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EMERGENCE AND CONTROL OF AUTORESONANT NONLINEAR WAVES

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OUTLINE

1. Autoresonance phenomenon: past and present research
2. Autoresonance of nonlinear waves
3. Wave modulations (pulses) in a channel
4. Excitation and control of multiphase nonlinear waves

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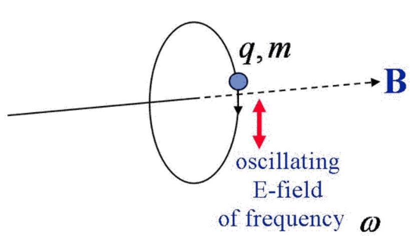
1. AUTORESONANCE PHENOMENON

Autoresonance: a property of driven nonlinear systems to stay in resonance when parameters vary in time and/or space

Example:



Cyclotron resonance:

$$\Omega_c = \frac{qB}{mc} = \omega$$

Cyclotron autoresonance:

$$\Omega_c = \frac{qB}{mc} \approx \omega(t)$$

$\omega(t)\searrow$ → $m(t)\nearrow$ (acceleration)

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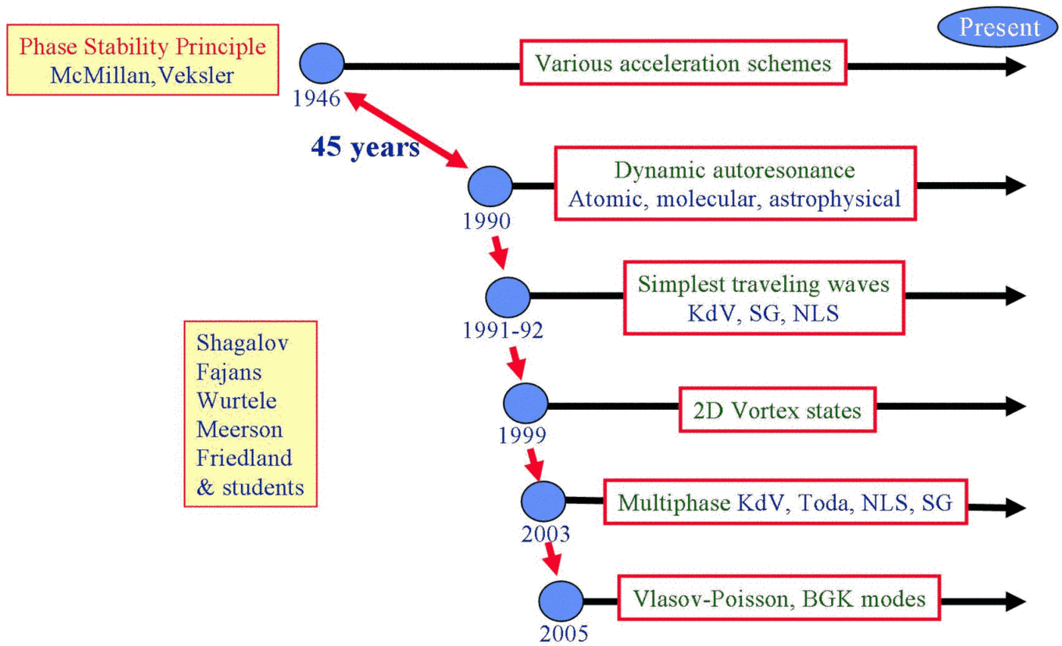
PAST AND PRESENT RESEARCH



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2. AUTORESONANCE OF NONLNEAR WAVES

Suppose the medium supports a slowly varying waveform

$$\boxed{\hat{N}(E)=0} \longrightarrow E(x,t)=E(\theta,A)$$

Slow amplitude $A(x,t)$, Fast phase $\theta(x,t)$

Slow $k(x,t)=\theta_x$, $\omega(x,t)=-\theta_t$

Nonlinear dispersion relation: $D(k,\omega,A)=0$

Drive by a small amplitude eikonal wave

$$\boxed{\hat{N}(E)=\varepsilon(x,t)\cos\theta_d}$$

Slow amplitude $\varepsilon(x,t)$, Fast phase $\theta_d(x,t)$

Slow $k_d(x,t)=(\theta_d)_x$, $\omega_d(x,t)=-(\theta_d)_t$

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WHY PHASE-LOCKING MEANS CONTROL?

Assume continuing phase-locking (autoresonance)

$$\theta(x,t) \approx \theta_d(x,t)$$

$$\downarrow$$

$$k(x,t) \approx k_d(x,t),\ \omega(x,t) \approx \omega_d(x,t)$$

Then nonlinear dispersion $D(k,\omega,A) = 0$ yields

$$D\big(k_d(x,t), \omega_d(x,t), A\big) \approx 0$$

$$\downarrow$$

$$A = A(x,t)$$

$$\downarrow$$

$$E \approx E(\theta_d, A)$$

The driven wave is fully controlled by the driving wave

QUESTIONS: 1. How to phase-lock?
2. Is the phase-locking stable?

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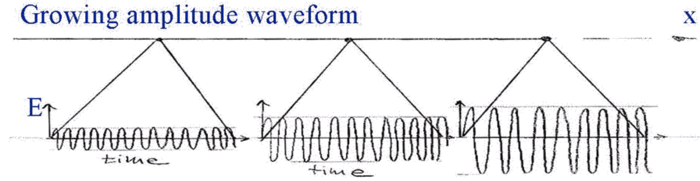
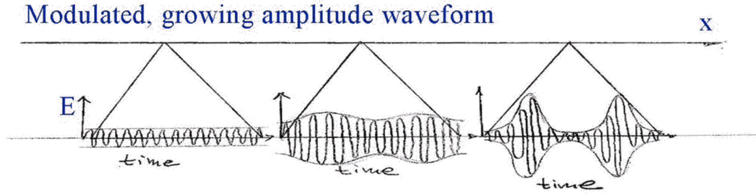
3. AUTORESONANT WAVES IN A CHANNEL (FIBER, PLASMA)

Q: How to excite one of the following waveforms?

Growing amplitude waveform



Modulated, growing amplitude waveform



FORMATION OF PULSES

A: Drive the system by a superposition of eikonal waves and pass trough resonances

$$\hat{N}(E) = \varepsilon f(x,t)$$

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WAVE ENVELOPE EQUATION

carrier frequency

$$E = \psi(x,t)\exp[\,i(k_0 x - \omega_0 t)] + cc$$

slow envelope — rapid phase

Nonlinear dispersion $k = K(\omega) + \beta|\psi|^2$

ENVELOPE EQUATION

$$i(\psi_x + K_0'\psi_t) - \tfrac{1}{2}K_0''\,\psi_{tt} + \beta|\psi|^2\psi = 0$$

Retarded time: $t - K_0'x \;\Rightarrow\; t$

+ Dimensionless variables

Nonlinear Schrödinger equation: $i\psi_x + \psi_{tt} + |\psi|^2\psi = 0$

for normal dispersion

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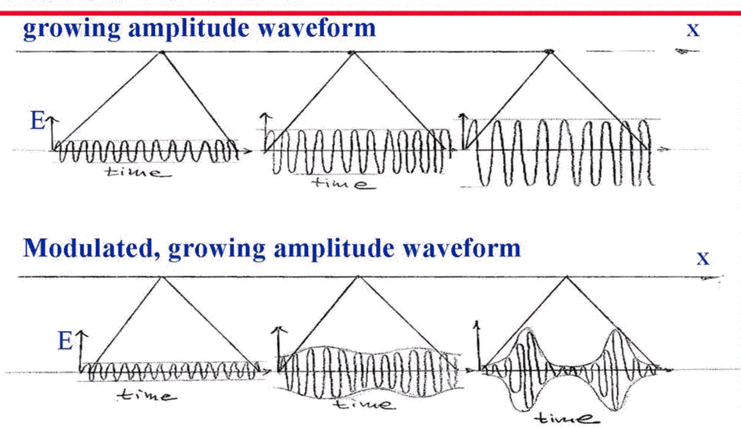
NLS EQUATION HAS MANY SOLUTIONS

$$i\psi_x + \psi_{tt} + |\psi|^2\psi = 0$$

1. time-independent envelope $\psi = a\exp(ia^2x)$
2. soliton solution $\psi = \sqrt{2}a\exp(ia^2x)\,\mathrm{sech}\,(at)$
3. infinitely many other solutions



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DRIVEN ENVELOPE EQUATION

Add driving:

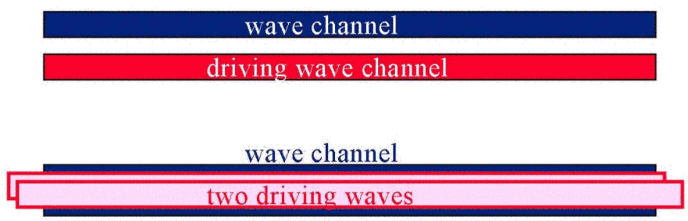
$$i\psi_x + \psi_{tt} + |\psi|^2\psi = \varepsilon \exp[i(\int \kappa dx - \nu t)]$$

$$\nu = \omega_d - \omega_0 \qquad \kappa = k_d(x) - k_0$$

Two types of couplings to external driving waves

Linear coupling

Resonant
3-wave coupling



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AUTORESONANT GROWING AMPLITUDE WAVES

$$i\psi_x + \psi_{tt} + |\psi|^2\psi = \varepsilon \exp[i(\int \kappa dx - \nu t)]$$

$$\nu = 0, \quad \kappa = \alpha x$$

$$\psi_t = 0$$

$$i\psi_x + |\psi|^2\psi = \varepsilon \exp(\tfrac{1}{2} i \alpha x^2)$$

$$\psi = \Psi \exp(\tfrac{i}{2}\alpha x^2) \qquad i\Psi_x + (|\Psi|^2 - \alpha x)\Psi = \varepsilon$$

Two-parameter system?

$$\xi = \alpha^{1/2} x, \quad \mu = \alpha^{-3/4}\varepsilon, \quad \tilde{\Psi} = \alpha^{-1/4}\Psi$$

Generic, single parameter NLS for passage through resonance

$$i\tilde{\Psi}_\xi + (|\tilde{\Psi}|^2 - \xi)\tilde{\Psi} = \mu$$

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PASSAGE THROUGH RESONANCE

$$i\tilde{\Psi}_{\xi} + (|\tilde{\Psi}|^2 - \xi)\tilde{\Psi} = \mu$$

$\xi = -\infty \rightarrow \xi = +\infty$

Two non-vanishing asymptotic solutions at $\xi \to +\infty$

Saturated solution
$\tilde{\Psi} = A\exp(-i\xi^2/2)$

Phase locked solution
$\tilde{\Psi} = \xi^{1/2}$

Single parameter μ controls the bifurcation

Phase locked solution for
$\mu > \mu_{th} = 0.41$

$$\varepsilon > \varepsilon_{th} = 0.41\alpha^{3/4}$$

First observed in experiments
on magnetized electron clouds
Fajans, Gilson, Friedland (1999)

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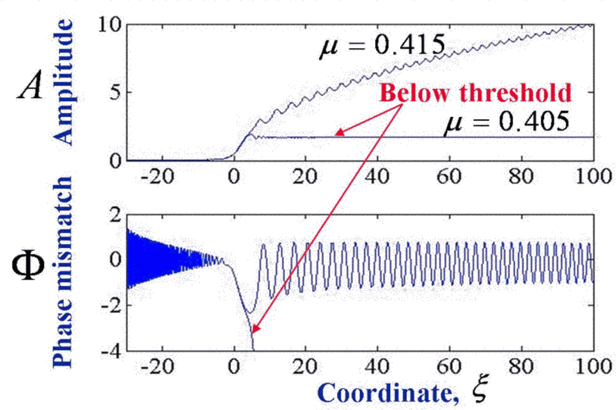
THRESHOLD PHENOMENON

$\tilde{\Psi} = Ae^{i\Phi}$



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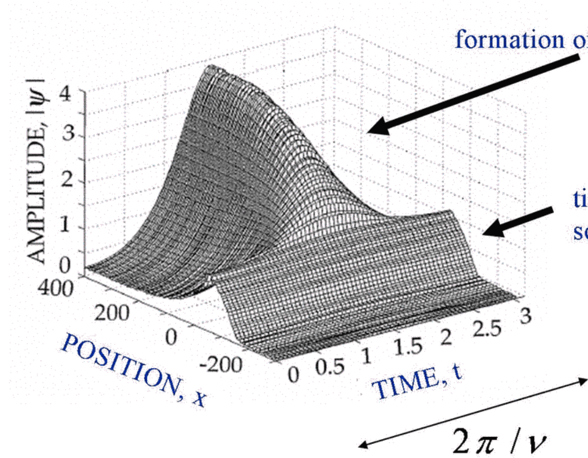
MODULATED, GROWING AMPLITUDE WAVES

$$i\psi_x + \psi_{tt} + |\psi|^2\psi = \varepsilon[1 + r\cos(\nu t)]\exp[\,i(\int \kappa dx)]$$

Friedland, Shagalov, 1998



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ESTIMATES FOR EM WAVES IN A PLASMA CHANNELS

Underdense case $\omega_p / \omega << 1$

Weakly relativistic (mass) nonlinearity

Condition for significant time compression of modulation (thin and tall solitons)

$$\omega_p / \omega < (\omega / \nu)(2\Delta k / k)^{1/2}$$

(ν — modulation frequency)

Threshold condition

$$a_{drive}^2 > 50(d/\lambda)^2(\omega/\omega_p)^2 N^{-3/2}$$

$a = \frac{v_{osc}}{c}$

d — distance between coupled channels

N — number of wavelengths in the channel

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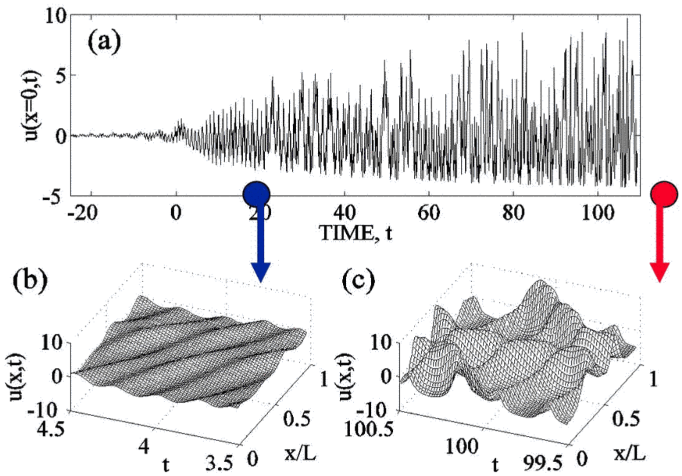
4. MULTIPHASE NONLINEAR WAVES $F(k_1 x - \nu_1 t,\, k_2 x - \nu_2 t,\, \ldots)$

Q: How to excite multiphase waves of, say, the KdV equation?

A: $u_t + uu_x + u_{xxx} = \Sigma\, \varepsilon_n \cos[k_n x - \int \omega_n(t)dt]$



Periodic KdV case
$u(x,t) = u(x+L,t)$
Friedland, Shagalov (2004)

3-phase wave

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IST Diagnostics

$u(x,t)$ → Associated linear eigenvalue problem

$$\psi_{xx} + [E + u(x,t)]\psi = 0$$
$$\psi(0) = 1, \quad \psi(x+L) = \pm\psi(x)$$

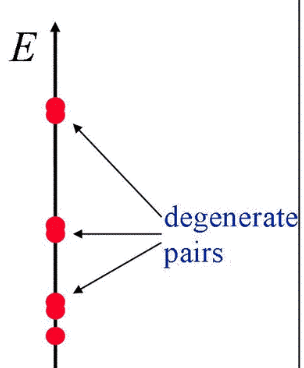
E - Main IST spectrum $\quad E = const(t)$

EXAMPLE 1

$u(x,t) = 0$

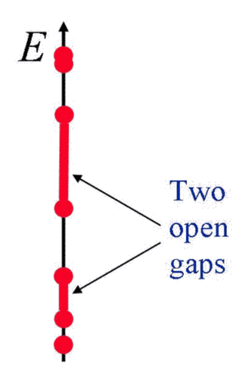
$\psi = \cos(kx)$

$E = k^2$
$k = \frac{\pi}{L} n$
$n = 0, \pm 1, \pm 2, \ldots$



EXAMPLE 2

A two-phase solution



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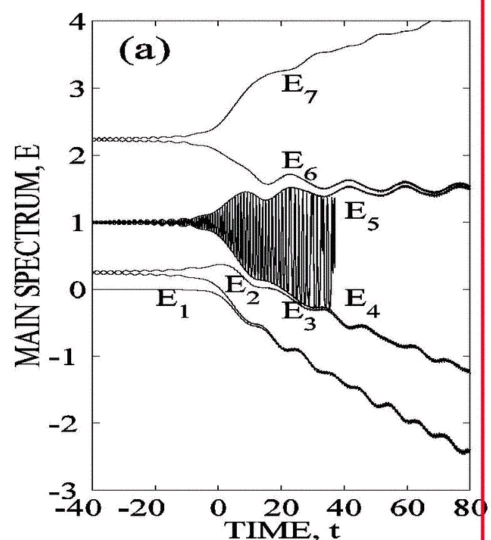
Synchronization is seen via spectral IST analysis

3-phase KdV wave

Evolution of main IST spectrum



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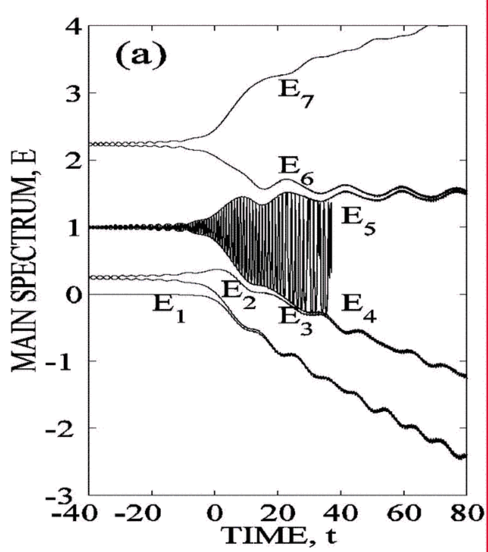
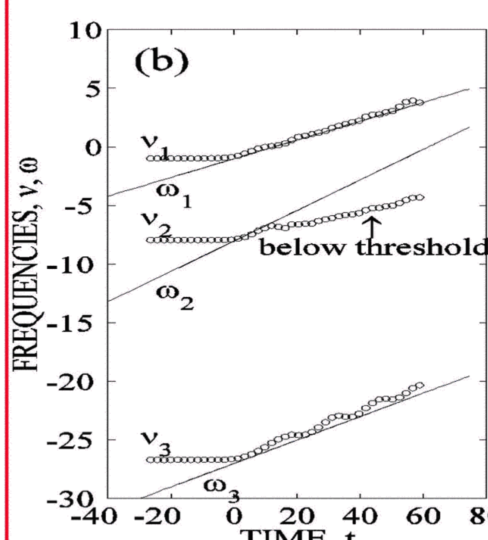


Frequency locking



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A general solution: Synchronization by passage through resonances.

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(3) **Mathematical methods for autoresonant waves**:
(1) Averaged variational principle for single phase waves
(work exists for a general case with application to SG, KdV, and NLS).
(2) Spectral IST approach for diagnostics of autoresonant multiphase waves
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(5) Applications in astrophysics, atomic/molecular physics, plasmas, fluids and nonlinear waves: www.phys.huji.ac.il/~lazar