One Hour of Stellar Structure

1. Hydrostatic Balance

The star is in a dynamical equilibrium with no accelerations

$$\Rightarrow \frac{d\underset{\sim}{v}}{dt} = 0 \Rightarrow \frac{dP}{dr} = -\rho \frac{G m(r)}{r^2}$$

(Force due to pressure gradient = Force due to gravity)

$$m(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$$

= mass interior to r

$\Rightarrow$ Gas must be hot at the center of the star in order to hold it up!

$\Rightarrow$ Cheap & Easy Way

$$\Rightarrow \frac{P}{R} \sim \rho \frac{GM}{R^2} \quad \& \quad P = \frac{\rho k T}{\mu m_p}$$

μ = mean molecular weight

$$\Rightarrow \frac{\rho k T}{\mu m_p} \sim \rho \frac{GM}{R}$$

or just $\boxed{k T_c \sim \mu \frac{G M m_p}{R}}$

So this already tells us much of what we need to know:

1. $R \downarrow$ $T_c \uparrow$ for fixed M
2. $\mu \uparrow$ $T_c \uparrow$ for fixed M & R (important for evolution)

There is one other critical point which is the Virial Thm.

Take $\frac{dP}{dr} = -\rho \frac{G m(r)}{r^2}$

multiply by $4\pi r^3 dr$ on both sides. & $\int$

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = -\int_0^R \frac{G m(r)}{r} (\rho \cdot 4\pi r^2) dr$$

$$4\pi r^3 P(r)\Big|_0^R - 3\int_0^R 4\pi r^2 P(r) dr = -\int_0^R \frac{G m(r)}{r} dm(r)$$

$$-3\int_0^R 4\pi r^2 P(r) dr = + E_{GR}$$

(As we set $E_{GR} = -\int$)

But we can write this as 3

$$\langle P\rangle = \frac{1}{V}\int_0^R 4\pi r^2 dr\, P(r) =$$ Volume Avg Pressure

So $$\boxed{-3V\langle P\rangle = E_{GR}}$$

This is what we refer to as the virial thm. The total energy is

$$E_{tot} = E_K + E_{GR}$$

But for non-relativistic particles

$$P = \frac{2}{3} n\langle K\rangle$$

$\langle K\rangle$ = Avg. kinetic energy

So

$$E_{tot} = \underbrace{N\langle K\rangle}_{E_K} - \underbrace{3N\frac{2}{3}\langle K\rangle}_{E_{GR}}$$

$$\Rightarrow \boxed{E_{tot} = -N\langle K\rangle}$$

⇒ So a hydrostatic object is bound ~~and~~ by − kinetic energy. Rel. particles have $P = \frac{1}{3}n\langle K\rangle \Rightarrow E_{tot} = 0$!

What Does This Mean? 4

$E_{tot} = -N\langle K\rangle$; $E_{GR} = -2N\langle K\rangle$

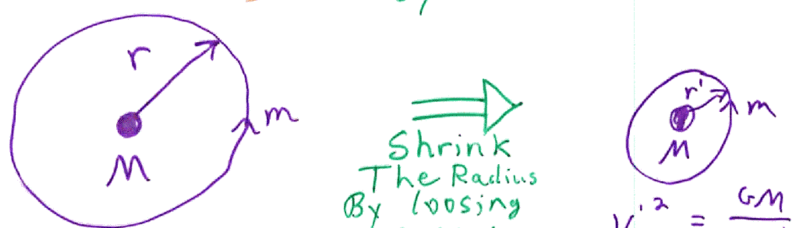
1.) Energy loss ⇒ E_{tot} gets more negative ⇒ $\langle K\rangle$ increases!

⇒ The central temperature rises as a star loses energy!

2) Energy Gain ⇒ E_{tot} gets less negative ⇒ $\langle K\rangle$ decreases!

⇒ T_c drops as a star gains energy! (This is why most nuclear burning in ✱'s is stable)

Analogy with Kepler Orbits



$$v^2 = \frac{GM}{r} \qquad v'^2 = \frac{GM}{r'}$$

$$E_{tot} = -\frac{GMm}{r} + \frac{1}{2}m\frac{GM}{r} = -\frac{1}{2}\frac{GMm}{r} \quad (-E_K)$$

$$E_{tot} = -\frac{1}{2}\frac{GMm}{r'}$$

So the particle lost energy and is now $v\uparrow$

A

$$\boxed{P_{rad}? \quad \text{or} \quad P_{Gas}?}$$

$$kT \sim \frac{GMm_p}{R}$$

$$\frac{P_{rad}}{P_{Gas}} \sim \frac{aT^4 m_p}{\rho kT}$$

$$a = \frac{8\pi^5}{15}\frac{k^4}{h^3c^3} \simeq \frac{k_B^4}{(\hbar c)^3}$$

$$\sim \frac{(kT)^3 m_p}{\rho\,(\hbar c)^3} = \left(\frac{GMm_p}{R}\right)^3 \frac{m_p R^3}{M\,(\hbar c)^3}$$

$$\sim \left(\frac{Gm_p^2}{\hbar c}\right)^3 \left(\frac{M}{m_p}\right)^2 = \alpha_G^3 \qquad \alpha_G = \frac{Gm_p^2}{\hbar c} = 6\times10^{-39}$$

$$\Rightarrow \quad M_{char} \approx m_p\left(\frac{1}{\alpha_G^{3/2}}\right)$$

$$= 10^{57}\, m_p \simeq 2M_\odot$$

[Real Answer is $\gtrsim 60 M_\odot$ or so]

B

Chandra Mass

$$P \sim \frac{GM^2}{R} \simeq n_e$$

$$E_F \sim \frac{GM}{R} m_p \simeq P_F c$$

$$P_F = \hbar k_F \simeq \hbar n_e^{+1/3} = \hbar\left(\frac{M}{R^3 m_p}\right)^{+1/3}$$

$$\frac{GMm_p}{\cancel{R}} \sim c\hbar\left(\frac{M}{m_p}\right)^{1/3}\frac{1}{\cancel{R}}$$

$$G^3M^3m_p^3 = (c\hbar)^3\frac{M}{m_p}$$

$$\frac{M^2}{m_p^2} = \frac{(\hbar c)^3}{G^3m_p^6} \Rightarrow M_{char} \sim m_p\frac{1}{\alpha_G^{3/2}}$$

Again... Same..

Just due to Virial Thm.

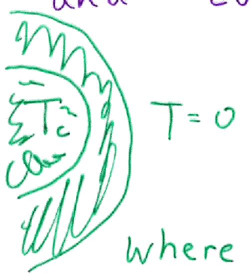
Stellar Luminosity 5

OK, so now we know that the star is hot in the center, roughly

$$kT_c \sim \frac{\mu G M m_p}{R}$$

and cold at the edge.

⇒ Heat Transport.



The flux is generally

$$F_x = -\frac{1}{3} v \ell \frac{d}{dx} u$$

where v = speed of particles carrying heat

ℓ = Their mean free path

u = " energy density.

Though low in u, the high $v + \ell$ of photons make them the best transporters of heat

$$\Rightarrow \quad F_x = -\frac{1}{3} c \frac{1}{n_e \sigma_{Th}} \frac{d}{dr} a T^4$$

(We only do Thomson scattering).

$$\Rightarrow \quad F_x = -\frac{4}{3} \frac{a c T^3}{n_e \sigma_{Th}} \frac{dT}{dr}$$

Again, Lets estimate via 6

$$L \sim 4\pi R^2 F \sim \frac{16\pi}{3} R^2 \frac{a c T_c^4 m_p}{\sigma_{Th} \rho} \frac{1}{R}$$

For this we drop all #'s & use $\rho \sim M/R^3$;

$$\Rightarrow \quad L \sim \frac{R^2 a c}{\sigma_{Th}} \frac{R^3}{M} m_p \frac{1}{R} \left(\frac{\mu G M m_p}{R k_B}\right)^4$$

$$\propto \frac{M^3 R^5}{R^5} \propto M^3 \mu^4 \text{ roughly}$$

Plugging in all the fund constants gives a prefactor not all that far from $L_\odot$ at $M = M_\odot$

Two Crucial Facts

1). When Thomson Scattering dominates ($M > M_\odot$) then L is independent of the stellar radius and $\boxed{\propto M^3}$!!

2.) Luminosity does care about the composition

Pure H $\quad P = \frac{2kT\rho}{m_p} \quad \mu = \frac{1}{2}$

Pure He $\quad P = n_\alpha kT + 2 n_\alpha kT = 3 n_\alpha kT = \frac{3}{4}\frac{\rho kT}{m_p}$

$\Rightarrow \mu = 4/3 \Rightarrow L \uparrow$ for He

Stellar Contraction

T

In the abscence of energy production a star of mass M & radius R collapses at a timescale (Kelvin-Helmholtz)

$$t \sim \frac{E_{tot}}{L} \sim \frac{GM^2/R}{4\times10^{33}(M/M_\odot)^3} \qquad M > M_\odot$$

$$t_{KH} \sim 3\times10^7\,\text{yrs}\left(\frac{R_\odot}{R}\right)\left(\frac{M_\odot}{M}\right) \qquad M > M_\odot$$

So that the central temperature rises as $R\downarrow$. Now, it is always in hydrostatic balance, as $t_{KH} \gg t_{dyn} \sim$ hrs.

So:

$$kT_c \sim \frac{GMm_p}{R} \;;\; \rho \sim \frac{M}{R^3} \Rightarrow R \sim \left(\frac{M}{\rho}\right)^{1/3}$$

$$\Rightarrow \quad kT_c \sim \frac{GMm_p}{M^{1/3}}\rho^{1/3} \propto M^{2/3}\rho^{1/3}$$

So as the star contracts, the central conditions more or less follows

$$T_c \approx 2\times10^6\,K\left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{\rho_c}{gr\,cm^{-3}}\right)^{1/3}$$

However, we also must be sure gas is ideal. Well, the e^- become degenerate when

$$\frac{h}{m_e v_{e,th}} = \lambda_{DeBroglie} \sim n_e^{-1/3} \sim \text{avg. } e^- \text{ spacing}$$

Degenerate when $E_F \sim kT$

$$\frac{p_f^2}{2m_e} \sim kT$$

but $n_e = \frac{8\pi}{3h^3}p_f^3 = \frac{\rho}{m_p}$

so $p_f^3 = \frac{3h^3\rho}{8\pi m_p}$

$$kT\cdot 2m_e \sim \left(\frac{3h^3\rho}{8\pi m_p}\right)^{2/3}$$

$$T \sim \frac{1}{2m_e k}\left(\frac{3h^3}{8\pi m_p}\right)^{2/3}\rho^{2/3}$$

$$T \simeq 3\times10^5\,K\,\rho^{2/3}$$

$$\rho = \frac{3M}{4\pi R^3}$$

$$kT \simeq \frac{1}{10}\frac{GMm_p}{R} \qquad R^3 = \frac{3M}{4\pi\rho}$$

$$= \frac{1}{10}\frac{GMm_p}{(3M)^{1/3}}(4\pi\rho)^{1/3}$$

$$= \left(\frac{4\pi}{3}\right)^{1/3}\frac{1}{10}\,GM^{2/3}m_p\,\rho^{1/3}$$

$$T = 2\times10^6\left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{\rho}{1}\right)^{1/3}$$

Thermonuclear Fusion

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You might have noticed that we have already built stars, but I have not yet generated any energy internally

$\Rightarrow$ Star's always contracting.

The only way to stop this is to find an energy source that will let the star stay at a fixed radius and thus

$$L_{out} = L_{in}.$$

Clearly we must do this, as the solar system is $\sim 5 \times 10^9$ yrs old

$$\gg t_{KH} \simeq 10^7 \text{ yrs.}$$

Star consists mostly of protons, so let's start with

$$p + p \rightarrow d + e^+ + \nu_e$$

This would be fast if it were not for the fact that we need a weak interaction. But to convert

$$p \rightarrow n$$

which is what we need anyway, we must do a weak interaction.

Two Routes

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PP Cycle	CN Cycle
$p + p \rightarrow d + e^+ + \nu_e$	$p + {}^{12}C \rightarrow {}^{13}N + \gamma$
$p + d \rightarrow {}^3He + \gamma$	${}^{13}N \rightarrow {}^{13}C + e^+ \ldots$ 800s
⋮	$p + {}^{13}C \rightarrow {}^{14}N + \gamma$
other strong reaction to make ${}^4He$	$p + {}^{14}N \rightarrow {}^{15}O + \gamma$
☐ = Weak Inter	${}^{15}O \rightarrow {}^{15}N + e^+$ 180s
☁ = Rate Limit Step	$p + {}^{15}N \rightarrow {}^4He + {}^{12}C$

In the CN cycle, weak interactions are done via β decays so that they do not directly enter into the time to go around the chain. However, they must overcome the Coulomb Barrier, as

$$\langle \sigma v \rangle \propto S \exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right)$$

where $E_G = (\pi \alpha Z_1 Z_2)^2 \, 2 m_r c^2$; $m_r = \frac{m_1 m_2}{m_1 + m_2}$

S = Nuclear Reaction S-factor $\propto$ cross-section $*$ E

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This first says that reactions are very T sensitive and secondly that it is harder to fuse high Z nuclei.

$\Rightarrow$ Low mass stars burn via pp cycle

At some T_c, the CN cycle wins, where

$$S_{pp}\exp\left(-3\left(\frac{E_{G,pp}}{4kT}\right)^{1/3}\right) \sim S_{p+Z}\exp\left(-3\left(\frac{E_{GZ}}{4kT}\right)^{1/3}\right)$$

$E_{G,pp} = 491$ keV ; $E_{G,p+Z} = 982\text{ keV } Z^2$

Lets ask for $T = 10^7 T_7$ K

$$\Rightarrow S_{pp}\exp\left(-\frac{15.6}{T_7^{1/3}}\right) \sim S_{p+Z}\exp\left(-\frac{19.7 Z^{2/3}}{T_7^{1/3}}\right)$$

$$\Rightarrow -\frac{15.6}{T_7^{1/3}} + \frac{19.7 Z^{2/3}}{T_7^{1/3}} \sim \ln\left(\frac{S_{p+Z}}{S_{pp}}\right) \sim \ln(10^{25}) \sim 57.6$$

Penalty for weak interaction

So we can calculate the required Z so that the CN cycle wins

$$Z_{Max} = \left(0.8 + 2.9\, T_7^{1/3}\right)^{3/2}$$

T_7	Z
0.1	3.1
0.5	5.5
1.0	7.1 ← ^{14}N
3.0	11

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Main Sequence

So, contraction halts whenever

$$L_{nuc} \equiv L$$

where obviously $L_{nuc} = \int \varepsilon\, 4\pi r^2 \rho\, dr$

and in reality it mostly scales with the Gamow exponent. Back of the envelope solution of this fixes the central T.

$\Rightarrow T \propto \frac{M}{R}$ so we know R! and T_{eff}!

For sun we find $T_c \simeq 10^7$ K, right at the pp/CNO boundary. Lets ask how T_c scales with M? (Above $M_\odot$)

$$L_{nuc} \propto \exp\left(-\frac{19.7 Z^{2/3}}{T_7^{1/3}}\right) \propto T_7^{\alpha}$$

where $\alpha = \frac{d\ln L_{nuc}}{d\ln T} = \frac{6.57 Z^{2/3}}{T_7^{1/3}} = \frac{24.}{T_7^{1/3}}$

Now we can say $L_* \propto M^3$ and the main sequence is $L_* \equiv L_{nuc}$

$$\Rightarrow \left(\frac{M}{R}\right)^{24} \propto M^3 \Rightarrow R_{MS} \propto M^{0.8}$$

Since $L = 4\pi R^2 \sigma T_{eff}^4 \Rightarrow T_{eff,MS} \propto M^{3/8}$

Evolution on Main Sequence

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For $M > M_\odot$: CNO cycle generates L

$$L \propto M^3 \mu^4$$

① ⇒ As central H depletes the μ increases so $L \uparrow$ What happens to T_c is curious. Since $L_{nuc} \propto T_c^{20}$ T_c needs to only change slightly to match.

⇒ T_c nearly constant ⇒ $kT_c \simeq \frac{\mu G M m_p}{R}$

$$\Rightarrow \quad R \propto \frac{\mu M}{T_c} \propto (\mu)\left(\frac{M}{T_c}\right)$$

so $\mu \uparrow$ then $R \uparrow$ ⇒ figure of HR.

② Once H is gone, what happens next depends on the stellar mass since He ignites at $\sim 10^8$ K.

⇒ ρ_c, T_c figure

③ Post He burning ⇒ c/o Ignition again depends on mass.

$2 M_\odot$ Star

① As the cores are completely convective they burn out once about $\simeq 0.2 M_\odot$ of Helium has built up ⇒ He Core Collapses

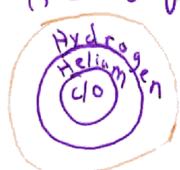
⇒ H Burning Ignites in a Shell

② Heads up the Red Giant Branch ⇒

③ He Flash (off-center) ignites

⇒ ? ⇒ Horizontal Branch (Central He Burning, Shell H)

④ C/O Core Develops & central He depletes ⇒ Asymptotic Giant Branch



⇒ Steady H Burning

⇒ He Shell Flashes

⇒ Would build up a $M_{ch} = 1.4 M_\odot$ C/O Core if it were not for winds

⇒ C/O WD formed at $\approx 0.6 M_\odot$

Red Giant Branch 14

Stars with $M < 2.5 M_\odot$ develop degenerate He cores after MS burning and burn H in a shell.

He Core $R_c \propto \frac{1}{M_c^{1/3}}$

Steady State

$L_{nuc} \propto M_{shell}\, \mathcal{E}_{nuc}$

$\equiv L$ rad transfer thru envelope.

But CN cycle is operating and the shell is always thick so $kT_s \sim \frac{G M_c m_p}{R_c}$

Now we find ρ_s in shell via

$L_{nuc} = L_{rad} \Rightarrow L_n \propto M_{sh} \mathcal{E} \propto \rho_s^2 R_c^3 T^\alpha \propto L_{nuc}$

$L_{rad} \propto R_c^2 \frac{}{\rho_s} \frac{T_s^4}{R_c} \propto R_c \frac{T_s^4}{\rho_s} \propto L_{rad}$

Equating $\Rightarrow \rho_s^3 \propto T_s^{4-\alpha} / R_c^2$ so we find

$L_{nuc} \propto R_c^3 T_s^\alpha \left[\frac{T_s^{4-\alpha}}{R_c^2}\right]^{\frac{2}{3}} \propto R_c^{5/3}\, T_s^{\frac{8}{3}}\, T_s^{\frac{\alpha}{3}}$

So we have found that 15

$$L \propto R_c^{5/3}\, T_s^{8/3}\, T_s^{\alpha/3}$$

and since it is CN cycle, we find that

$$\alpha = \frac{24}{T_7^{1/3}}$$

Burning occurs at $T_7 \sim 5 \Rightarrow \alpha = 15$
$T_7 \sim 6 \Rightarrow \alpha = 13$

Lets take $\alpha \simeq 13$ then

$$L \propto R_c^{5/3}\, T_s^7 \propto R_c^{5/3} \left(\frac{M_c}{R_c}\right)^7$$

But $R_c \propto M_c^{-1/3}$ so

$$L \propto \frac{M_c^7}{R_c^{16/3}} \propto M_c^{7 + 16/9} \propto M_c^9$$

The actual slope of the relation changes with the He core mass as $T_c \uparrow$ as $M_c \uparrow$ along the RGB. This all ends when the He core ignites at $M_c = 0.42\, M_\odot \Rightarrow$ Horiz. Branch

$M < M_\odot$ Stars

For these, the opacity on MS is ~~#~~ Kramers' opacity, so

$$L \propto M^{5.5} \frac{1}{R^{0.5}}$$

and nuclear burning is pp limited. Typical T's are $< 10^7$ and the Gamow piece

$$\exp\left(-3\left(\frac{E_G}{4kT}\right)^{1/3}\right) \Rightarrow \exp\left(-\frac{15.6}{T_7^{1/3}}\right)$$

so $$\alpha = \frac{d\ln L_{nuc}}{d\ln T} = \frac{5.2}{T_7^{1/3}} \sim 6.5 \text{ at } T = 5\times10^6$$

for $0.5 M_\odot < M < M_\odot$

$$\Rightarrow \quad L_{nuc} \propto T_c^6 \propto \frac{M^{5.5}}{R^{0.5}}$$

$$\Rightarrow \left(\frac{M}{R}\right)^{6.5} \propto \frac{M^{5.5}}{R^{0.5}} \Rightarrow M \propto R^6$$

So there are quite different $R \propto M^{1/6}$ (i.e. nearly the same radius) so

$$T_c \propto \frac{M}{R} \propto M \Rightarrow$$ Lower Mass Stars burn at lower T's between