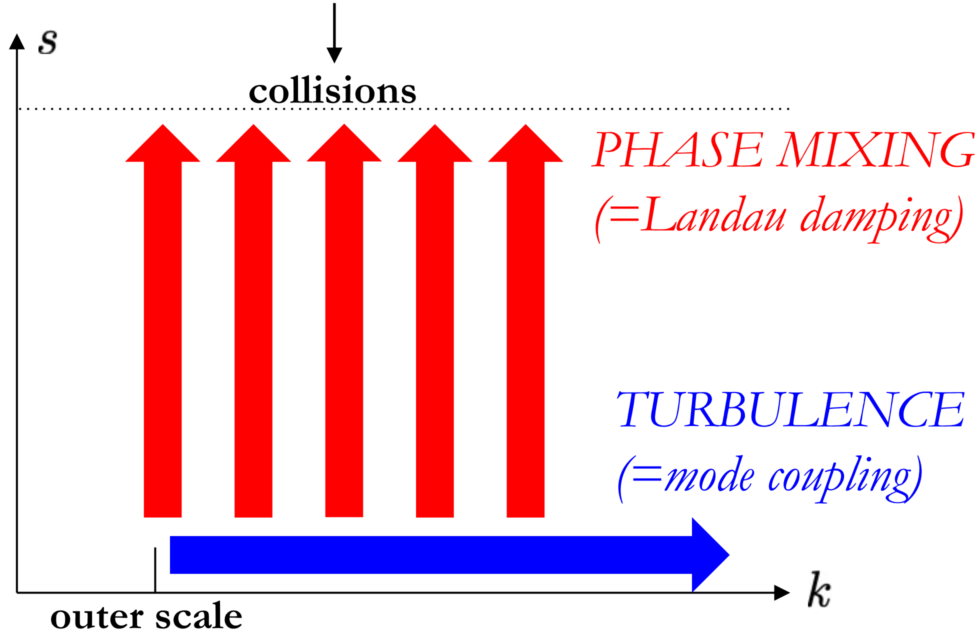

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Phase mixing (**k.s** > 0)
takes energy to higher *s*
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Anti-phase mixing (**k.s** < 0)
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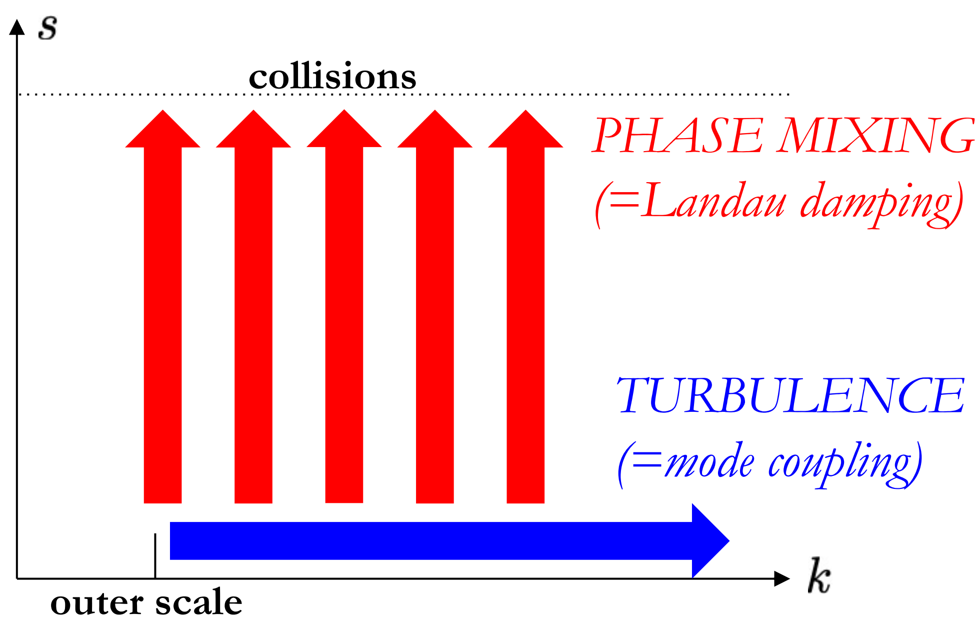

Phase Mixing vs. Stochastic Echo

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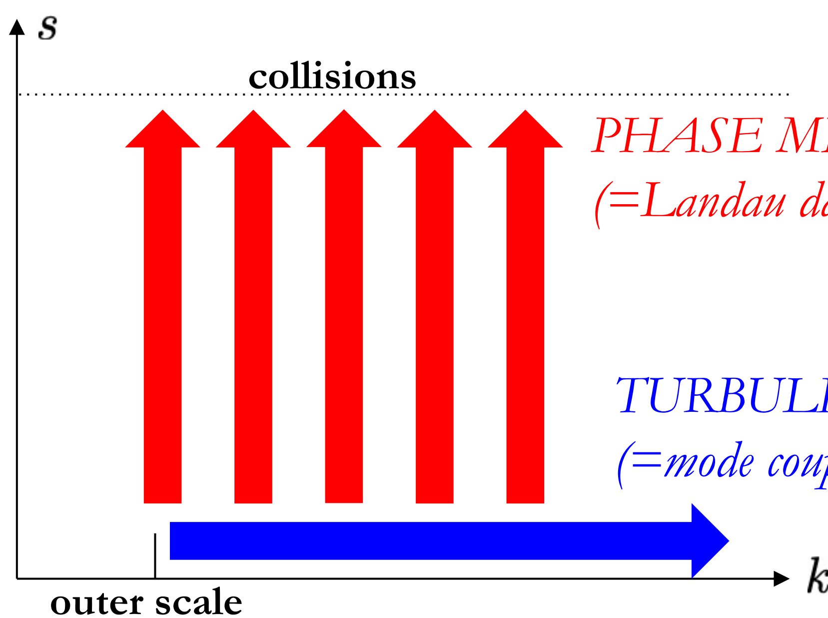
Phase mixing (**k.s** > 0) takes energy to higher *s* at velocity *k*.
Anti-phase mixing (**k.s** < 0) does the opposite, but there is no energy source at high *s*.

A **phase-mixing** perturbation (**k.s** > 0) can turn around and come back (**anti-phase-mix**) if the **nonlinearity** couples it to a **k** of opposite sign (to **k.s** < 0) – **PLASMA ECHO** effect.



Is there Landau damping in a turbulent plasma?
(there is certainly turbulence!)

Kinetic Turbulence in 1D

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$$\boxed{\partial_t C_{k,s} + k\partial_s C_{k,s} \approx S_k \delta(s) - 2s^2 \sum_p p^2 \varkappa_p \left(C_{k,s} - C_{k-p,s}\right)}$$

Two further simplifications to get to an **exactly solvable model**:

– One spatial dimension

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Solvable Model of Kinetic Turbulence

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steady state

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$$S_k = 2k^2 \varkappa_k [f_0'(u)]^2$$

(in fact, there is a boundary layer at small *s*...)

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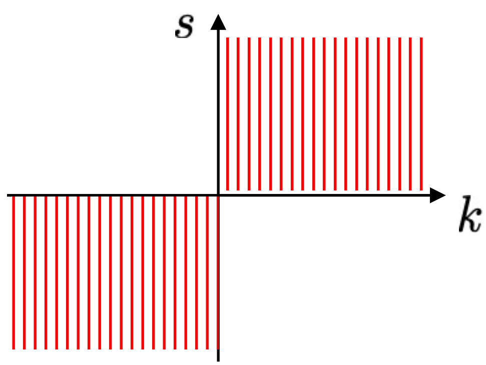
Spectrum of linear phase mixing:

$$C_{k,s} \propto H[s\,\mathrm{sgn}(k)]$$

This is the same as the $m^{-1/2}$ Hermite spectrum [Zocco & AAS, *PoP* **18**, 102309 (2011)]



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"time evolution" forward or backward depending on sign of k (eff. time is $\tau = \gamma s^3/3$)

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diffusion or antidiffusion (echo) depending on sign of k (eff. space coordinate is $\xi = 2k^{3/2}/3$)

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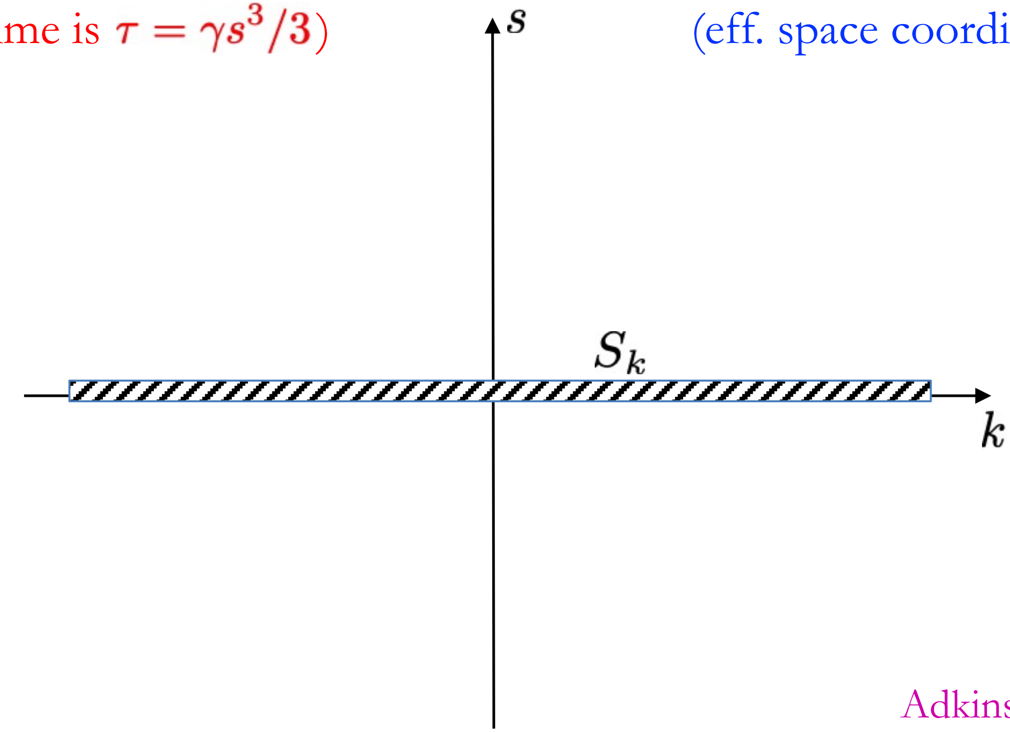

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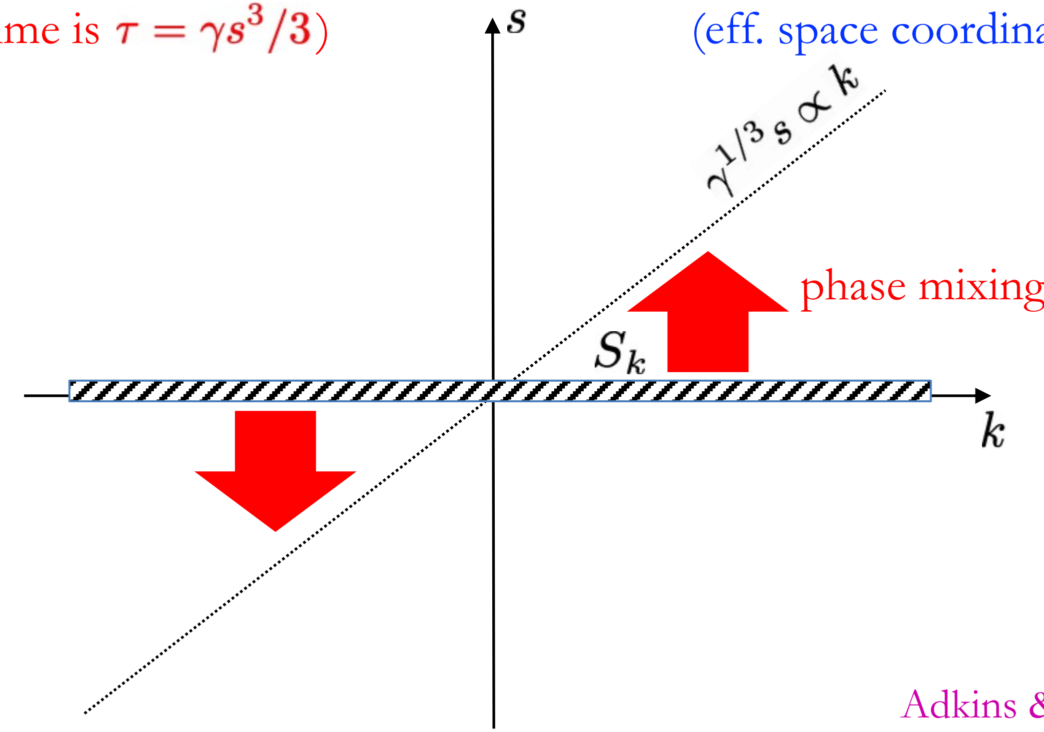

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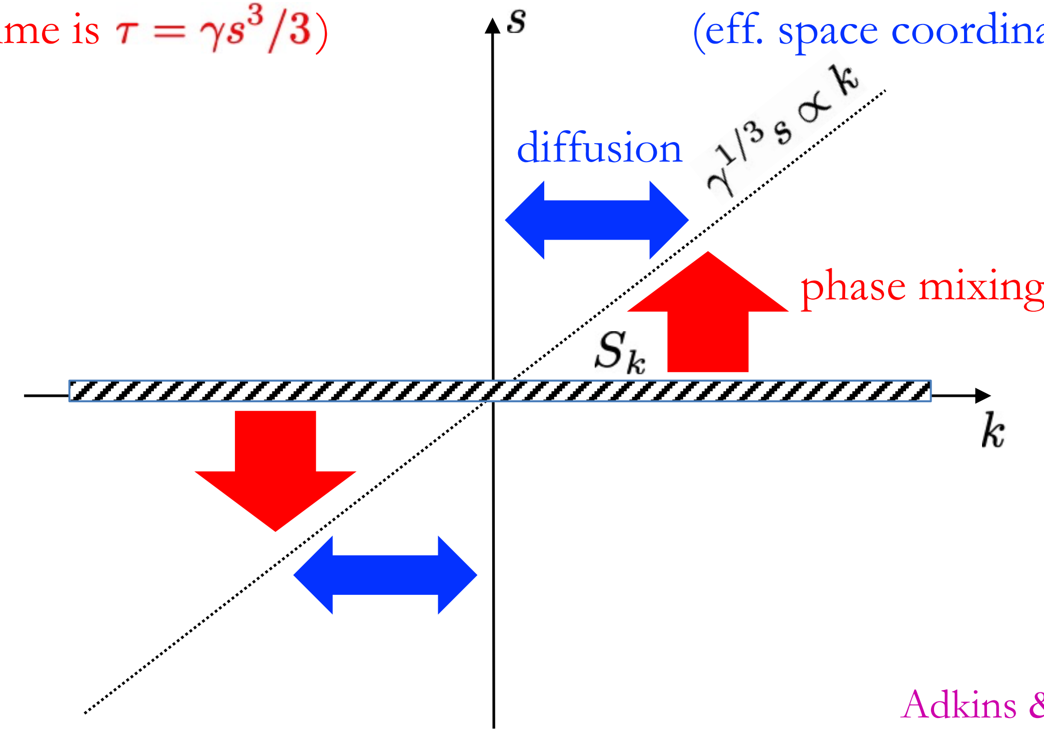

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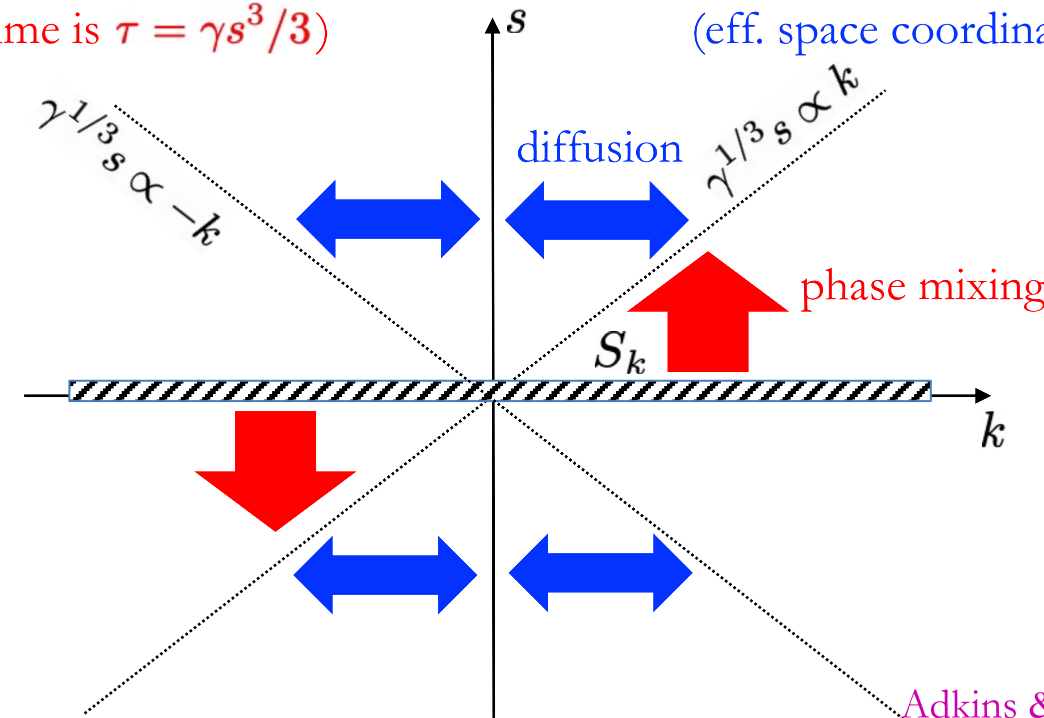

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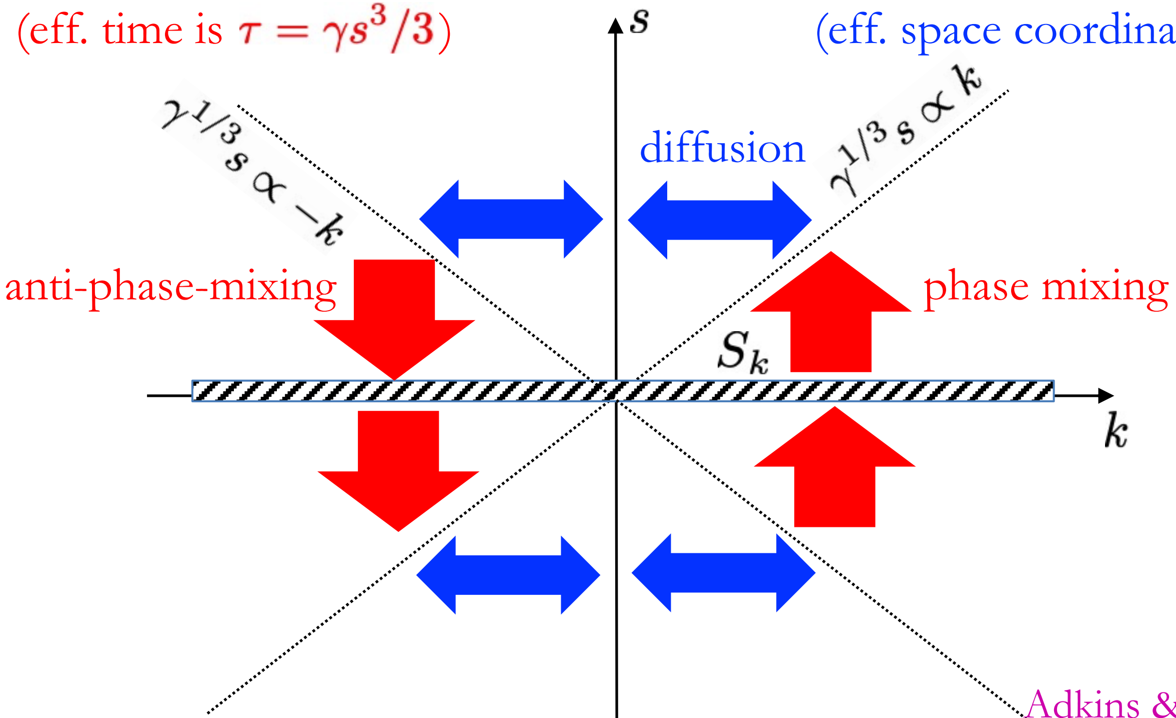

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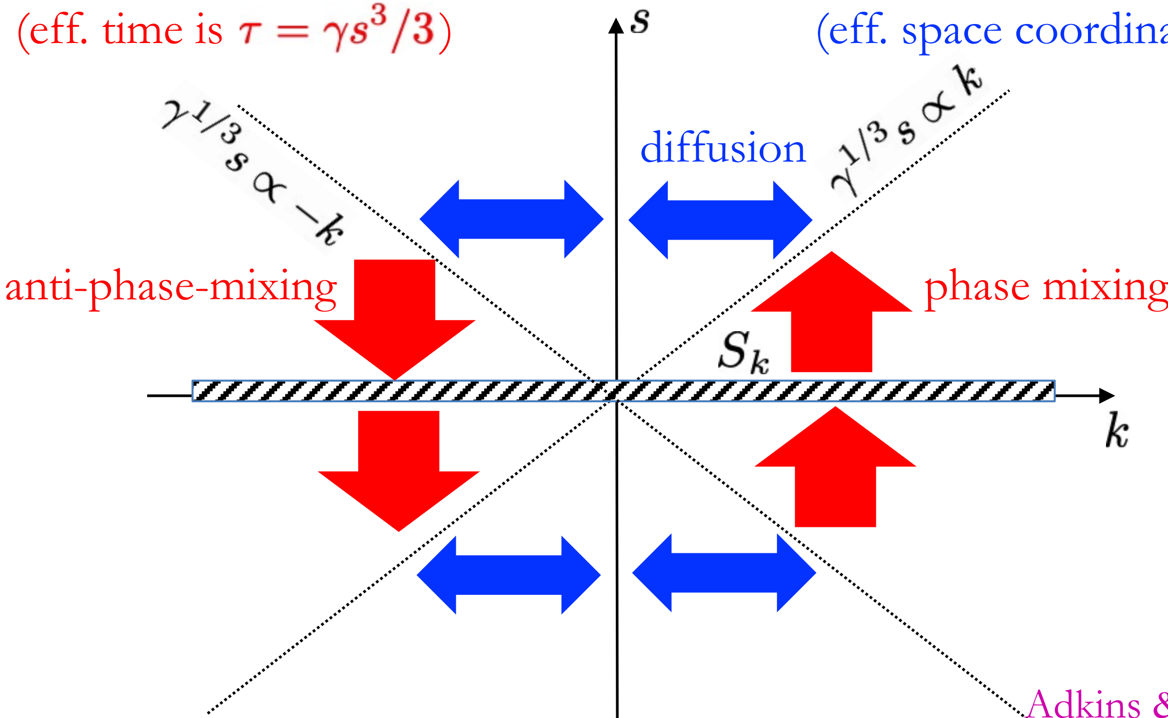

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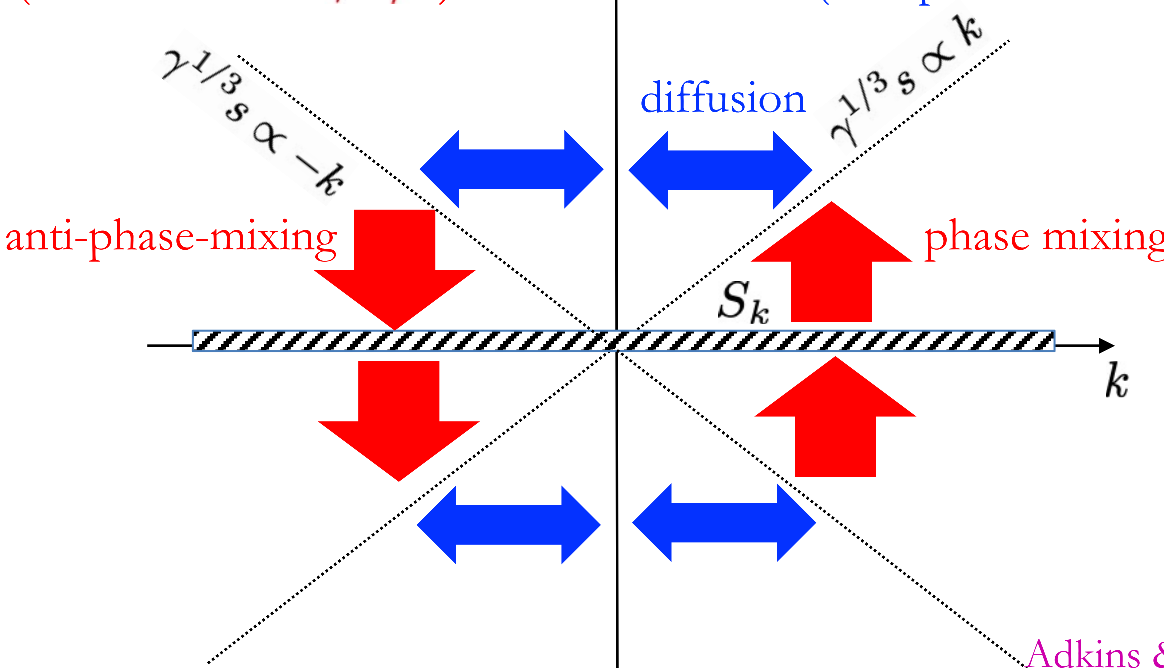

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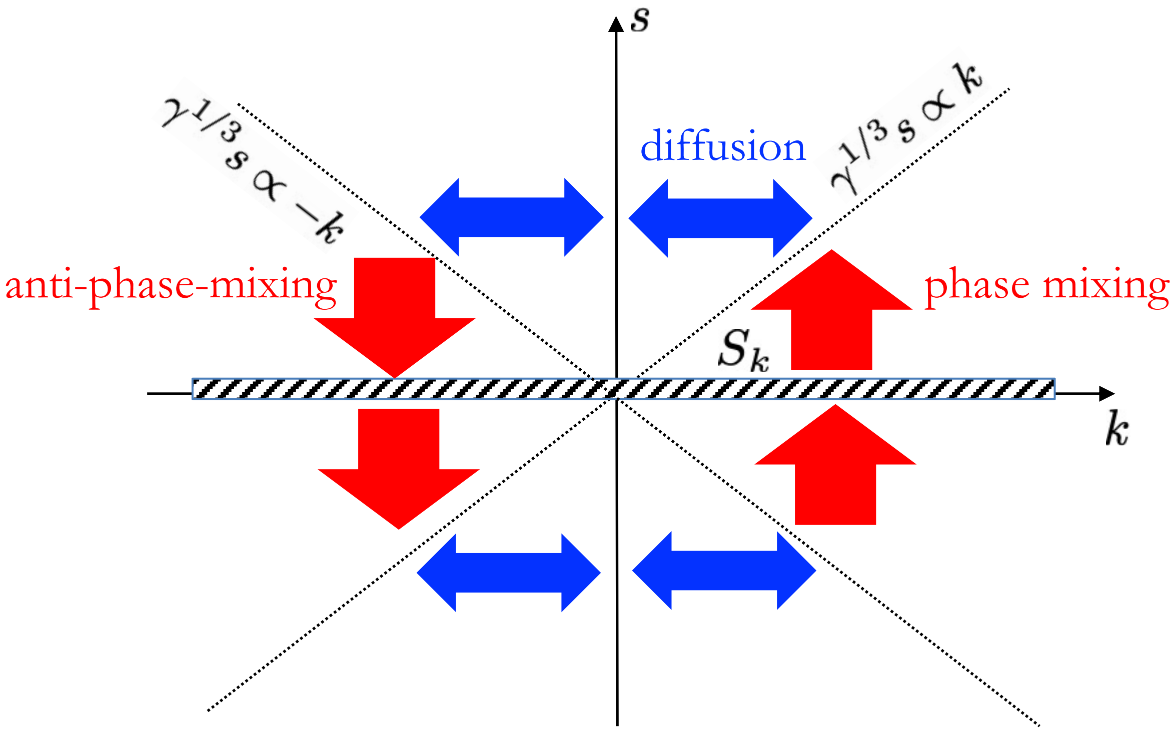
Mathematically, Green's function $\mathcal{G}$ for this equation can be derived and...

$$\boxed{C_{k>0,s=0} \to \mathcal{G} \to C_{0,s>0} \to \mathcal{G} \to C_{0,k<0}}$$

Details: Adkins & Schekochihin, *Journal of Plasma Physics* **84**, 905840107 (2018)



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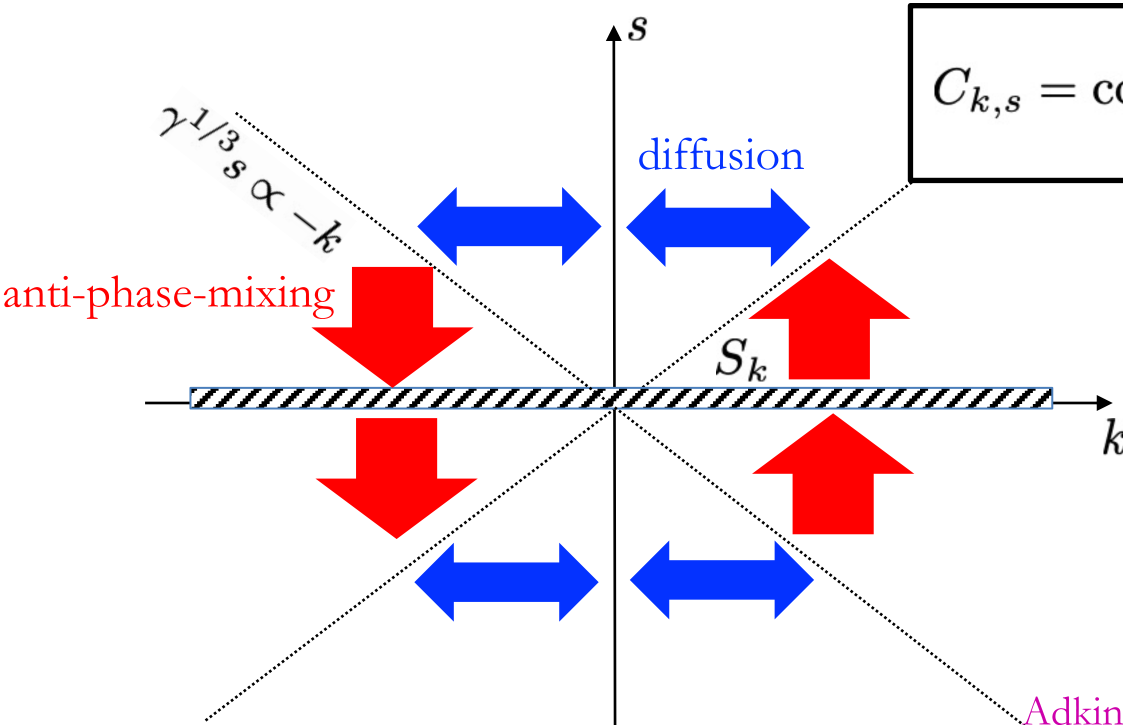
...this fixes the solution everywhere:

$$\boxed{C_{k,s} = \text{const} \int_0^\infty dz \frac{e^{-z}}{|k^3 - 3\gamma s^3 z|^{2/3}}}$$



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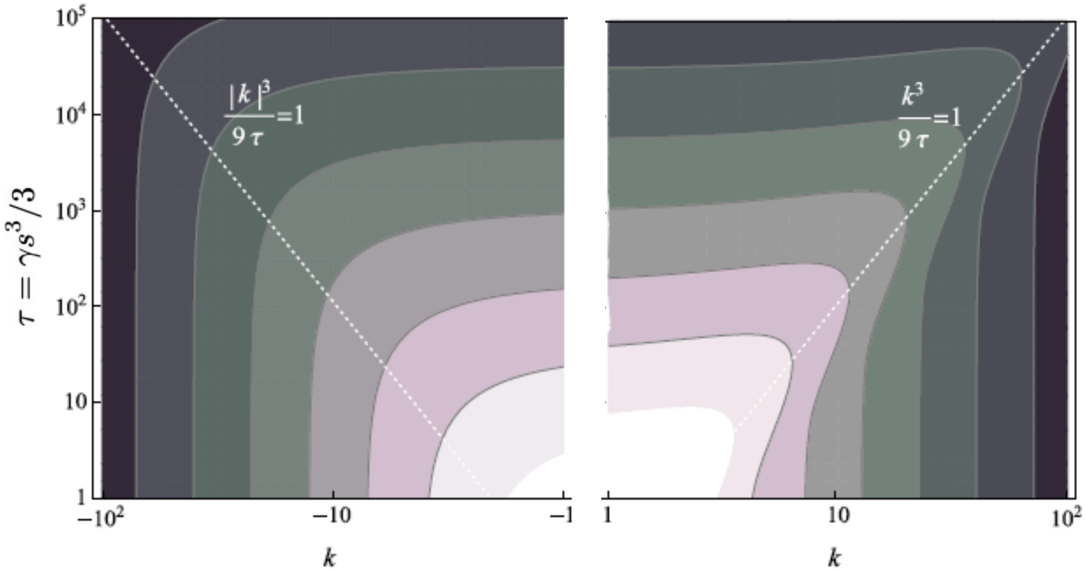


Universal phase-space spectrum

Adkins & AAS, *JPP* **84**, 905840107 (2018)

Phase-Space Spectrum of Kinetic Turbulence

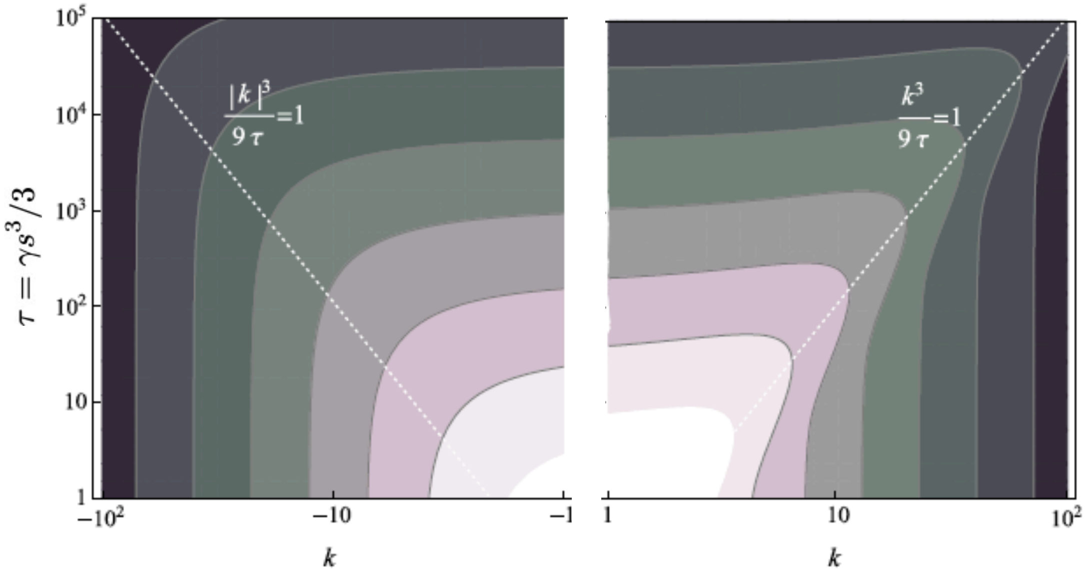
OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f^*_{\mathbf{k}}(\mathbf{v}') \rangle$

$$\cancel{\partial_t C_{k,s}} + k\partial_s C_{k,s} \approx S_k \delta(s) + \gamma s^2 \partial_k^2 C_{k,s} \qquad \gamma = \sum_p p^4 \varkappa_p$$

Mathematically, Green's function $\mathcal{G}$ for this equation can be derived and...

$$C_{k>0,s=0} \to \mathcal{G} \to C_{0,s>0} \to \mathcal{G} \to C_{0,k<0}$$

...this fixes the solution everywhere:



Universal phase-space spectrum

$$\boxed{C_{k,s} = \mathrm{const} \int_0^\infty dz \frac{e^{-z}}{|k^3 - 3\gamma s^3 z|^{2/3}}}$$

$$\to \begin{cases} s^{-2}, & \gamma^{1/3} s \gg k \\ k^{-2}, & k \gg \gamma^{1/3} s \end{cases}$$

Adkins & AAS, *JPP* **84**, 905840107 (2018)

R.I.P. Landau Damping?

OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f^*_{\mathbf{k}}(\mathbf{v}')\rangle$

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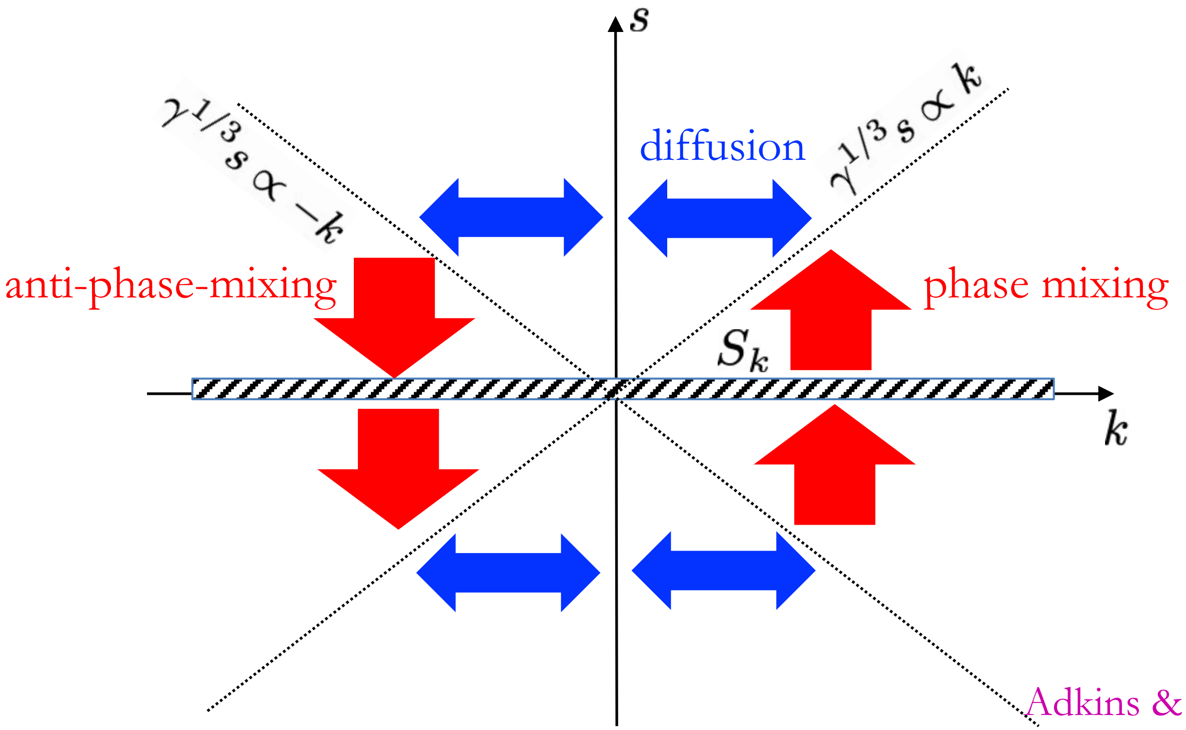
$$\boxed{\textbf{Thus, Landau damping is suppressed}}$$

...really?



Adkins & AAS, *JPP* **84**, 905840107 (2018)

Collisional Cutoff

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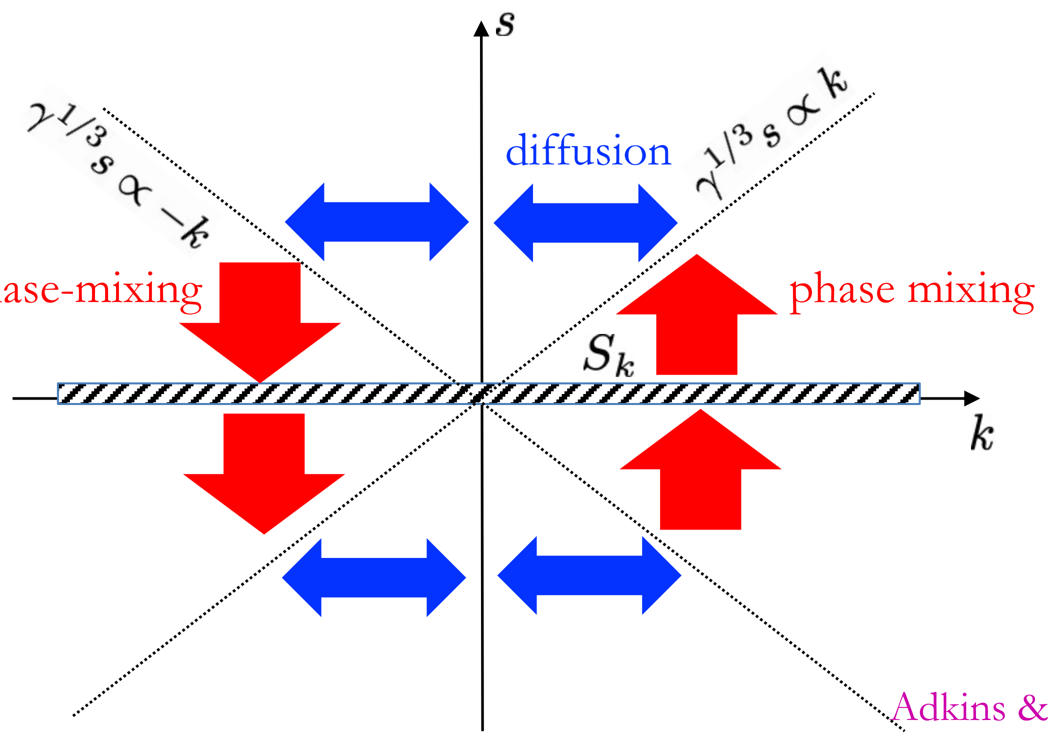
What is the role of collisions in all this? (Fluctuation energy must be thermalised!)



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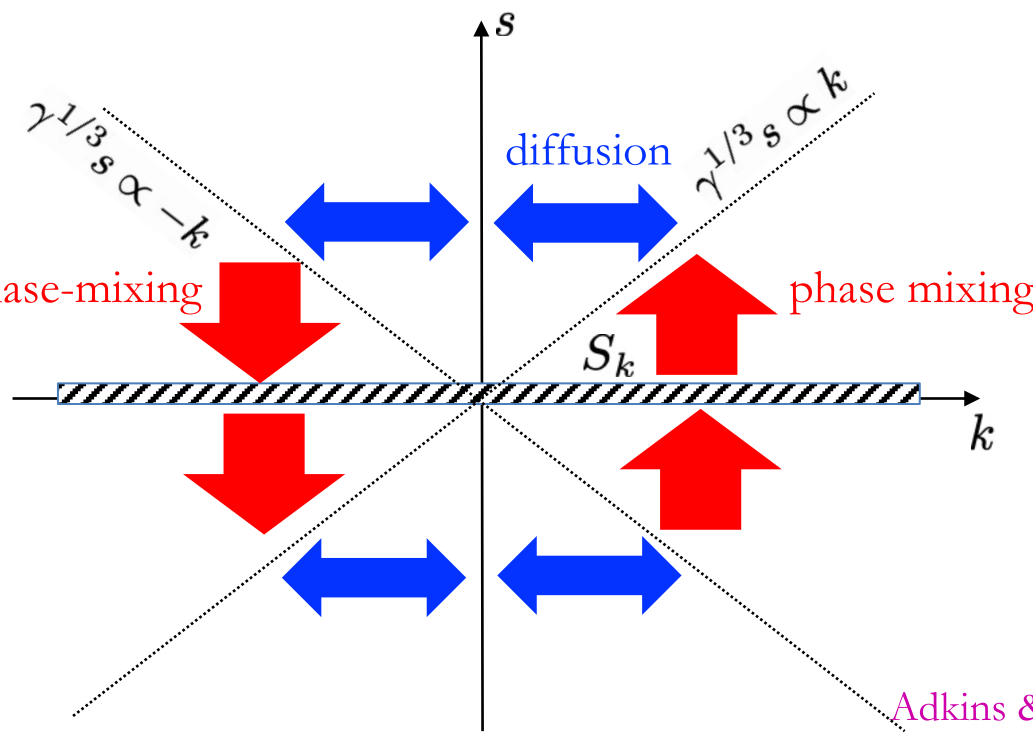

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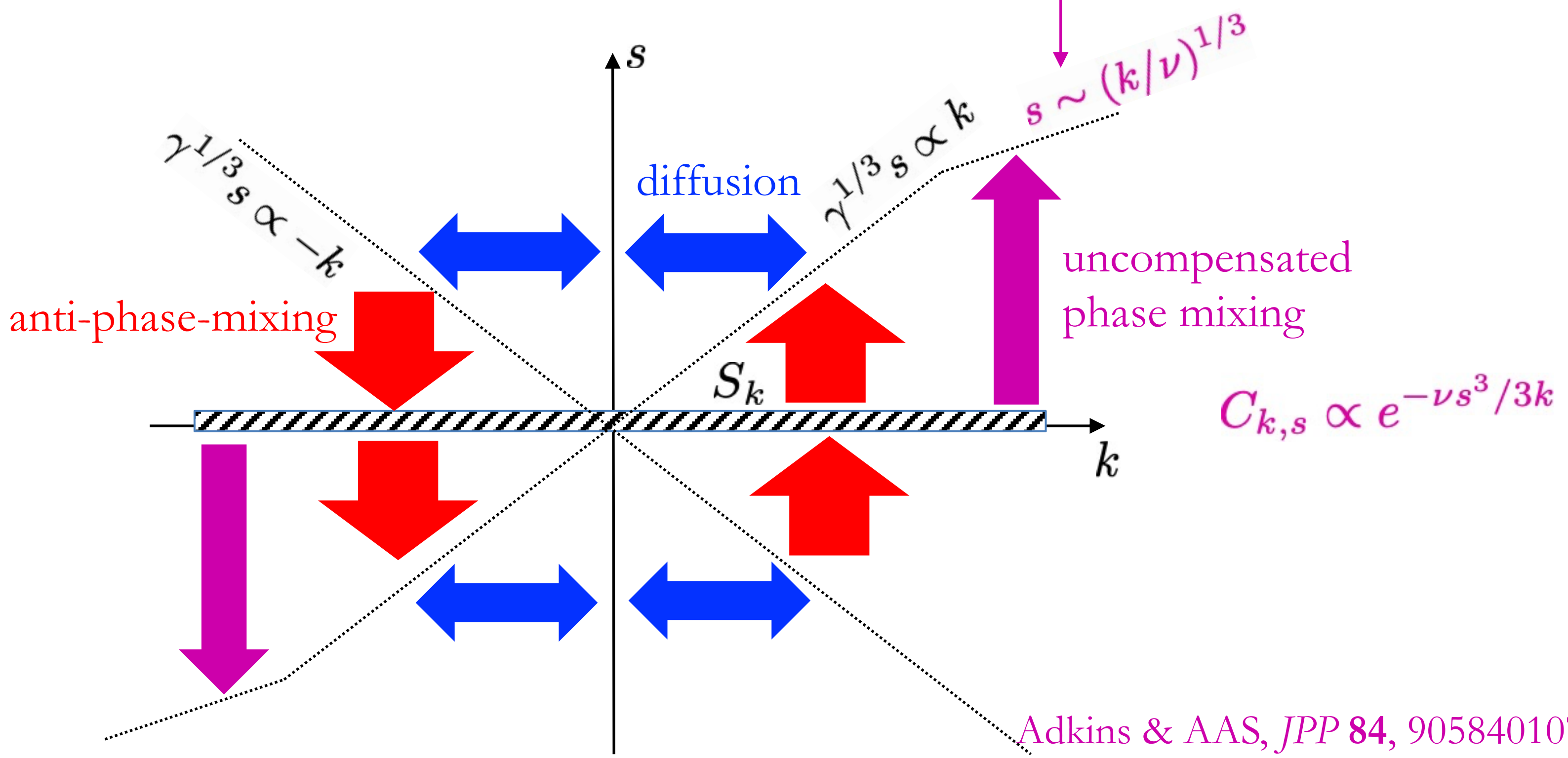

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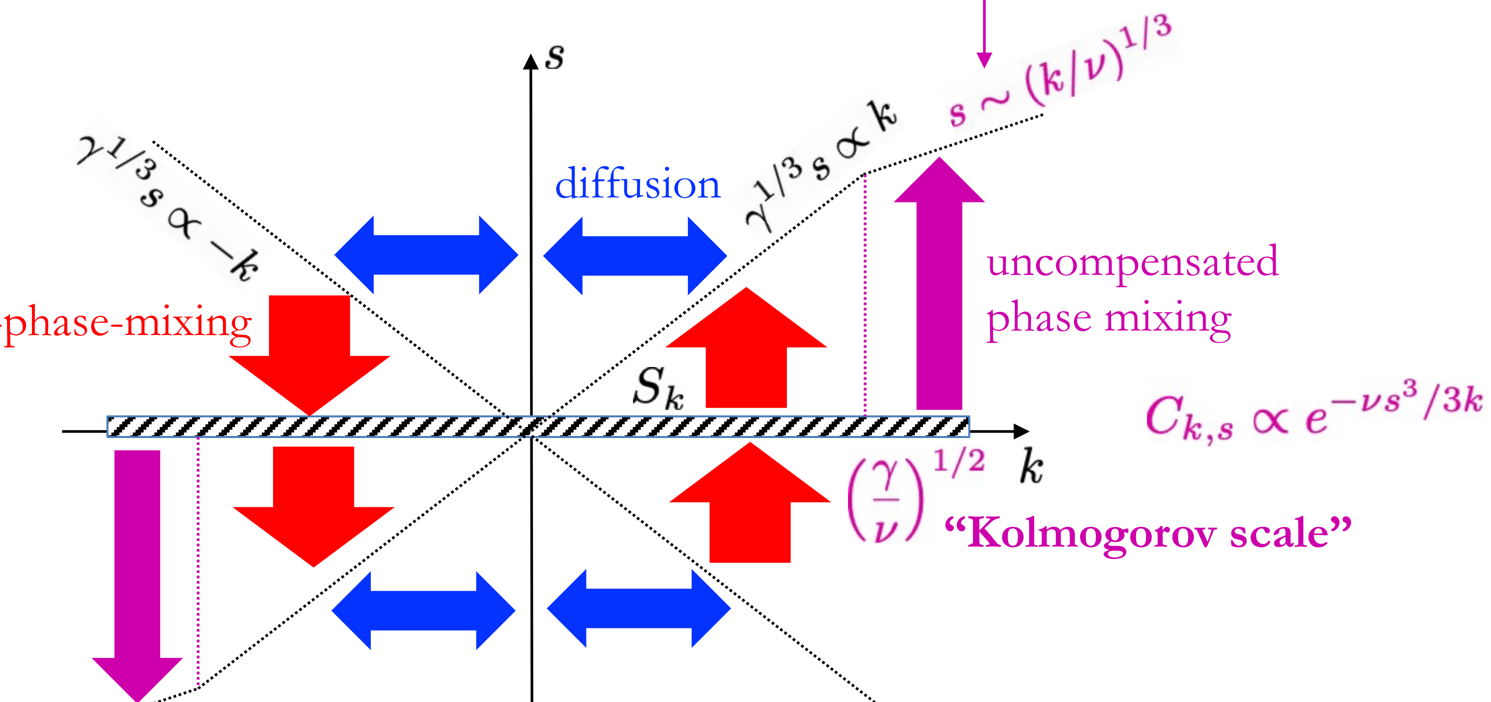

Adkins & AAS, *JPP* **84**, 905840107 (2018)

Landau Damping Resurrected (a bit)

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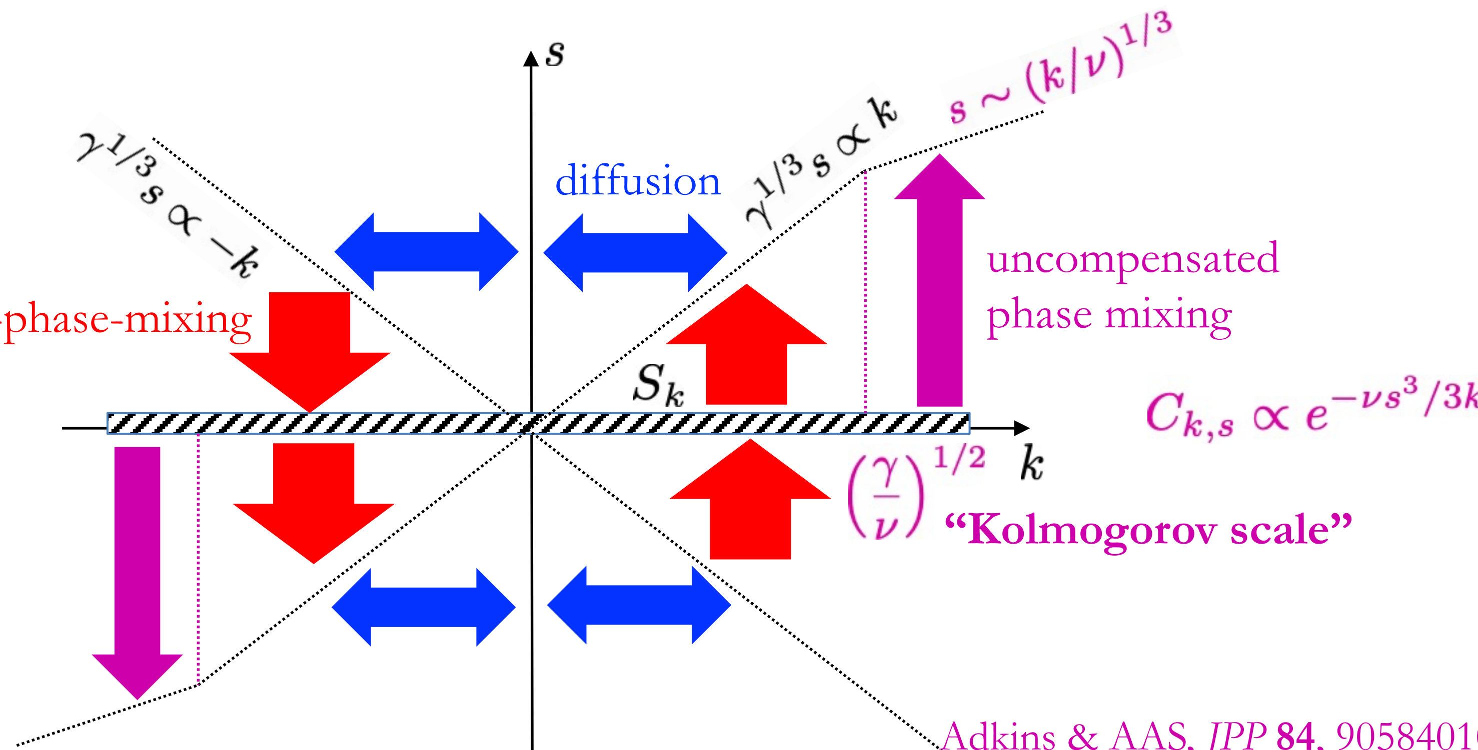
$$\cancel{\partial_t C_{k,s}} + k\partial_s C_{k,s} \approx S_k \delta(s) + \gamma s^2 \partial_k^2 C_{k,s} - \nu s^2 C_{k,s}$$

Thus, Landau damping is suppressed only at $k \ll (\gamma/\nu)^{1/2}$
(analogue of "inertial range")



Adkins & AAS, *JPP* **84**, 905840107 (2018)

Velocity-Space Correlations

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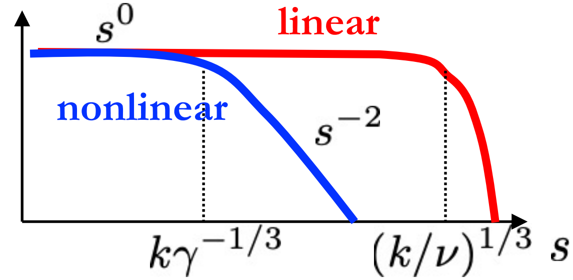
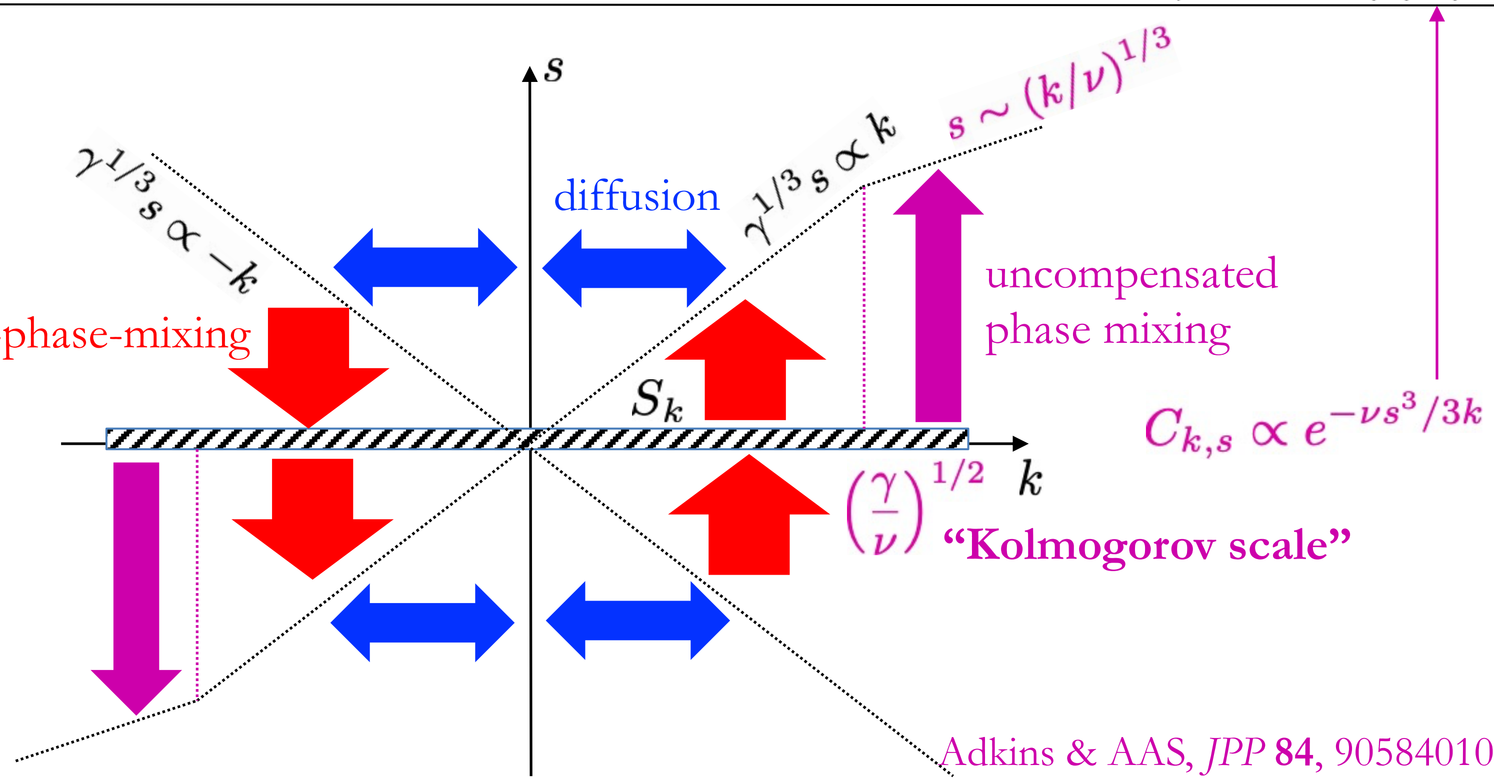
Velocity-space spectrum is $C_{k,s} \sim \dfrac{1}{s^2}$ at $s \gg k\gamma^{-1/3}$

steeper than for Landau damping: $C_{k,s} \sim \text{const}$





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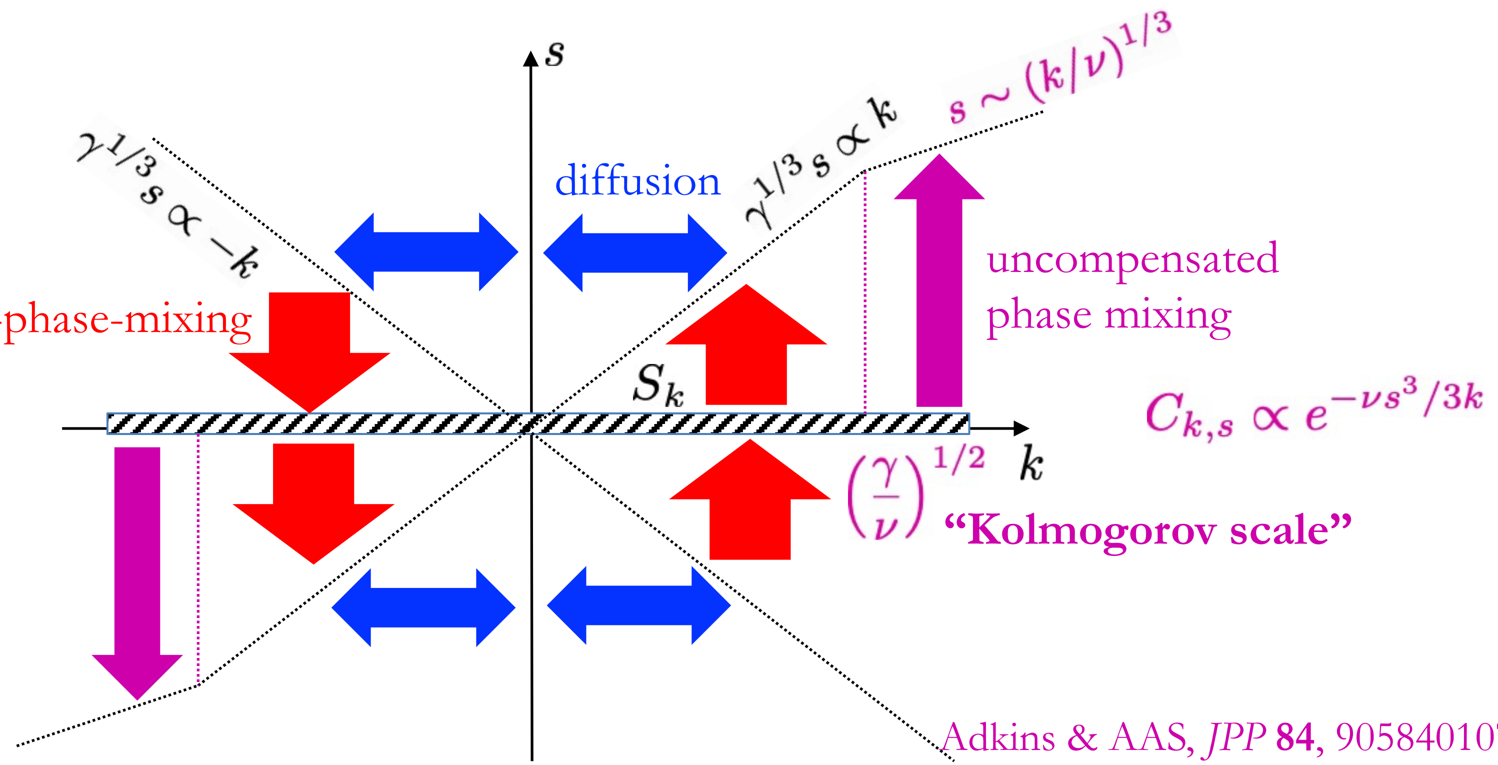
Velocity-space scale

$$\boxed{\delta v \sim \frac{\gamma^{1/3}}{k}} \gg \left(\frac{\nu}{k}\right)^{1/3}$$



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$\Rightarrow$

Velocity-space scale

$$\boxed{\delta v \sim \frac{\gamma^{1/3}}{k}} \gg \left(\frac{\nu}{k}\right)^{1/3}$$

What can we learn from this about collisionless relaxation to universal equilibria?

$$\boxed{f = f_0(\mathbf{v}) + \delta f(\mathbf{r},\mathbf{v})}$$

Is there a universal collisionless equilibrium? $\longleftrightarrow$ Fine structure in phase space. **What is the structure of this "phase-space turbulence"?**

(or classes of equilibria independent of precise initial conditions)

Back to Relaxation

OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f_{\mathbf{k}}^*(\mathbf{v}')\rangle$

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Velocity-space scale

$$\boxed{\delta v \sim \frac{\gamma^{1/3}}{k}} \gg \left(\frac{\nu}{k}\right)^{1/3}$$

$$\partial_t f_0 = \partial_{\mathbf{v}} \cdot \int d\mathbf{v}'' \left[\mathsf{D}(\mathbf{v},\mathbf{v}'')\cdot\partial_{\mathbf{v}} f_0(\mathbf{v}) - \mathsf{D}(\mathbf{v}'',\mathbf{v})\cdot\partial_{\mathbf{v}''} f_0(\mathbf{v}'')\right]$$

$$\mathsf{D}(\mathbf{v}'',\mathbf{v}) = -\frac{16\pi^2 e^4}{m^2}\sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4}\mathrm{Im}\iint \frac{d\omega' d\omega}{(2\pi)^2}\frac{e^{i(\omega'-\omega)t}}{\epsilon^*_{\mathbf{k}\omega'}\epsilon_{\mathbf{k}\omega}}\int d\mathbf{v}' \frac{\langle h^*_{\mathbf{k}\omega'}(\mathbf{v}') h_{\mathbf{k}\omega}(\mathbf{v})\rangle}{\omega - \mathbf{k}\cdot\mathbf{v}'' + i0}$$

$$\boxed{f = f_0(\mathbf{v}) + \delta f(\mathbf{r},\mathbf{v})}$$

Is there a universal collisionless equilibrium? (or classes of equilibria independent of precise initial conditions) $\longleftrightarrow$ Fine structure in phase space. **What is the structure of this "phase-space turbulence"?**

Effective Collision Integral

OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f^*_{\mathbf{k}}(\mathbf{v}')\rangle$

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$\Rightarrow$ **Velocity-space scale**

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two-time correlation function of h

Effective Collision Integral

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$$\int d\mathbf{s}\, e^{i\tau[\mathbf{k}\cdot(\mathbf{v}''-\mathbf{v}) - Ds^2] + \tau^2 D\mathbf{s}\cdot\mathbf{k} - \tau^3 Dk^2/3 - i\mathbf{s}\cdot(\mathbf{v}-\mathbf{v}')} C_{\mathbf{k},\mathbf{s}}(\mathbf{v}')$$

all this stuff follows from the equation
for the two-time correlation function,
which is easily derived, boring, and can be solved exactly;
take-away: $\tau_c \sim (Dk^2)^{-1/3}$, where $D = \dfrac{1}{d}\sum_{\mathbf{p}} p^2 \varkappa_p$

our friend the one-time
correlation function
(phase-space spectrum)

Effective Collision Frequency

OK, let us focus just on turbulence. Quantity of interest: $C_{\mathbf{k}}(\mathbf{v},\mathbf{v}') = \langle f_{\mathbf{k}}(\mathbf{v}) f_{\mathbf{k}}^*(\mathbf{v}')\rangle$

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$$\int d\mathbf{s}\, e^{i\tau[\mathbf{k}\cdot(\mathbf{v}''-\mathbf{v}) - Ds^2] + \tau^2 D\mathbf{s}\cdot\mathbf{k} - \tau^3 Dk^2/3 - i\mathbf{s}\cdot(\mathbf{v}-\mathbf{v}')} C_{\mathbf{k},\mathbf{s}}(\mathbf{v}')$$

In quasilinear theory, $D = 0$, $C_{\mathbf{k},\mathbf{s}}$ is independent of $\mathbf{s}$, so the above turns into

$$\propto \delta v^d \delta(\mathbf{v}-\mathbf{v}')\delta\big(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'')\big)$$

Hence an effective "collisionless collision integral" with $\nu_{\text{eff}} \propto \delta v^d \to 0$ as $\nu \to 0$.
In contrast, now we can have an effective collision frequency independent of ν!

Effective Collision Frequency

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Velocity-space spectrum is $C_{k,s} \sim \frac{1}{s^2}$ at $s \gg k\gamma^{-1/3}$

steeper than for Landau damping: $C_{k,s} \sim \text{const}$

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$$\boxed{\delta v \sim \frac{\gamma^{1/3}}{k}} \gg \left(\frac{\nu}{k}\right)^{1/3}$$

Thus, "collisionless collisions" may be able to cause (turbulent) plasma to relax to some universal equilibrium on a time scale independent of Coulomb collisionality.

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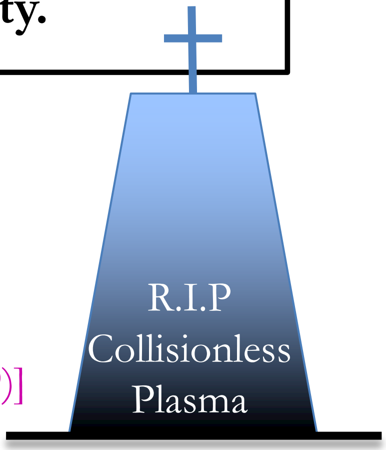
Conclusions (tentative)

Thus, "collisionless collisions" may be able to cause (turbulent) plasma to relax to some universal equilibrium on a time scale independent of Coulomb collisionality.

Turbulent plasma is effectively collisional?

NB: It might be (and most likely is) collisional in a different way than the Landau-Lenard-Balescu-collisional plasma. [Adkins & AAS, in prep. (2019)]

***To do*: work out the new collision integral and find its solutions.**



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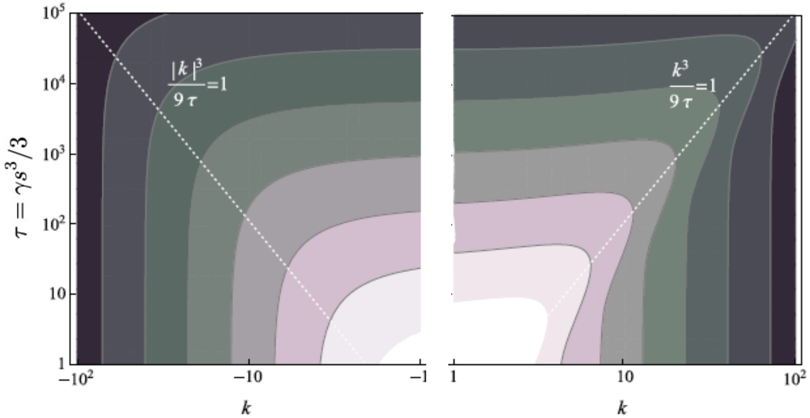
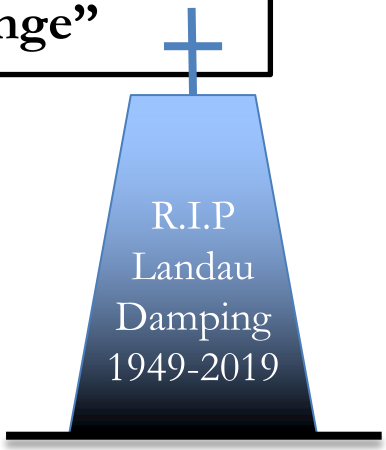


Thus, Landau damping is suppressed in a certain "inertial range"

$$C_{k\to 0,s} \propto s^{-2}$$
$$C_{k,s\to 0} \propto k^{-2}$$



Universal phase-space spectrum (1D), steeper than the linear phase-mixing one

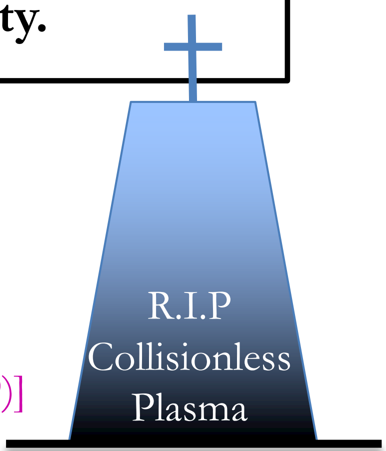
[Adkins & AAS, *JPP* **84**, 905840107 (2018)]

<u>To do</u>: 3D, less brutalist approximations, different regimes etc.

Thus, "collisionless collisions" may be able to cause (turbulent) plasma to relax to some universal equilibrium on a time scale independent of Coulomb collisionality.

⇓

Turbulent plasma is effectively collisional?



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A solvable model of Vlasov-kinetic plasma turbulence in Fourier–Hermite phase space

T. Adkins[1,2] **and A. A. Schekochihin**[1,2,†]

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A class of simple kinetic systems is considered, described by the one-dimensional Vlasov–Landau equation with Poisson or Boltzmann electrostatic response and an energy source. Assuming a stochastic electric field, a solvable model is constructed for the phase-space turbulence of the particle distribution. The model is a kinetic analogue of the Kraichnan–Batchelor model of chaotic advection. The solution of the model is found in Fourier–Hermite space and shows that the free-energy flux from low to high Hermite moments is suppressed, with phase mixing cancelled on average by anti-phase-mixing (stochastic plasma echo). This implies that Landau damping is an ineffective route to dissipation (i.e. to thermalisation of electric energy via velocity space). The full Fourier–Hermite spectrum is derived. Its asymptotics are $m^{-3/2}$ at low wavenumbers and high Hermite moments (m) and $m^{-1/2}k^{-2}$ at low Hermite moments and high wavenumbers (k). These conclusions hold at wavenumbers below a certain cutoff (analogue of Kolmogorov scale), which increases with the amplitude of the stochastic electric field and scales as inverse square of the collision rate. The energy distribution and flows in phase space are a simple and, therefore, useful example of competition between phase mixing and nonlinear dynamics in kinetic turbulence, reminiscent of more realistic but more complicated multi-dimensional systems that have not so far been amenable to complete analytical solution.

Key words: plasma nonlinear phenomena

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