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Developments of Holographic Entanglement Entropy

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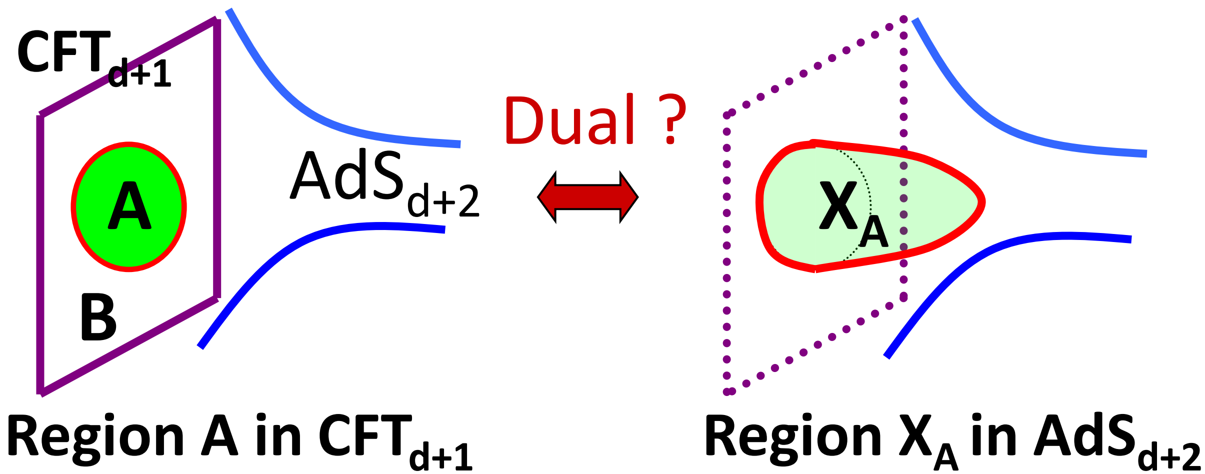
① Introduction

To generalize the AdS/CFT to holography in more general spacetimes, we need to better understand the basic mechanism of holography.

A basic question: Which region in the AdS does encode the `information in a certain region' of the CFT ?



[For recent proposals on closely related problems , see also Bousso-Leichenauer-Rosenhaus , Czech-Karczmarek-Nogueira-Raamsdonk, Hubeny-Rangamani 12]

The entanglement entropy (EE) can measure the amount of effective information in A.

Example: Spin Chain

$$H_{tot} = H_A \otimes H_B \quad .$$

Define the reduced density matrix ρ_A for A by

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

Finally, the entanglement entropy (EE) S_A is defined

by $$\boxed{S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .}$$ (von-Neumann entropy)

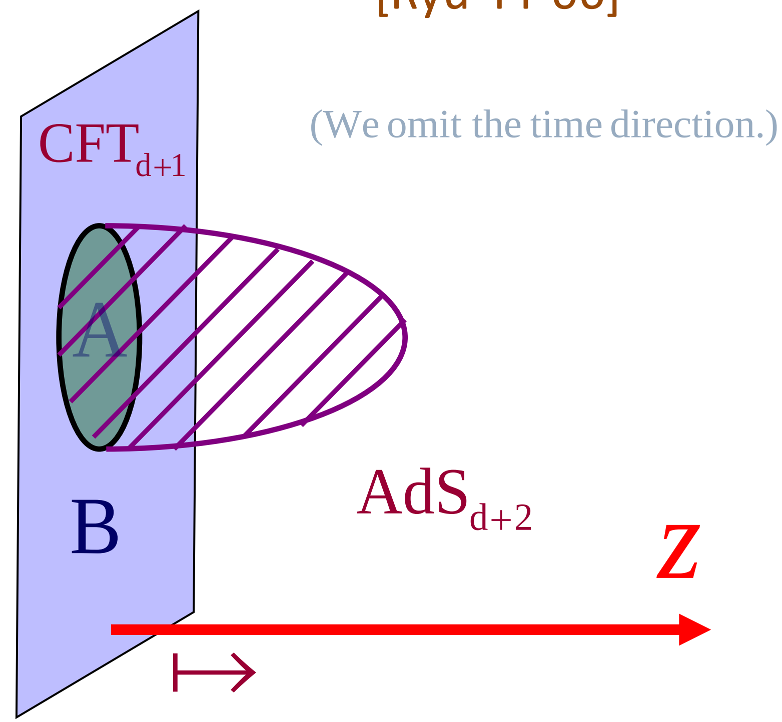
Consider the holographic entanglement entropy (HEE).

② Holographic Entanglement Entropy (HEE)

(2-1) Holographic Entanglement Entropy Formula

[Ryu-TT 06]

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$



(We omit the time direction.)

γ_A is the minimal area surface (codim.=2) such that

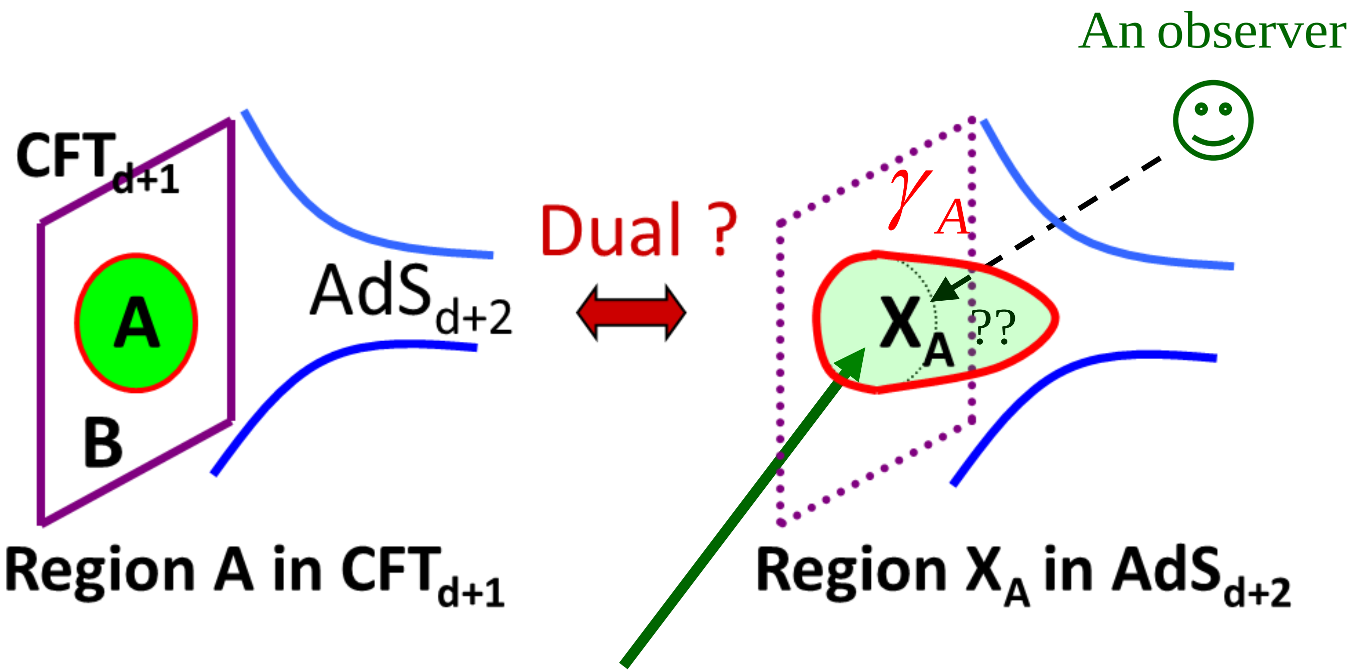
$$\partial A = \partial \gamma_A \text{ and } A \sim \gamma_A .$$

homologous

$z > a$ (UV cut off)

$$ds^2_{AdS} = R^2_{AdS} \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2} .$$

Motivation of this proposal



The information in A is encoded here.

Cf. Bousso bound

Comments

- A complete proof of HEE formula is still missing, there has been many evidences and no counter examples.

- If backgrounds are time-dependent, we need to employ extremal surfaces in the Lorentzian spacetime instead of minimal surfaces.

  [Hubeny-Rangamani-TT 07]

- In the presence of black hole horizons, the minimal surfaces wraps the horizon as the subsystem A grows enough large.

  $\Rightarrow$ Reduced to the Bekenstein-Hawking entropy, consistently.

  [Eternal BH as an entangled state, Maldecana 01]

Higher derivative corrections to HEE

⇒ A precise formula was found for Lovelock gravities.

[Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11]

Ex. Gauss-Bonnet Gravity

$$S_{GBG} = -\frac{1}{16 G_N}\int dx^{d+2}\sqrt{g}\left[R - 2\Lambda + \lambda R_{AdS}^2 L_{GB}\right]$$

$$L_{GB} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 .$$

⇒ $$S_A = \underset{\gamma_A}{\mathrm{Min}}\left[\frac{1}{4G_N}\int_{\gamma_A} dx^d\sqrt{h}\,(1 + 2\lambda R_{AdS}^2 R)\right].$$

[But for general higher derivative theories, this is hard !]

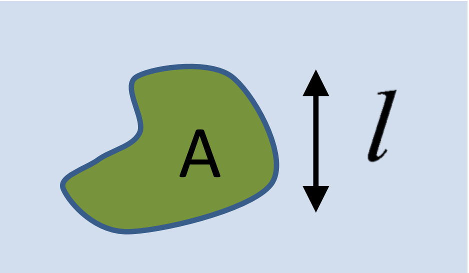
⇒ However, Any HEE formula is not known in more general cases.

[A Partial List of Evidences]

- Area law follows straightforwardly [Ryu-TT 06]
- Agreements with analytical 2d CFT results for AdS3 [Ryu-TT 06]
- Holographic proof of strong subadditivity [Headrick-TT 07, Allais-Tonni 11] Callan-He-Headrick 12]
- A proof of HEE for A=round spheres [Casini-Hueta-Myers 11]
- Cadney-Linden-Winter inequality (monogamy) [Hayden-Headrick-Maloney 11]
- Consistency of 2d CFT results for disconnected subsystems [Calabrese-Cardy-Tonni 09] with our holographic formula [Headrick 10]
- Agreements on the coefficients of log term in 4d CFT (~a+c) [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

General Behavior of HEE

[Ryu-TT 06]



$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{a}\right) + p_d & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{a}\right)^2 + q\log\left(\frac{l}{a}\right) & (\text{if } d = \text{odd}) \end{cases} \right],$$

where $p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$

$\ldots.. \quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$

Area law divergence

A universal quantity which characterizes odd dim. CFT

⇒ Analogue of c-function

[Myers-Sinha 10, Liu-Mezei 12, Proof vial SSA ⇒Casini-Huerta 12]

Conformal Anomaly (central charge)

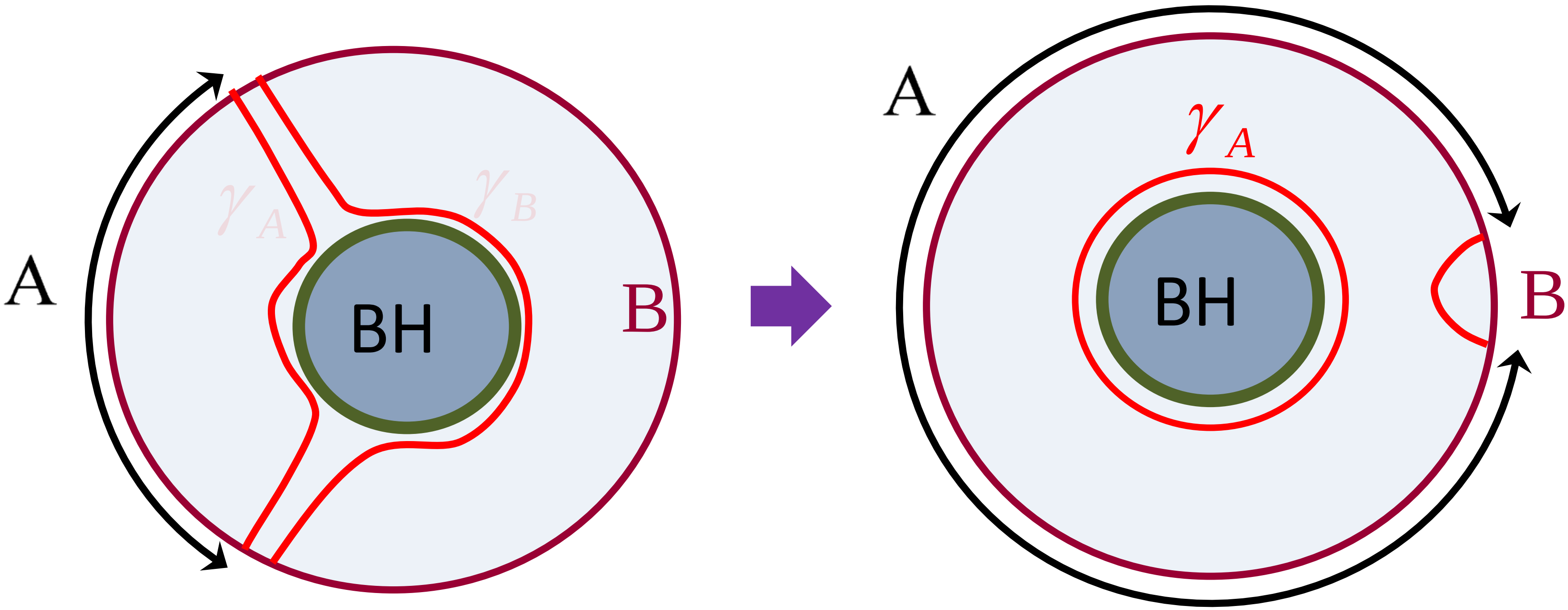
2d CFT c/3・log(l/a)

4d CFT -4a・log(l/a)

[Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11,…]

③ HEE and Black Holes

(3-1) Eternal BH (Mixed State)



Note: $S_A \neq S_B$ due to the horizon.

$\Leftrightarrow$ mixed state

(3-2) BH Formation (Pure State)

Black hole formation in AdS $\Leftrightarrow$ Thermalization in CFT

[See e.g. Chesler-Yaffe 08, Bhattacharyya-Minwalla 09,…]

An Entropy Puzzle

(i) Von-Neumann entropy remains vanishing under a unitary evolutions of a pure state.

$$\rho_{tot}(t) = U(t,t_0)|\Psi_0\rangle\langle\Psi_0|U(t,t_0)^{-1}$$

$$\Rightarrow \quad S(t) = -Tr\,\rho_{tot}(t)\,\log\rho_{tot}(t) = S(t_0).$$

(ii) In the gravity dual, its holographic dual inevitably includes a black hole at late time and thus the entropy looks non-vanishing !

Clearly, (i) and (ii) contradicts !

Resolution of this Puzzle via Entanglement Entropy

[Arrastia-Aparicio-Lopez 10, Ugajin-TT 10]

Upshot: The non-vanishing entropy appears only after coarse-graining. The von-Neumann entropy itself is always vanishing.

First, notice that the (thermal) entropy for the total system can be found from the entanglement entropy via the formula

$$S_{tot} = \lim_{|B| \to 0} (S_A - S_B).$$

This is indeed vanishing if we assume for the pure state: $S_A = S_B$.

➡ Instead, we can regard **S_A as the coarse-grained entropy**.

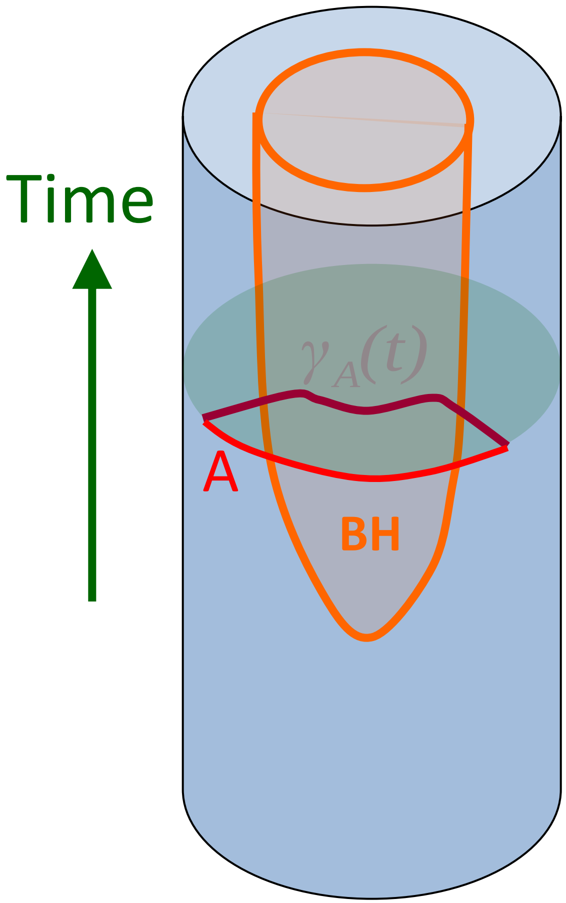
Indeed, we can holographically show this as follows:

$$S_A(t) = \min\left[\frac{\mathrm{Area}(\gamma_A(t))}{4G_N}\right],$$

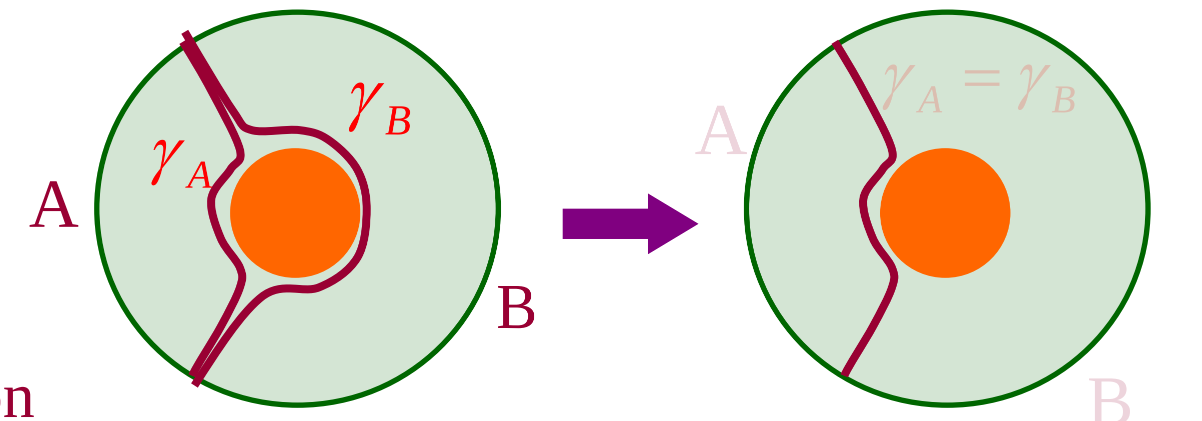
$$\gamma_A(t) = \text{extremal surfaces homotopic to } A(t)$$

$$\text{such that } \partial\gamma_A(t) = \partial A(t).$$

[Hubeny-Rangamani-TT 07]



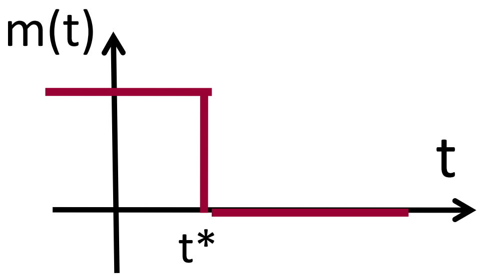
Black hole formation in global AdS_{d+2}

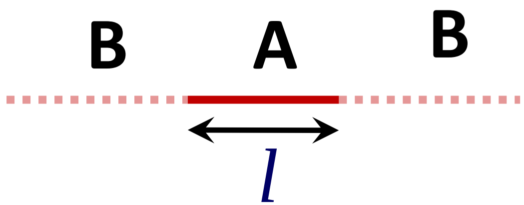


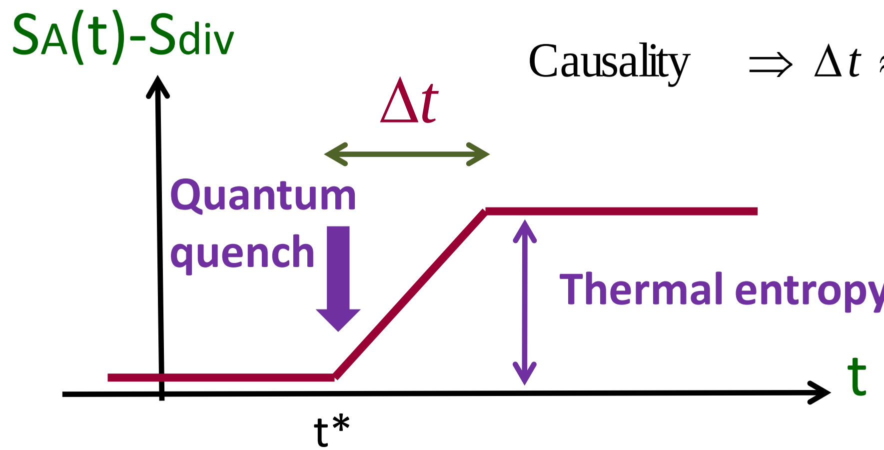
Continuous deformation

Time Evolutions of HEE under Quantum Quenches

In 1+1 dim. CFTs, we expect a linear growth of EE after a quantum quench. [Calabrese-Cardy 05]



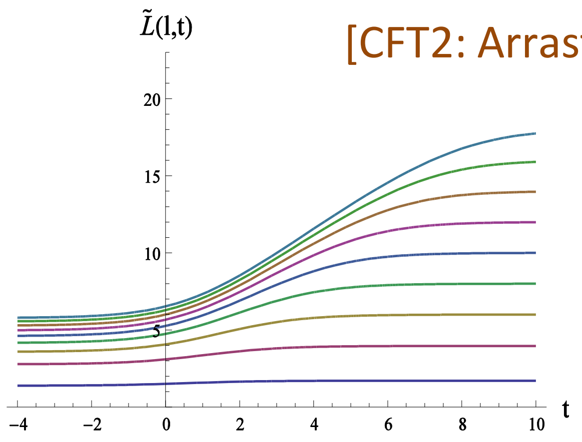




Causality $\Rightarrow \Delta t \approx \frac{l}{2}$

Evolutions of HEE in Vaidya BH $ds^2 = -(r^2 - m(v))dv^2 + 2drdv + r^2 dx^2$

[CFT2: Arrastia-Aparicio-Lopez 10]



[Higher dim. CFTs: Albash-Johnson 10, Balasubramanian-Bernamonti-de Boer-Copland-Craps- Keski-Vakkuri-Müller-Schäfer-Shigemori-Staessens 10, 11, ...]

④ Emergent Metric from Entanglement

[Nozaki-Ryu-TT, work in progress]

(4-1) Entanglement and Emergent Metric

In principle, we can obtain a metric from a CFT as follows:

a CFT state ⇒ Information (~EE) = Minimal Areas ⇒ metric

$$|\Psi\rangle \qquad S_A \qquad \mathrm{Area}(\gamma_A) \qquad g_{\mu\nu}$$

One candidate of such frameworks is so called the entanglement renormalization (MERA) [Vidal 05 (for a review see 0912.1651)] as pointed out by [Swingle 09]. [cf. Emergent gravity: Raamsdonk 09, Lee 09]

(4-2) Emergent Metric in a (d+1) dim. Free Scalar Theory

Hamiltonian: $H = \frac{1}{2}\int dk^d [\pi(k)\pi(-k) + (k^2 + m^2)\phi(k)\phi(-k)].$

Ground state $|\Psi\rangle$ **:** $a_k|\Psi\rangle = 0.$

Moreover, we introduce the `IR state' $|\Omega\rangle$ which has no real space entanglement.

$$a_x|\Omega\rangle = 0, \qquad a_x = \sqrt{M}\,\phi(x) + \frac{i}{\sqrt{M}}\pi(x),$$

$$\text{i.e.} \quad |\Omega\rangle = \prod_x |0\rangle_x \qquad a_x^+ = \sqrt{M}\,\phi(x) - \frac{i}{\sqrt{M}}\pi(x).$$

$$\Rightarrow \quad S_A = 0.$$

By including a class of excited states, let us assume:

$$(A_k a_k + B_k a^+_{-k})\left|\Psi\right\rangle = 0, \qquad (|A_k|^2 - |B_k|^2 = 1).$$

Note: This includes the time-evolution after quantum quenches.

The IR state satisfies $(\alpha_k a_k + \beta_k a^+_{-k})\left|\Omega\right\rangle = 0,$

$$\alpha_k = \frac{1}{2}\left(\sqrt{\frac{k}{M}} + \sqrt{\frac{M}{k}}\right), \quad \beta_k = \frac{1}{2}\left(\sqrt{\frac{k}{M}} - \sqrt{\frac{M}{k}}\right).$$

The strength of quantum entanglement is measured by a `**distance**' between $\left|\Psi\right\rangle$ and $\left|\Omega\right\rangle$.

There is a natural metric which respects SU(1,1) symmetry of the Bogoliubov transf., which preserves $[a_k, a_p^+] = \delta^d (k - P)$.

We can parameterize as follows:

$$A_k = \cosh a_k \cdot e^{ib_k}, \qquad B_k = \sinh a_k \cdot e^{ic_k}.$$

This leads to AdS3 metric:

$$ds^2 = da^2 - (\cosh a)^2 db^2 + (\sinh a)^2 dc^2.$$

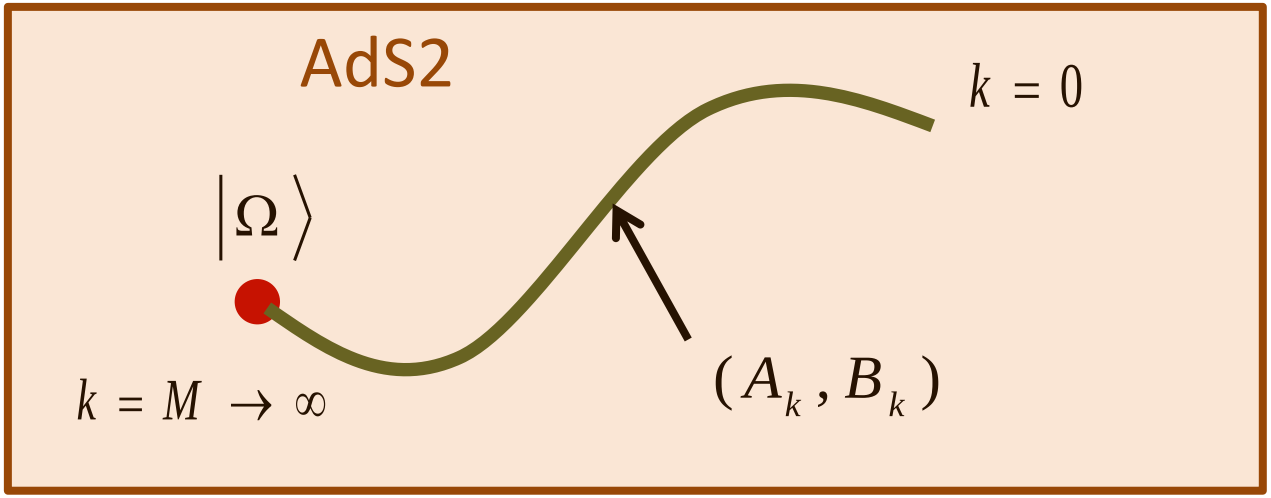
Moreover, we impose the identification $(A_k, B_k) \approx (e^{i\theta} A_k, e^{i\theta} B_k)$.

In this way, we finally reach the AdS2 metric:

$$ds^2 = da^2 + \frac{1}{4}(\sinh 2a)^2 (db - dc)^2.$$

Note: This AdS2 is nothing related to the AdS of AdS/CFT.

We can regard the vector $(A_k, B_k) \in AdS_2$ for various energy scales k as a RG-flow of the state $|\Psi\rangle$.



Our claim: we can define a measure of entanglement for the state $|\Psi\rangle$ using the AdS2 metric:

$$\text{Length} = \int_0^M dk \cdot \sqrt{|(\partial_k \tilde{A}_k, \partial_k \tilde{B}_k)|}.$$

(A, B) is rotated into $(\tilde{A}, \tilde{B})$ so that (α, β) goes to (1,0).

We argue that this is related to the **entanglement entropy**:

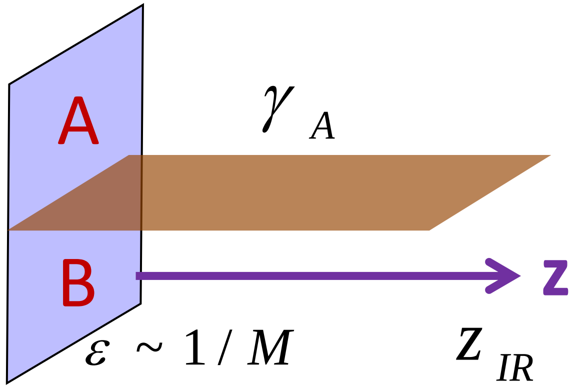
$$S_A \sim L^{d-1} \int_0^M \underline{k^{d-1}} (dk) \sqrt{|(\partial_k \tilde{A}_k, \partial_k \tilde{B}_k)|^2} \quad ,$$

DOS

where the subsystem A is assumed to be **the half of total system**.

Compare this with the HEE:

$$S_A^{Hol} \propto L^{d-1} \int_\varepsilon^{z_{IR}} \frac{dz}{z^{d-1}} \sqrt{g_{zz}} \,.$$



By identifying z=1/k , we obtain the correspondence:

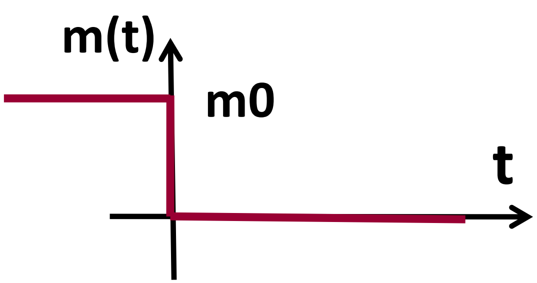
$$\sqrt{g_{zz}} \propto k^2 \sqrt{|(\partial_k \tilde{A}_k, \partial_k \tilde{B}_k)|^2} \,.$$

Example 1: The ground state of a massless free scalar

$$\sqrt{g_{zz}} \propto k^2 \sqrt{|(\partial_k \tilde{A}_k, \partial_k \tilde{B}_k)|^2} = \frac{1}{2z}.$$

$$\Rightarrow \quad AdS_{d+2} \ \text{metric} \ : ds^2 = \frac{dz^2}{z^2} + \frac{-dt^2 + (d\vec{x})^2}{z^2}.$$

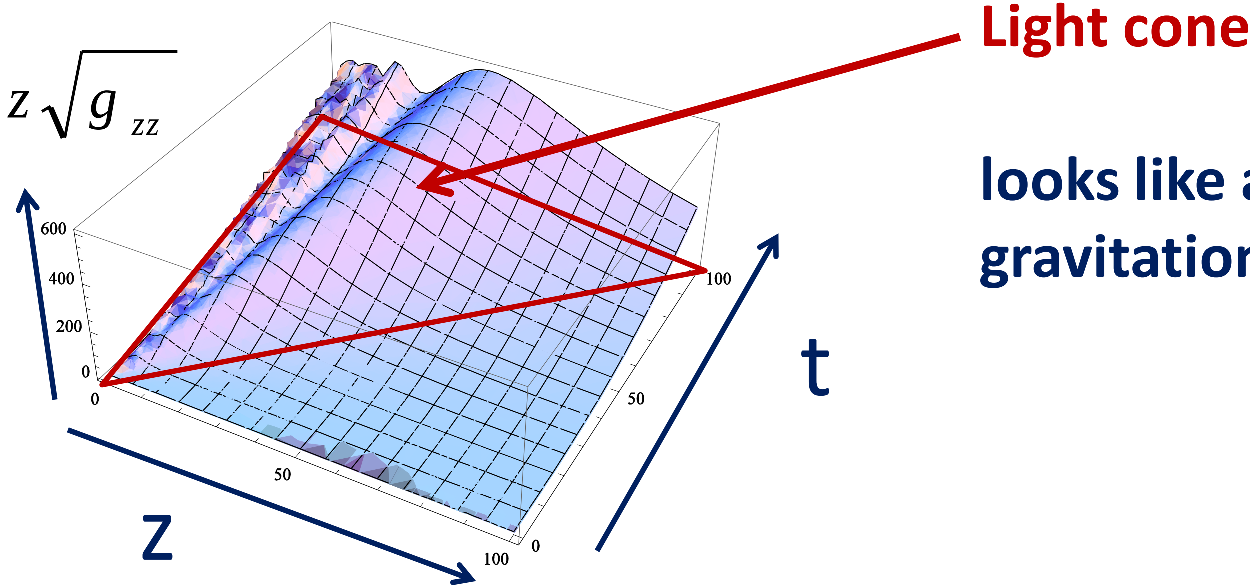
Example 2: The excited state after a quantum quench



$$A_k = \frac{1}{2}\left(\left(\frac{k^2 + m0^2}{k^2}\right)^{1/4} + \left(\frac{k^2}{k^2 + m0^2}\right)^{1/4}\right) \cdot e^{ikt},$$

$$B_k = \frac{1}{2}\left(\left(\frac{k^2 + m0^2}{k^2}\right)^{1/4} - \left(\frac{k^2}{k^2 + m0^2}\right)^{1/4}\right) \cdot e^{-ikt}.$$

Time dependent metric from the Quantum Quench



looks like a propagation of gravitational wave.

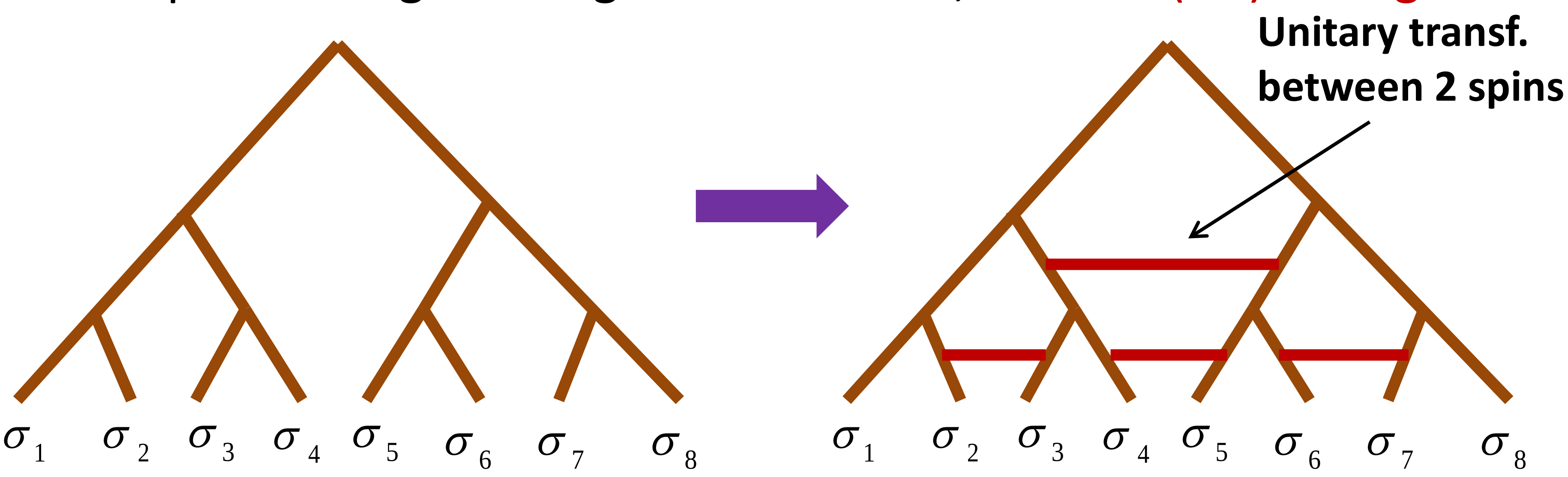
We can also confirm the linear growth of EE: SA∝t.
This is consistent with the known CFT (2d) and HEE results (any dim.).

(4-3) Relation to MERA and cMERA

<u>MERA</u> (Multiscale Entanglement Renormalization Ansatz):

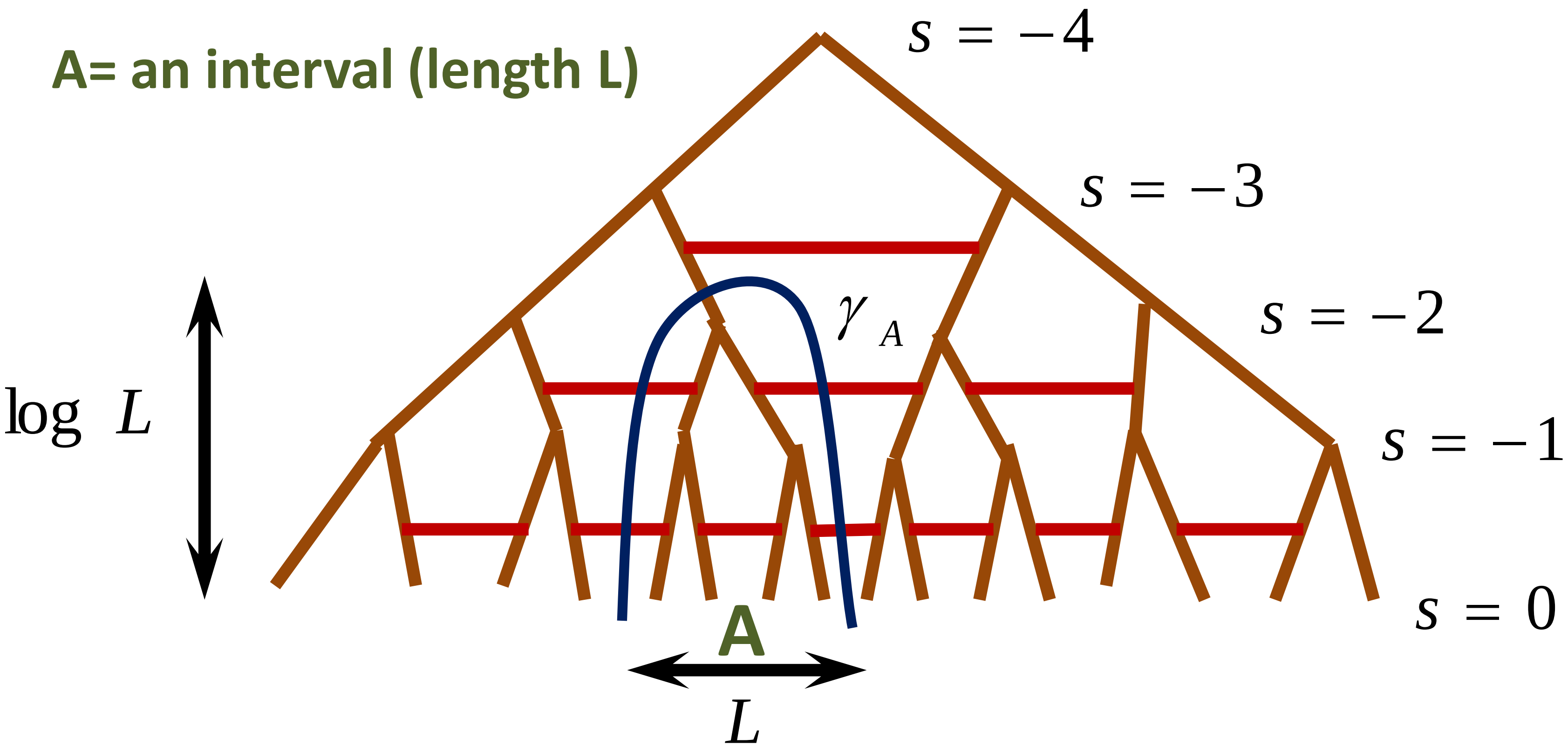
An efficient variational ansatz to find CFT ground states have been developed recently. [Vidal 05 (for a review see 0912.1651)].

To respect its large entanglement in a CFT, we add (dis)entanglers.



Calculations of EE in 1+1 dim. MERA

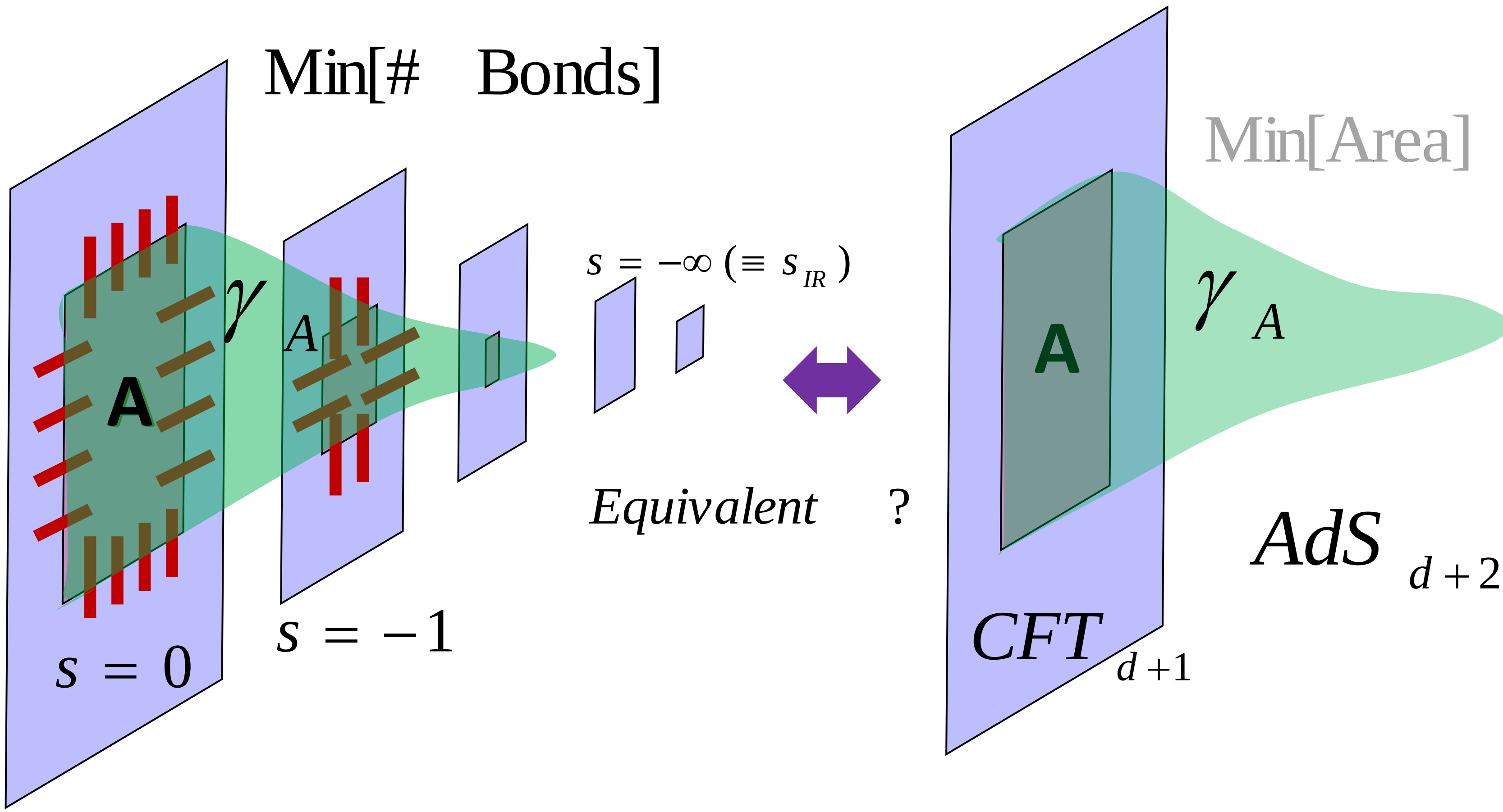
A= an interval (length L)



$$S_A \propto \mathrm{Min}\,[\#\ \mathrm{Bonds}\ ] \propto \log L$$

$\Rightarrow$ agrees with 2d CFTs.

A conjectued relation to AdS/CFT [Swingle 09]



$$\text{Metric} = ds^2 + \frac{e^{2s}}{\varepsilon^2}\left(-dt^2 + d\vec{x}^2\right) = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2},$$

where $z = \varepsilon \cdot e^{-s}$.

Now, to make the connection to AdS/CFT clearer, we would like to consider the MERA for quantum field theories.

Continuous MERA (cMERA)

[Haegeman-Osborne-Verschelde-Verstraete 11]

$$\underbrace{|\Psi\rangle}_{\substack{\text{True ground state} \\ \text{(highly entangled)}}} = P \cdot \exp\left(-i\int_{s_{IR}}^{s} ds\, \hat{K}(s)\right) \cdot \underbrace{|\Omega\rangle}_{\substack{\text{IR state} \\ \text{(no entanglement)}}},$$

$$\Rightarrow \text{Real space renormalization flow : length scale} \sim \varepsilon \cdot e^{-s}.$$

Our Conjecture

$$d+1 \text{ dim. cMERA} = \text{gravity on AdS}_{d+2} \qquad z = \varepsilon \cdot e^{-s}.$$

For a free scalar theory, the ground state corresponds to

$$\hat{K}(s) = \frac{i}{2}\int dk^{d}\left[\chi(s)\Gamma\left(ke^{-s}/M\right)a_k^{+}a_{-k}^{+} + (h.c.)\right],$$

where $\Gamma(\mathrm{x})$ is a cut off function : $\Gamma(\mathrm{x}) = \theta(1-|\mathrm{x}|)$.

$$\chi(s) = \frac{1}{2}\cdot\frac{e^{2s}}{e^{2s} + m^2/M^2}, \quad (\text{for } m = 0,\ \chi(\mathrm{s}) = 1/2.)$$

For the excited states, $\chi(s)$ becomes time-dependent.

One might be tempting to guess

$$ds^2_{Gravity} = g_{ss}ds^2 + \frac{e^{2s}}{\varepsilon^2}\cdot d\vec{x}^2 - g_{tt}dt^2 \quad \Rightarrow \quad \sqrt{g_{ss}} \propto |\chi(s)| \ ?$$

Indeed, our previous metric based on the SU(1,1) sym. can be regarded as a refinement of this naïve guess.

④ Conclusions and Discussions

- HEE can be used as a probe of black hole formation, while the microscopic total entropy remains vanishing.

  ⇒ Analysis of time-dependent EE in higher dim. ?

- We can construct an emergent metric via the quantum entanglement just from QFTs. We confirmed its linear growth after a quantum quench.

  ⇒ How to calculate gtt ? Derive Einstein eq. ?

  Free scalars → Higher spin gauge theories ?

  Large N gauge theories ?

- EE is `something' between the wave function and the metric.