The phase diagram of polar condensates

Taking the square root of a vortex

Austen Lamacraft [*with Andrew James*]
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KITP, UCSB

Outline

1. Magnetism in Bose condensates
2. Order parameters and topology in polar condensates
3. Domain walls, disclinations, and the phase diagram in 2D
4. Generalized XY model: view from field theory

Outline

Magnetic Bose condensates

BEC $\Longrightarrow$ magnetism for bosons with spin!

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$$\phi = \begin{pmatrix} \phi_{1/2} \\ \phi_{-1/2} \end{pmatrix} = e^{i\psi} \begin{pmatrix} e^{-i\varphi/2} \cos\theta/2 \\ e^{i\varphi/2} \sin\theta/2 \end{pmatrix}$$

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Does magnetism $\implies$ BEC?

or can ordering happen sequentially as temperature lowered?

Higher spin gives additional possibilities

Consider the spin-1 state

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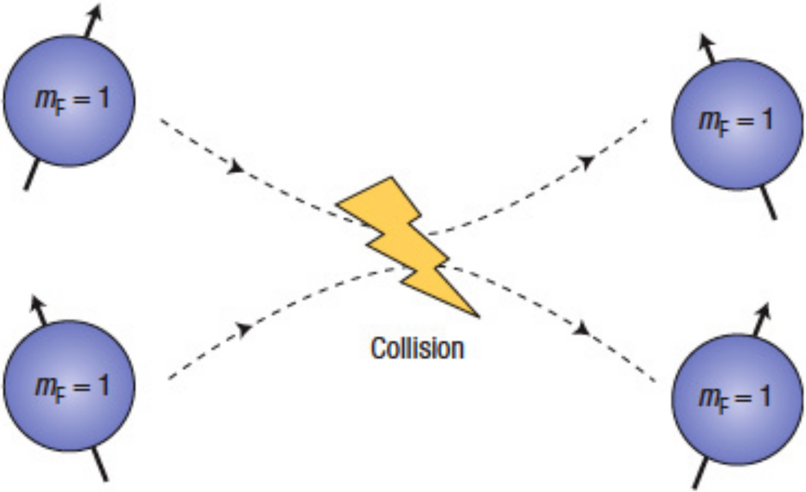
- Fixing the nature of the BEC requires some dynamical input (interactions)

Contact interactions in a spin-1 gas

Total spin 2



Contact interactions in a spin-1 gas

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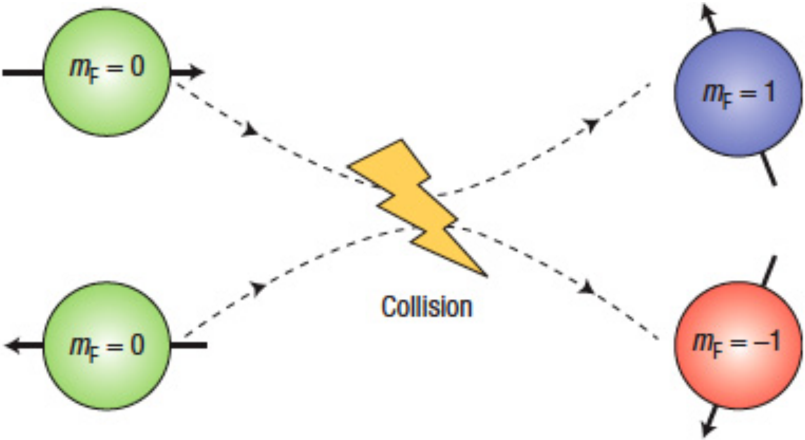

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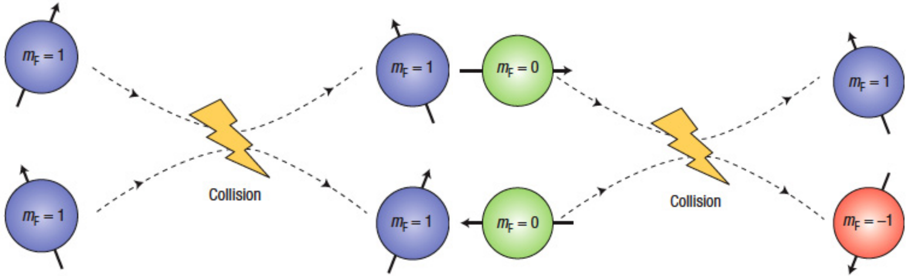

$$H_{\text{int}} = \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j) \left[g_0 \mathcal{P}_0 + g_2 \mathcal{P}_2 \right]$$
$$= \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j) \left[c_0 + c_2 \mathbf{S}_i \cdot \mathbf{S}_j \right]$$

$$c_0 = (g_0 + 2g_2)/3 \qquad c_2 = (g_2 - g_0)/3$$

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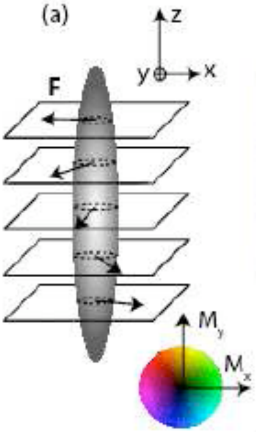
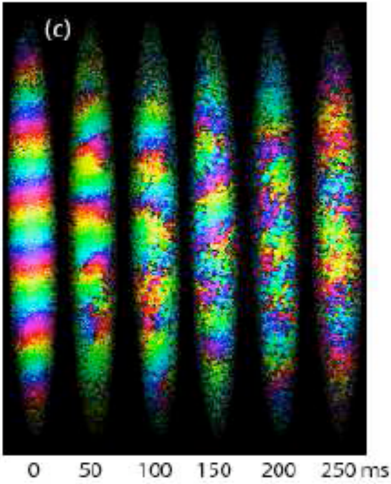
The Bose Ferromagnet





Stamper-Kurn group, Berkeley

Outline

Order parameter manifold for polar condensates

Mean field ground state written $\phi = \mathbf{n}e^{i\theta}$
There is an evident redundancy: $(\mathbf{n}, \theta)$ and $(-\mathbf{n}, \theta + \pi)$ are the *same*

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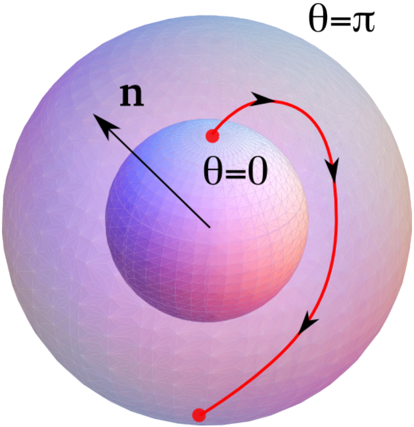

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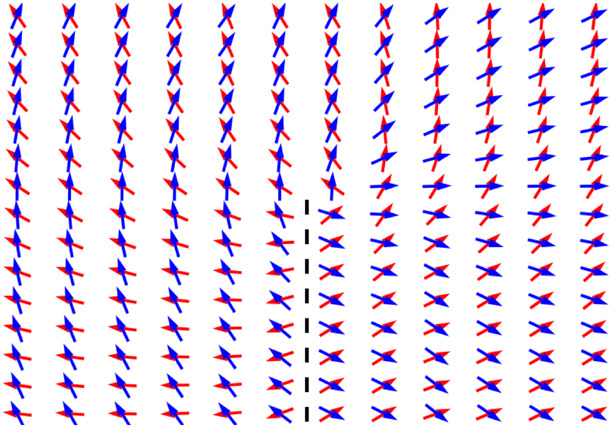
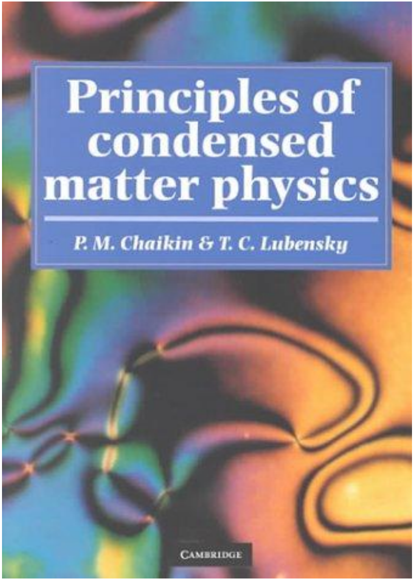


Disclinations in nematic liquid crystals



Consequences for Kosterlitz–Thouless transition

Circulation quantum is *halved*

$$\oint \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{m} \oint \nabla\theta = \frac{\hbar}{m}\pi n = \frac{h}{2m} n, \qquad n \in \mathbb{Z}$$

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Consider free energy of a single half vortex / disclination

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Possible Experiments on Two-dimensional Nematics

BY P. G. DE GENNES

Physique du Solide, Faculté des Sciences, 91 Orsay

Received 9th July, 1971

Vanishes for

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KT transition mediated by half-vortices / disclinations

Jump is $4\times$ bigger than usual! (Korshunov, 1985)

$$\Delta n_{KT/2} = 4\Delta n_{KT} = \frac{8}{\pi}\frac{mk_B T_c}{\pi\hbar^2}$$

KT transition mediated by half-vortices / disclinations

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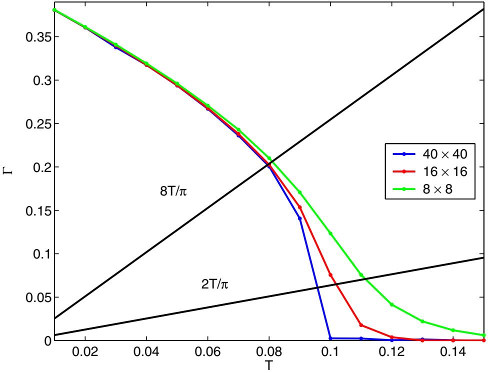
Topological Defects and the Superfluid Transition of the $s = 1$ Spinor Condensate in Two Dimensions

Subroto Mukerjee,[1,2] Cenke Xu,[1] and J. E. Moore[1,2]



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$$H \to a^{2-d} t \int d\mathbf{r} \left[(\nabla\theta)^2 + (\nabla\mathbf{n})^2 \right]$$

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- Beneath KT transition, half vortices are absent and one can treat the **n** degrees of freedom as Heisenberg spins
- Mermin–Wagner theorem says they do not order at finite temperature.

Outline

The Zeeman effect

Within the spin-1 multiplet, the Zeeman energy is

$$H_{Z,m} = pm + qm^2$$

$p \propto B$ *linear* and $q \propto B^2/A_{\mathrm{HF}}$ *quadratic* Zeeman effects

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Basic problem:

How does the phase diagram evolve with q from the $\frac{1}{2}$KT transition mediated by half vortices to the usual KT transition?

Consequences of quadratic Zeeman effect

$$H_{QZ} = q \sum_i \phi_i^\dagger S_z^2 \phi_i = q \sum_i (1 - n_{z,i}^2)$$

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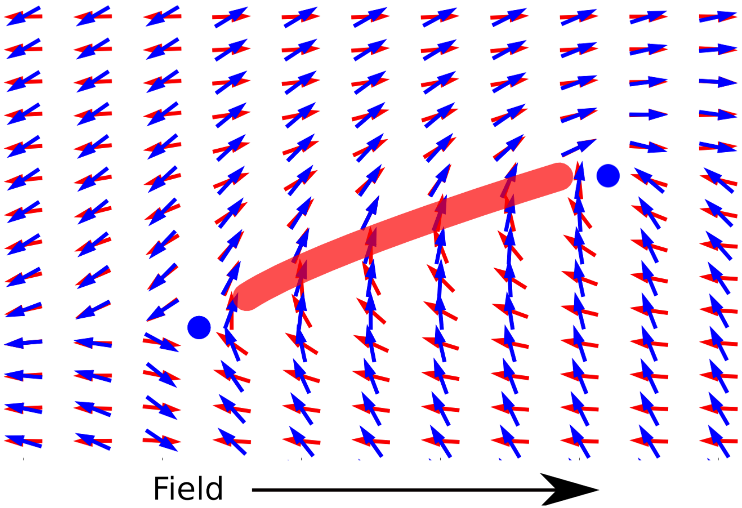
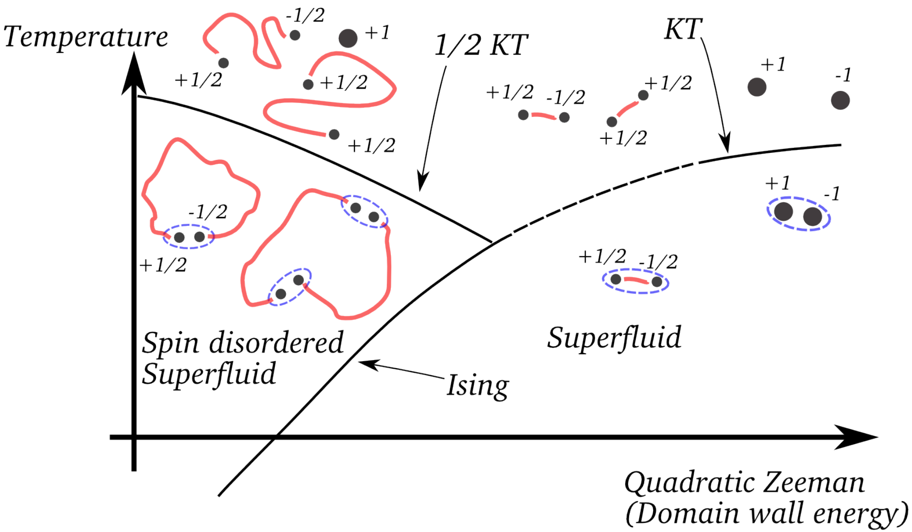
Large q fixes $\mathbf{n}$: regular KT transition

Half vortices terminate soliton walls



Conjectured phase diagram



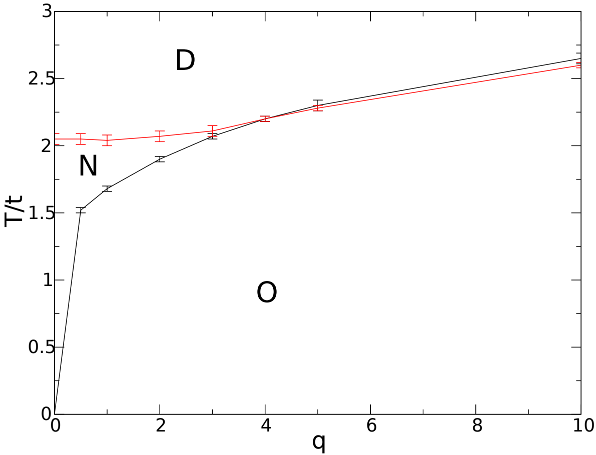
Monte Carlo simulation

Include hopping of singlet pairs $\phi \cdot \phi = \cos(2\theta)$

$$H = -\sum_{ij} \left[2t\, \mathbf{n}_i \cdot \mathbf{n}_j \cos\left(\theta_i - \theta_j\right) + u \cos(2\left[\theta_i - \theta_j\right])\right] - q \sum_i n_{z,i}^2$$

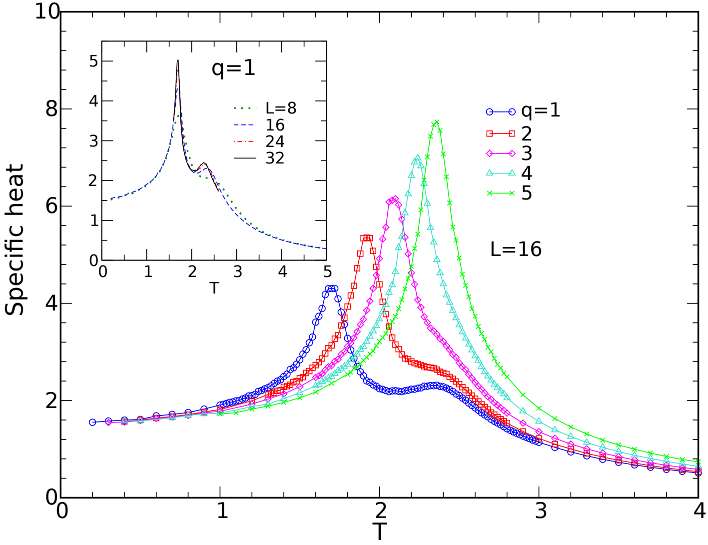
Monte Carlo simulation

Intermediate phase with singlet pair quasi-long range order



(data at $u = 1$)

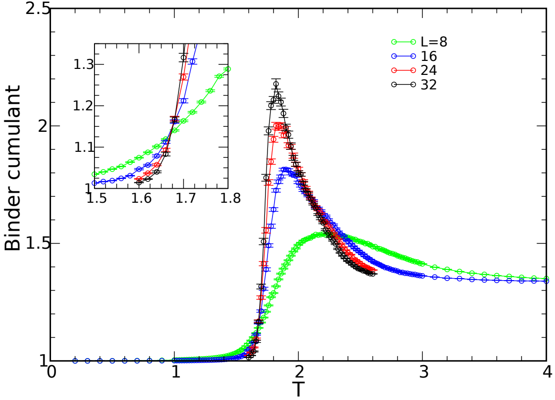
Heat Capacity



Binder cummulant

$$\Phi = \sum_i \phi_i \qquad U_4 = \frac{\langle \left(\Phi^\dagger \Phi\right)^2 \rangle}{\left(\langle \Phi^\dagger \Phi \rangle\right)^2}$$

q=1



Ising scaling for the lower transition

$$U_4 = f\left(\frac{L}{\xi}\right) = f\left(L\left|\frac{T-T_c}{T_c}\right|^{\nu}\right)$$

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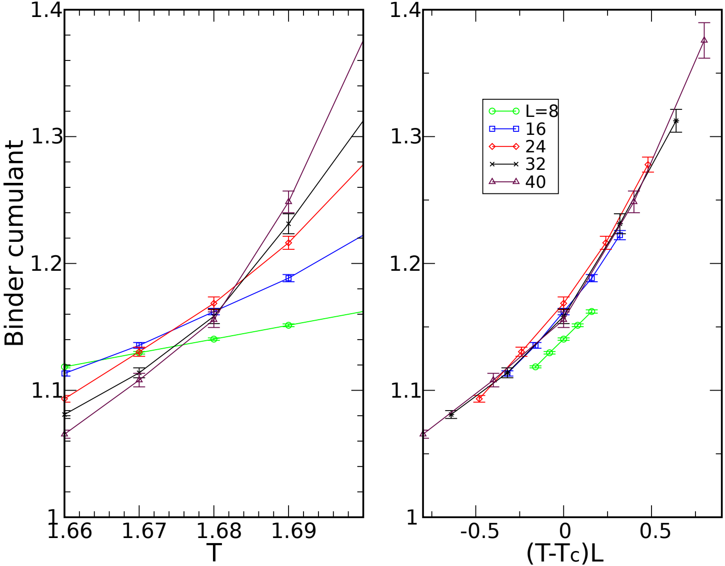

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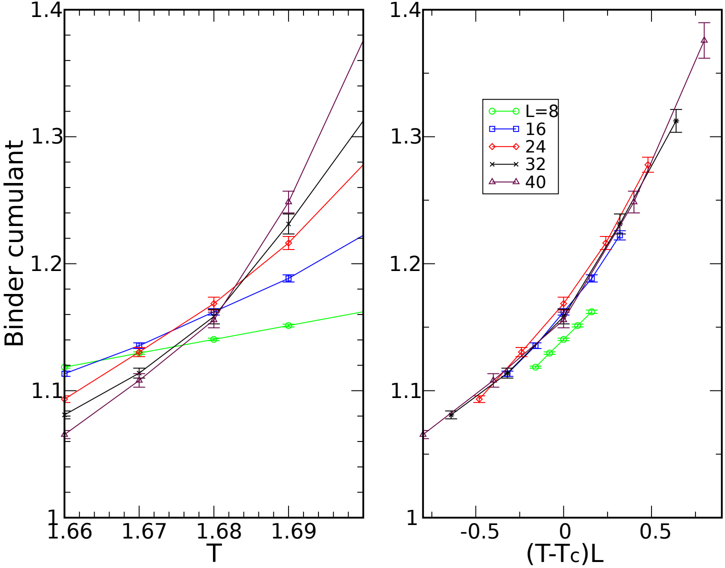
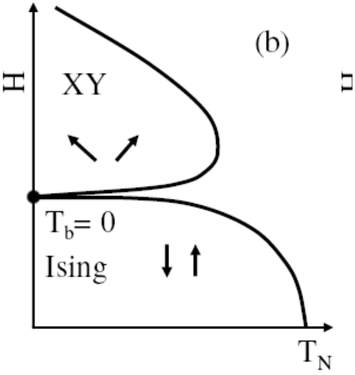


What about finite magnetization?

At finite magnetization have *spin-flop* transition: $\mathbf{n}$ flops into $x - y$ plane
Analogous to antiferromagnet



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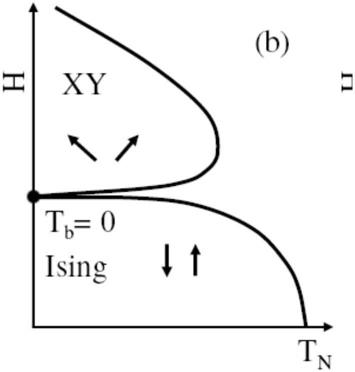


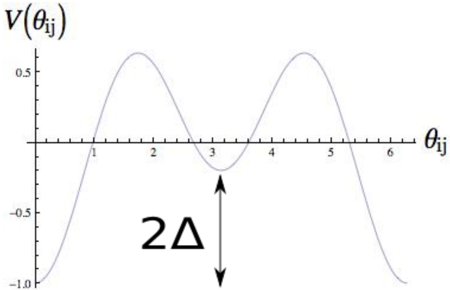
Zero temperature Heisenberg fixed point

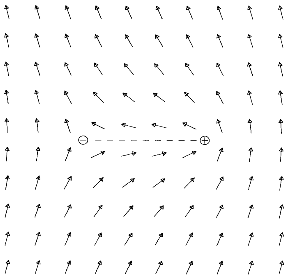
Outline

Generalized XY model

$$H_{\text{gen}} = -\sum_{\langle ij \rangle} \Big(\Delta \cos(\theta_i - \theta_j) + (1-\Delta)\cos(2\theta_i - 2\theta_j) \Big)$$

Korshunov (1985), Grinstein & Lee (1985)





Field theoretic view: how can domain walls end?[1]

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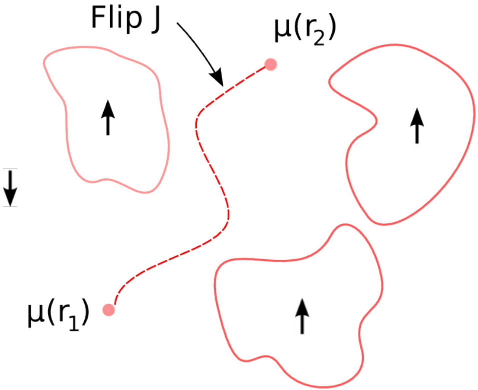

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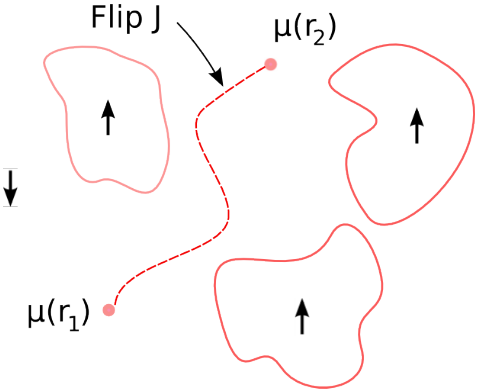

Half vortex insertion tied to disorder operator

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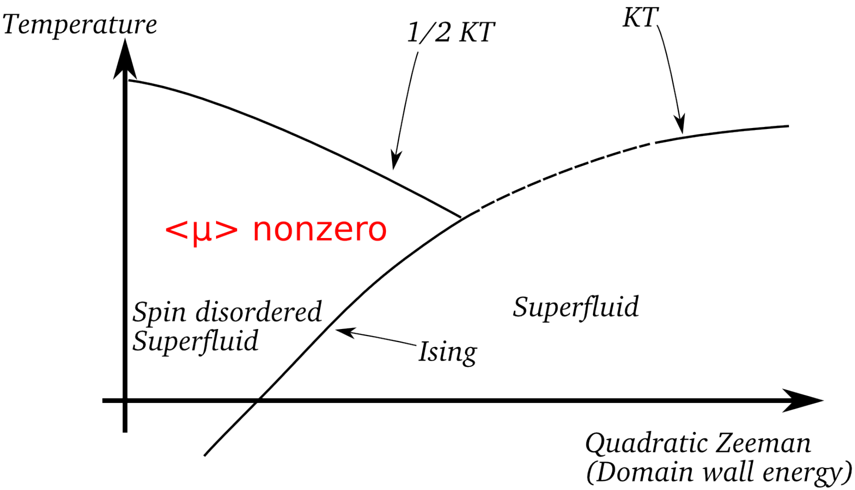
Phase diagram from $\mu \cos(\phi/2)$ perturbation



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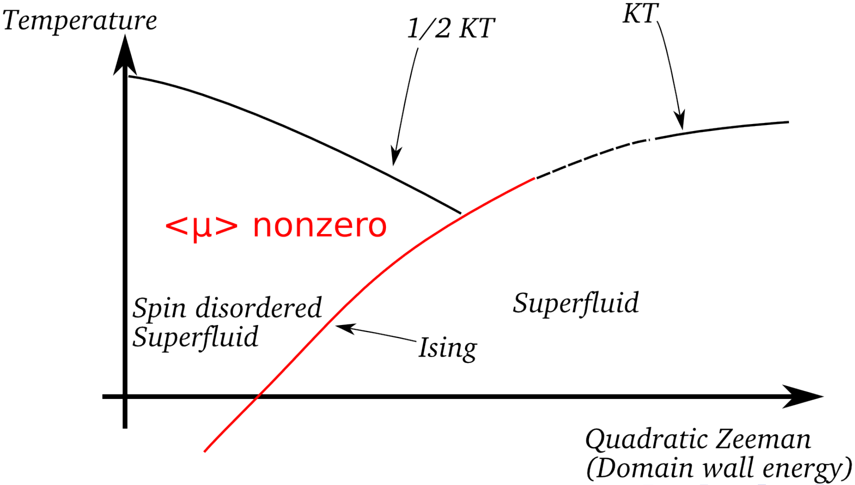
- $\frac{1}{2}$KT occurs when $\cos(\phi/2)$ becomes relevant if $\langle\mu\rangle \neq 0$
- Along Ising line: continuous transition until $\mu \cos(\phi/2)$ relevant



Scaling with $\mu\cos(\phi/2)$ perturbation

$$H_{\text{vortex}} = \lambda\,\mu\cos\phi/2$$

Three couplings to keep track of

1. λ, half vortex fugacity
2. t_I, energy operator for Ising
3. Stiffness K of superfluid

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$$\frac{d\lambda}{d\ell} = \left(2 - \frac{1}{8} - \frac{\pi K}{4}\right)\lambda + \frac{1}{2}\lambda t_I$$
$$\frac{dt_I}{d\ell} = t_I + \frac{\lambda^2}{2}$$
$$\frac{dK}{d\ell} = \lambda^2$$

RG flow for $K > 15/2\pi$