The Resurgent Bootstrap and the 3D Ising Model

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KITP

Motivation

Why return to the bootstrap?

- Conformal symmetry very powerful tool that *goes largely unused* in $D > 2$.
- Completely non-perturbative tool to study field theories
  1. Does not require SUSY, large N, or weak coupling.
- Map out "landscape of CFTs"
  1. Constraints on spectrum and interactions with few or no assumptions.
  2. Possibly classify CFTs as in D=2?
- Universality
  1. Fixed points *universal* $\Rightarrow$ isolate them with minimal input.

Applications

- **The Three-dimensional Ising Model.**
- 4D phenomenology applications to (walking) technicolor [Rattazzi et al].
- Constructive Holography: deriving AdS from CFT.
  [Heemskerk et al, Fitzpatrick et al, SE and Papadodimas]
- M5-theory? (0,2) SCFT in 6D.

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Outline

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- Motivation
- Lightning Ising model refresher.
- CFT Review
  - Correlators from OPE
  - Crossing Symmetry
  - Conformal Blocks
- The Bootstrap: solving theories by consistency alone
  - Expanding the bootstrap around $z = \bar{z} = 1/2$.
  - Why does this work?
  - Linear Programming
- The 3D Ising Model
  - Constraints from $\langle\sigma\sigma\sigma\sigma\rangle$ correlator
  - The landscape of 3D CFTs
- Other Applications

What exactly is the Ising model?

The Ising Model

Original Formulation

Basic Definition

- Lattice theory with nearest neighbor interactions

$$H = -J \sum_{<i,j>} s_i s_j$$

with $s_i = \pm 1$ (this is $O(N)$ model with $N = 1$).

Relevance

- Historical: 2d Ising model solved exactly. [Onsager, 1944].
- Relation to ϵ-expansion.
- "Simplest" CFT (universality class)
  1. Only $\mathbb{Z}_2$ symmetry
  2. Not multi-critical: only one relevant operator.
- Describes:
  1. Ferromagnetism
  2. Liquid-vapour transition
  3. ...

Continuum Limit

- To study fixed point can take continuum limit (and $\sigma(x) \in \mathbb{R}$)

$$H = \int d^D x \, \left[(\nabla \sigma(x))^2 + t \, \sigma(x)^2 + a \, \sigma(x)^4 \right]$$

- Interaction generated by Guassian "$\mathbb{Z}_2$" constraint: $(\sigma(x)^2 - 1)^2$.
- In $D < 4$ coefficient a is relevant and theory *flows* to a fixed point.

ϵ-expansion

Wilson-Fisher set $D = 4 - \epsilon$ and study critical point perturbatively.

3d Ising model: take $\epsilon = 1$ expect CFT with:

Field:	σ	ϵ	ϵ'	$T_{\mu\nu}$	$C_{\mu\nu\rho\lambda}$
Dim (Δ):	0.5182(3)	1.413(1)	3.84(4)	3	5.0208(12)
Spin (l):	0	0	0	2	4
$\mathbb{Z}_2$:	-	+	+	+	+

CFT Refresher

Conformal Symmetry in $D > 2$

Primary Operators

Conformal symmetry:

$$\underbrace{SO(1, D-1) \times \mathbb{R}^{1,D-1}}_{Poincare} \quad + D \quad \text{(Dilatations)} + K_\mu \quad \text{(Special conformal)}$$

Representations built on:

$$\text{Primary operators:} \qquad K_\mu \mathcal{O}(0) = 0$$
$$\text{Descendents:} \qquad P_{\mu_1} \dots P_{\mu_n} \mathcal{O}(0)$$

All dynamics of *descendants* fixed by those of primaries.

Clarifications vs 2D

- Primaries $\mathcal{O}$ called *quasi-primaries* in $D = 2$.
- Descendents are with respect to "small" conformal group: $L_0, L_{\pm 1}$.
- Viraso descendents $L_{-2}\mathcal{O}$ are *primaries* in our language.
- In this talk we always mean *small conformal group* (i.e. for descendants, conformal blocks, primaries, ...).

On the uses of Conformal Symmetry

Definition

- Abstract CFT defined by:
  - OPE coefficients C_{ijk}.
  - Conformal dim, spin of primary operators (Δ_i, l_i).
- This data formally *defines* CFT non-perturbatively.
- Unlike general QFT this formulation is well-defined and convergent.
- Unfortunately until recently has not been a practical definition (in $D > 2$).

Simple constraints:

- Conformal invariance imposes constraints on the above data.
- *Unitarity* bound on dimensions:

$$L = 0 : \quad \Delta \geq \frac{D-2}{2}, \qquad L > 0 : \quad \Delta \geq L + D - 2$$

- Two-point functions fixed up to normalization.
- Three point function $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle \sim C_{ijk}$

CFT defined by specifying:

- Spectrum $\mathcal{S} = \{\mathcal{O}_i\}$ of primary operators dimensions, spins: (Δ_i, l_i)
- Operator Product Expansion (OPE)

$$\mathcal{O}_i(x) \cdot \mathcal{O}_j(0) \sim \sum_k C_{ij}^k D(x, \partial_x) \mathcal{O}_k(0)$$

$\mathcal{O}_i$ are primaries. Diff operator $D(x, \partial_x)$ encodes *descendent* contributions.

Higher point functions contain no new dynamical information!

- Can be reconstructed from above data:

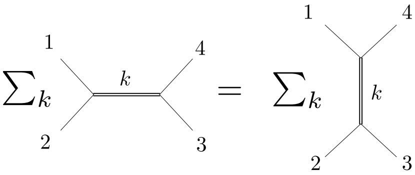
$$\langle \underbrace{\underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}_{\sum_k C_{12}^k D(x_{12},\partial_{x_2})\mathcal{O}_k(x_2)} \quad \underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}_{\sum_l C_{34}^l D(x_{34},\partial_{x_4})(x_3)\mathcal{O}_l(x_4)}}_{\sum_{k,l} C_{12}^k C_{34}^l D(x_{12},x_{34},\partial_{x_2},\partial_{x_4})\langle \mathcal{O}_k(x_2)\mathcal{O}_l(x_4)\rangle} \rangle$$

- Operators $D(x, \partial_x)$ fixed kinematically: no dynamical info.
- OPE coefficients C_{ij}^k are *constants*: encode full dynamics.

Crossing Symmetry

CFT Background

This procedure is not unique: $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$



Consistency requires equivalence of two different contractions

$$\sum_k C_{12}^k C_{34}^k \, G_{\Delta_k, l_k}^{12;34}(x_1, x_2, x_3, x_4) = \sum_k C_{14}^k C_{23}^k \, G_{\Delta_k, l_k}^{14;23}(x_1, x_2, x_3, x_4)$$

Functions $G_{\Delta_k, l_k}^{ab;cd}$ are *conformal blocks* (of “small” conformal group):

- Encode contribution of operator $\mathcal{O}_k$ to double OPE contraction.
- Entirely *kinematical*: all dynamical information is in C_{ij}^k.
- Crossing sym. give non-perturbative constraints on (Δ_k, C_{ij}^k).

Conformal Blocks in all their Glory

CBs eigenfunctions of quadratic and quartic conformal casismirs:

$$\Box^{(2)} G_{\Delta,l} = \lambda^{(2)}_{\Delta,l} G_{\Delta,l} \qquad \Box^{(4)} G_{\Delta,l} = \lambda^{(4)}_{\Delta,l} G_{\Delta,l}$$

In $D = 2, 4$ Dolan-Osborn have computed conformal blocks, e.g. $D = 4$:

$$G^{12;34}_{\Delta,l}(x_1, x_2, x_3, x_4) = \frac{1}{l+1} \frac{z\bar{z}}{(z-\bar{z})} \left[k_{\Delta+l}(z) k_{\Delta-l}(\bar{z}) - (z \leftrightarrow \bar{z})\right]$$

with

$$k_\beta(z) = z^{\beta/2} \, {}_2F_1\left(\frac{\beta - \Delta_{12}}{2}, \frac{\beta + \Delta_{34}}{2}, \beta, z\right)$$

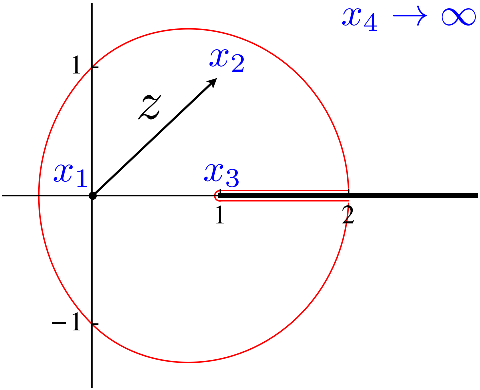
with $\Delta_{ij} = \Delta_i - \Delta_j$ and u, v conformal cross-ratios

$$u = \frac{x_{12} x_{34}}{x_{13} x_{24}}, \qquad v = \frac{x_{14} x_{23}}{x_{13} x_{24}}$$

and $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$.

Conformal Blocks in $z, \bar{z}$ coords

- Via conformal transform can map x_1, x_2, x_3, x_4 to a plane.
- $(z, \bar{z})$ then complex coords on this plane.



- In general D no compact expression but double-infinte sum.
- At $z = \bar{z}$ sum simplifies so we work in a neighborhood of $z = \bar{z}$.

CBs at $z = \bar{z}$

- $l = 0, 1$ blocks exact expression in terms of ${}_3F_2$ hypergeometrics.
- Recursion relations for higher spin (at $z \neq \bar{z}$ involve higher derivatives).

Derivative Recursion Relations

- ${}_3F_2$ satisfies cubic equation.
- Combine with casimir eqns to get derivative recursion relations.
- Take

$$z = \frac{a + \sqrt{b}}{2}, \qquad \bar{z} = \frac{a - \sqrt{b}}{2}$$

and expand around $(a, b) = (1, 0)$.

Can now compute CBs in arbitrary dim expanded around $z = \bar{z}$!

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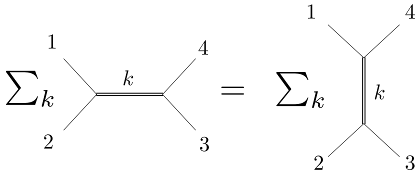
Imposing Crossing Symmetry

So how do we enforce crossing symmetry in practice?

Consider four *identical* scalars: $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad \dim(\phi) = \Delta_\phi$

Crossing symmetry:

$$\sum_k (C^k_{\phi\phi})^2\, G^{12;34}_{\Delta_k,l_k}(x_1,x_2,x_3,x_4) = \sum_k (C^k_{\phi\phi})^2\, G^{14;23}_{\Delta_k,l_k}(x_1,x_2,x_3,x_4)$$



Crossing Symmetry Nuts and Bolts

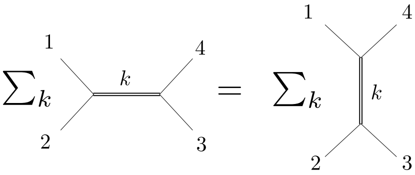
Bootstrap

So how do we enforce crossing symmetry in practice?

Consider four *identical* scalars: $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad \dim(\phi) = \Delta_\phi$

Move everything to LHS:

$$\sum_k (C^k_{\phi\phi})^2\, G^{12;34}_{\Delta_k,l_k}(x_1,x_2,x_3,x_4) - \sum_k (C^k_{\phi\phi})^2\, G^{14;23}_{\Delta_k,l_k}(x_1,x_2,x_3,x_4) = 0$$



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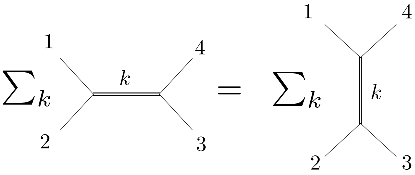
Express as sum with positive coefficients:

$$g(z,\bar{z}) = \sum_k (C^k_{\phi\phi})^2 \left[G^{12;34}_{\Delta_k,l_k}(x_1,x_2,x_3,x_4) - G^{14;23}_{\Delta_k,l_k}(x_1,x_2,x_3,x_4)\right] = 0$$



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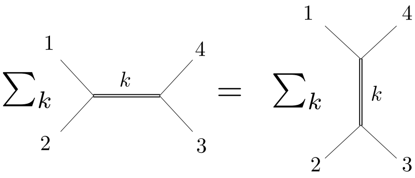
$\mathcal{F}^{\phi}_{\Delta_k,l_k}(z,z)$ are combined s-t channel CBs:

$$g(z,\bar{z}) = \sum_k \underbrace{(C^k_{\phi\phi})^2}_{p_{\Delta_k,l_k}} \underbrace{\left[u^{\Delta_\phi} G_{\Delta,l}(u,v) - v^{\Delta_\phi} G_{\Delta,l}(v,u)\right]}_{\mathcal{F}^{\phi}_{\Delta_k,l_k}(z,\bar{z})} = 0$$



Combined blocks $\mathcal{F}^{\phi}_{\Delta_k,l_k}(z,z)$ depend on:

- External scalar dimension: Δ_ϕ.
- Exchanged operators spin, dimension: l_k, Δ_k.
- Coordinates $z, \bar{z}$ in entirely kinematical way.

So how do we enforce crossing symmetry in practice?

Consider four *identical* scalars: $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ $\quad \dim(\phi) = \Delta_\phi$

$$g(z,\bar{z}) = \sum_k \underbrace{(C^k_{\phi\phi})^2}_{p_{\Delta_k,l_k}} \underbrace{\left[u^{\Delta_\phi} G_{\Delta,l}(u,v) - v^{\Delta_\phi} G_{\Delta,l}(v,u)\right]}_{\mathcal{F}^{\phi}_{\Delta_k,l_k}(z,\bar{z})} = 0$$

1. Expand in derivatives around $z = \bar{z} = 1/2$

$$g(1/2,1/2) = 0, \qquad \partial_z^2 g(1/2,1/2) = 0$$
$$\partial_{\bar{z}}^2 g(1/2,1/2) = 0, \qquad \ldots$$

2. If can find any constant vector $\Lambda = (\lambda_{2,0}, \lambda_{0,2}, \lambda_{2,2}, \lambda_{4,0}, \ldots)$ such that

$$\lambda_{m,n}\, \partial_z^m \partial_{\bar{z}}^n\, g(z,\bar{z})|_{z=\bar{z}=1/2} > 0$$

then crossing symmetry has no solutions.

3. Can reformulate in terms of vectors (derivatives at $z = \bar{z} = 1/2$):

$$\vec{f}_{\Delta,l} = (\mathcal{F}^{(0,0)}_{\Delta,l}, \mathcal{F}^{(1,0)}_{\Delta,l}, \mathcal{F}^{(0,1)}_{\Delta,l}, \ldots)$$

If $\{\vec{f}_{\Delta,l}\}$ form a cone cannot solve crossing symmetry!

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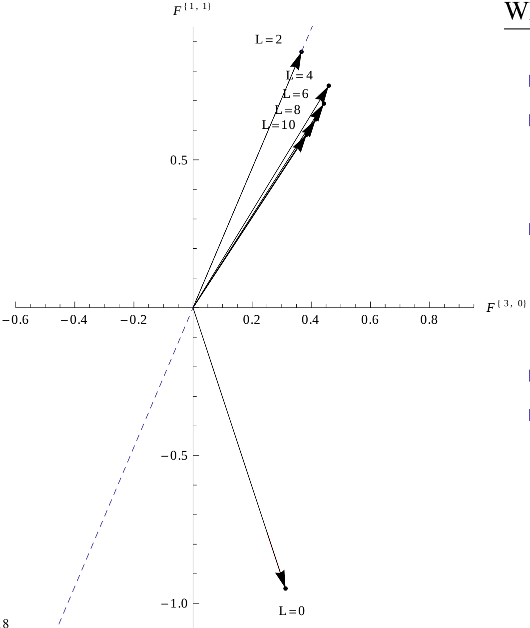
Cones in Derivative Space



Why does this work?

- Consider $\langle \phi\phi\phi\phi \rangle$ with $\Delta(\phi) = 0.515$.
- Project $\vec{f}_{\Delta,l}$ to plane:

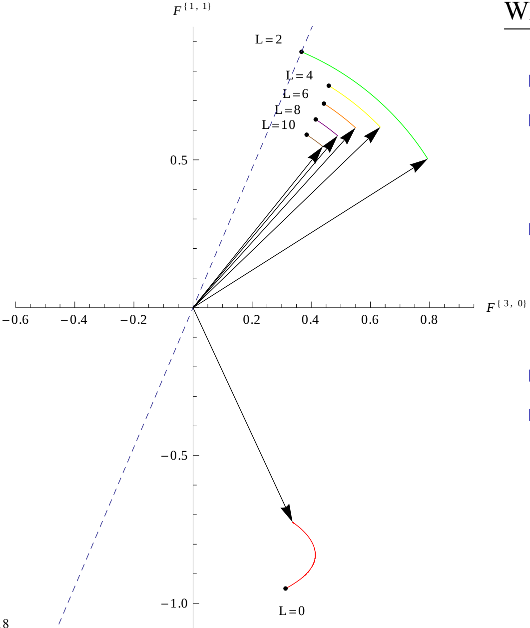
$$(\partial_a^1 \partial_b^1 \mathcal{F}_{\Delta,l},\ \ \partial_a^3 \mathcal{F}_{\Delta,l})$$

- Plot

$$\Delta = \Delta_{unitarity} \text{ to } \Delta_{unitarity} + \epsilon$$
$$l = 0 \text{ to } 10$$

- ϵ parametrized range of Δ we consider.
- Take $\epsilon = 0$ so CBs at unitarity bound.
  $\Rightarrow$ vectors in "cone"
  $\Rightarrow$ no crossing symmetry.

Cones in Derivative Space



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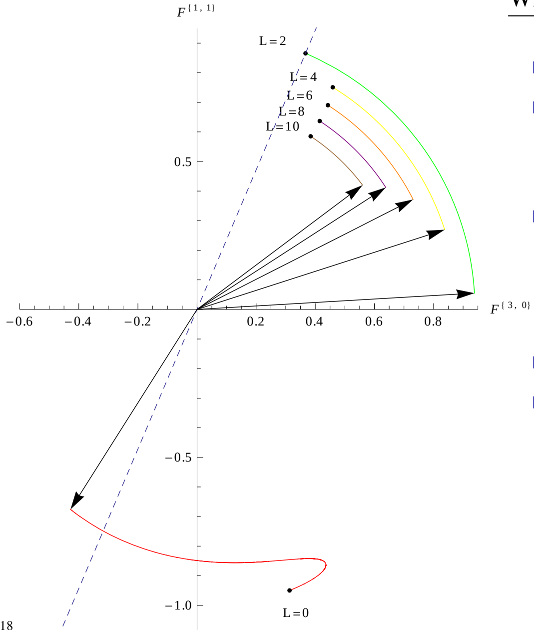
- ϵ parametrized range of Δ we consider.
- For ϵ small
  $\Rightarrow$ vectors *still* in "cone"
  $\Rightarrow$ no crossing symmetry.

Cones in Derivative Space



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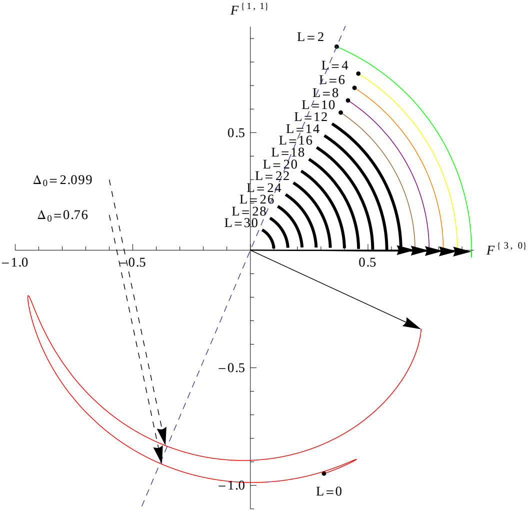
- ϵ parametrized range of Δ we consider.
- For ϵ large enough
  $\Rightarrow$ vectors *span* plane.
  $\Rightarrow$ In particular can find $p_{\Delta,l} > 0$

$$\sum p_{\Delta,l}\vec{f}_{\Delta,l} = 0$$

$\Rightarrow$ crossing sym. can be satisfied!!

Cones in Derivative Space



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- ϵ parametrized range of Δ we consider.
- When ϵ big enough
  $\Rightarrow$ vectors *no longer* in "cone"
  $\Rightarrow$ crossing sym. can be satisfied.
  $\Rightarrow$ Requires $0.76 \le \Delta_0 \le 2.099$.

Linear Programming

Putting Crossing Symmetry on a (big) Computer

- Plots visually intuitive but hard to work with.
- Want to systematically check crossing symmetry.

Algorithm

1. Fix a putative spectrum $\mathcal{S} = \{(\Delta, l)\}$.
2. If there exists a vector $\Lambda = (\lambda_{(0,0)}, \lambda_{(1,0)}, \lambda_{(0,1)}, \lambda_{(1,1)}, \ldots)$ such that

$$\Lambda(\mathcal{F}) := \sum_{m,n} \lambda_{(m,n)} \partial_a^m \partial_b^n \mathcal{F}_{\Delta,l} > 0$$

for *all* $(\Delta, l) \in \mathcal{S}$ then:

$\mathcal{S}$ cannot be the spectrum of a consistent CFT.

- To make this tractable *discretize* possible Δ.
- Then finding such Λ is a *linear optimization problem*[1].
- Efficient algorithms and implementations: e.g. IBM's Cplex.

[1] Without the optimization :-)

Solving the 3d Ising Model with Crossing Symmetry??

Is the putative spectrum of 3d Ising consistent with crossing symmetry?

Field:	σ	ϵ	ϵ'	$T_{\mu\nu}$	$C_{\mu\nu\rho\lambda}$
Dim (Δ):	0.5182(3)	1.413(1)	3.84(4)	3	5.0208(12)
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Constraining the spectrum

- Consider crossing symmetry of

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$$

- What are possible values of Δ_ϵ as a function of Δ_σ?
- Argue by exclusion: show certain values *inconsistent* with crossing.
- How do we determine this?
  1. Fix Δ_σ.
  2. Check crossing symmetry assuming the *next* scalar has $\Delta_\epsilon > 1$. (Note: we do not fix Δ_ϵ to its Ising model value.)

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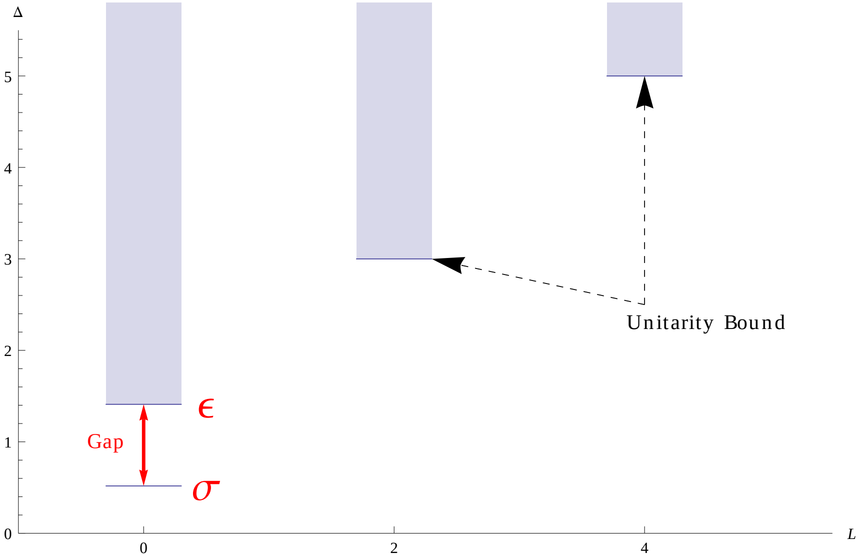
$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$$

- What are possible values of Δ_ϵ as a function of Δ_σ?
- Argue by exclusion: show certain values *inconsistent* with crossing.
- How do we determine this?
  1. Fix Δ_σ.
  2. Check crossing symmetry assuming the *next* scalar has $\Delta_\epsilon > 1$. (Note: we do not fix Δ_ϵ to its Ising model value.)

Putative Spectrum: Gapped Scalar Sector

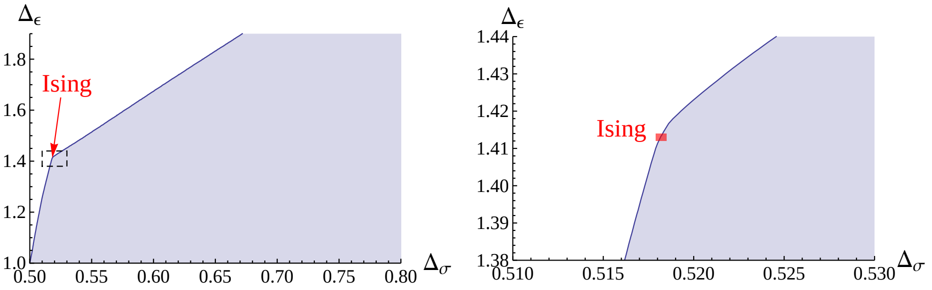
Allow any spectrum but impose "Gap" in scalar sector



Spectrum of the Ising Model

Assuming gap in scalar spectrum between Δ_σ and Δ_ϵ

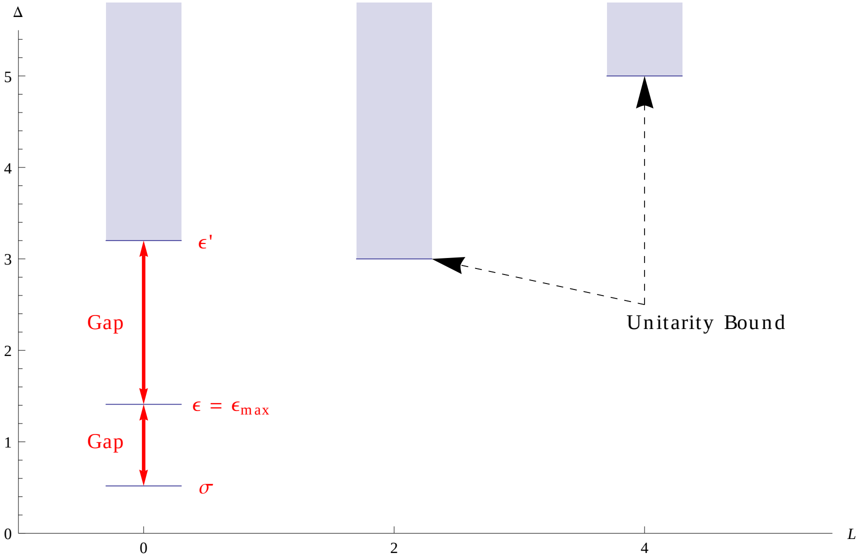
Plot: possible values of second lightest operator, Δ_ϵ, as function of Δ_σ.



1. Valid range of $(\Delta_\sigma, \Delta_\epsilon)$ restricted by crossing symmetry.
2. Ising model values seem to sit at a "kink".
3. Note: this plot is *completely* general. Only a "gap" is assumed.
4. Crossing symmetry excludes $\approx \frac{1}{3}$ error bar region.

Putative Spectrum: Only One Relevant $\mathbb{Z}_2$ Singlet

Allow any spectrum but allow only one relevant $\mathbb{Z}_2$ operator, ϵ



Spectrum of the Ising Model

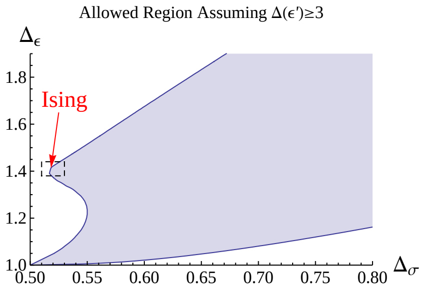
Assuming *only one* relevant scalar (i.e. ϵ with $\Delta_\epsilon < 3$)

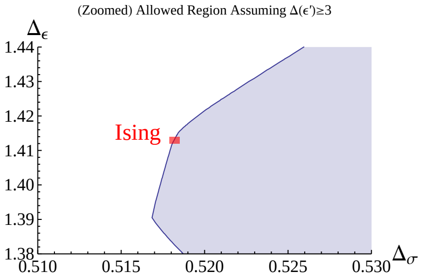
Ising model has *only one* irrelevanat scalar so lets try:

- Impose gap between ϵ and next scalar, ϵ'.
- ϵ' irrelevant so $\Delta_{\epsilon'} > 3$ (but we also consider $> 3.4, 3.8$).

Plot of allowed $(\Delta_\sigma, \Delta_\epsilon)$ region assuming:

Next scalar in spectrum ϵ': $\Delta_{\epsilon'} > 3$





Spectrum of the Ising Model

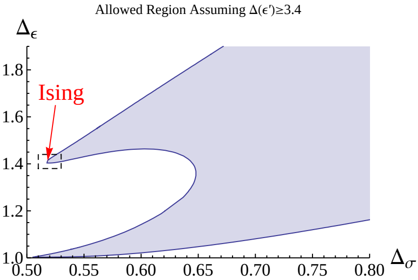
Assuming *only one* relevant scalar (i.e. ϵ with $\Delta_\epsilon < 3$)

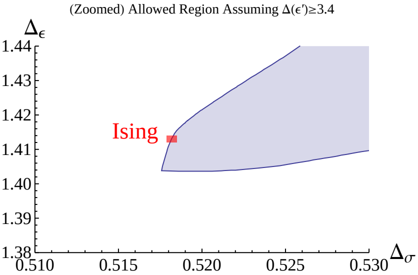
Ising model has *only one* irrelevanat scalar so lets try:

- Impose gap between ϵ and next scalar, ϵ'.
- ϵ' irrelevant so $\Delta_{\epsilon'} > 3$ (but we also consider $> 3.4, 3.8$).

Plot of allowed $(\Delta_\sigma, \Delta_\epsilon)$ region assuming:

Next scalar in spectrum ϵ': $\Delta_{\epsilon'} > 3.4$





Spectrum of the Ising Model

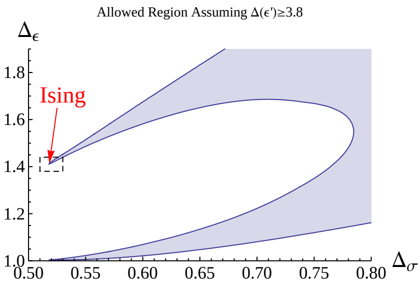
Assuming *only one* relevant scalar (i.e. ϵ with $\Delta_\epsilon < 3$)

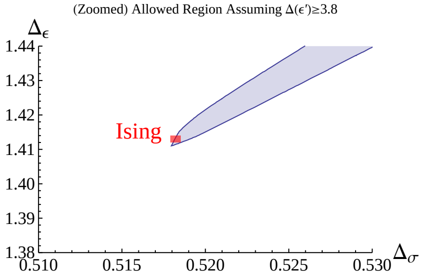
Ising model has *only one* irrelevanat scalar so lets try:

- Impose gap between ϵ and next scalar, ϵ'.
- ϵ' irrelevant so $\Delta_{\epsilon'} > 3$ (but we also consider $> 3.4, 3.8$).

Plot of allowed $(\Delta_\sigma, \Delta_\epsilon)$ region assuming:

Next scalar in spectrum ϵ': $\Delta_{\epsilon'} > 3.8$



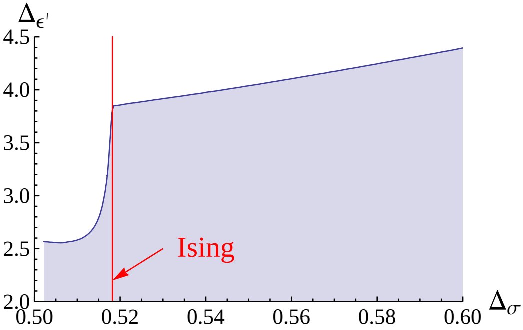


Spectrum of the Ising Model

Inverting the Logic: Bounding ϵ' assuming ϵ has maximal dimension.

Assuming Δ_ϵ takes maximal allowed value (as function of Δ_σ):

Plot: possible values of $\Delta_{\epsilon'}$ vs. Δ_σ



1. Again Ising model seems to stand out.
2. At Ising point CFT third scalar ϵ' can be irrelevant.
3. "Kink" or "cusp" in (ϵ, σ) plot due to rapid rearrangement of spectrum.

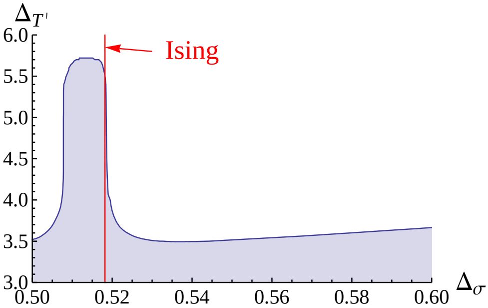
Spectrum of the Ising Model

Spin 2 sector.

Higher spin?

- Stress-tensor $T_{\mu\nu}$ fixed by symmetry: $\Delta = 3$.
- What about *next* spin 2 field: $T'_{\mu\nu}$.

Plot $\Delta_{T'}$ vs Δ_σ (i.e. *maximal* gap in spin 2 spectrum):



Again Ising *region* seems very special!

Central Charge of the Ising Model

Going beyond the Spectrum.

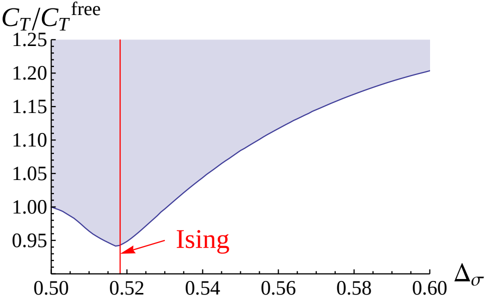
What else?

- Putting the *optimization* back in linear optimization can constrain OPE coefficients.
- Coefficient of stress-tensor CB, $\mathcal{F}_{3,2}$, fixed by conf sym to be:

$$p_{3,2} = \frac{\Delta_\sigma^2}{C_T} \qquad \text{with} \quad C_T \sim \langle T_{\mu\nu} T_{\rho\lambda} \rangle$$

Plot Min(C_T / C_T^{free}) vs Δ_σ:

- Compare C_T to "free" value ($\Delta_\sigma = 0.5$).
- No assumptions in this plot!
- Again Ising *region* very special!



Summary

Results so far.

So what have we shown?

Conformal Blocks in Any Dimension

General Stuff

- A way to efficiently compute (tabulate) CBs in any dim around $z = \bar{z}$.
- Although a general expression would be nice this suffices for crossing symmetry.

The 3d Ising Model

- Crossing symmetry applied to $\langle \sigma\sigma\sigma\sigma \rangle$ already very constraining.
- Even without assumptions Ising model stands out.
- With a few simple assumptions:
  1. Gap in scalar spectrum with: $\sigma, \epsilon < 3$ and $\epsilon' > 3$.
  2. Gap in spin 2 spectrum $T' > 4$.

  can restrict "landscape" of CFTs to neighborhood of Ising point.
- From this follows the hope:

  Could crossing symmetry allow us to classify & solve CFTs in any dim?

The Future

What's left to do?

Honing in on the Ising model?

- Lets add another correlator: $\langle \sigma\sigma\epsilon\epsilon \rangle$.
- C_T and $C_{\sigma\sigma\epsilon}$ appear in both correlators $\Rightarrow$ should give strong constraints.
- "Saturation" bounds seems to give unique answers close to Ising model.
- Suggests strategy:
  1. For each $\mathcal{O}$ find max $\Delta_{\mathcal{O}}$ as function of Δ_σ.
  2. Fixing $\Delta_{\mathcal{O}}$ to its max look for *next* operator $\mathcal{O}'$ as function of Δ_σ.
  3. Iterate over-and-over to get full spectrum.
  4. Iterate over spins imposing bounds from lower spins.

Finding new CFTs

- 3d Ising model follows largely from minimal constraints on spectrum.
- Adding symmetries (e.g. $O(N)$) expect stronger constraints
  $\Rightarrow$ isolate more CFTs.
- Can we use this to classify CFTs using only global symmetry and crossing symmetry (as in $D = 2$)?

The Future

What's left to do?

AdS/CFT Applications

- Generalized Free Field CFTs are dual to free ($N \sim \infty$) fields in AdS [Heemskerk et al, SE and Papadodimas]
- Higher spin GFFs are "multi-particle states" in bulk:

$$\mathcal{O} \sim \phi \partial_{\{\mu_1} \dots \partial_{\mu_n\}} \phi$$

  with $\Delta_{\mathcal{O}} = n + 2\Delta_\phi$ and $\Delta_\phi > \frac{D-2}{2}$.
- Tentative result: Bound on gap for *any* spins is *saturated* by GFFs.
- If true then: leading $1/N^2$ always negative!

Other stuff

- Technology still begin refined $\Rightarrow$ lots to do!
- Seem to get new bounds/results all the time.
- Only just begun to take advantage of conformal symmetry in $D > 2$.
- Lots to do...

Thanks