# *Conventional and Unconventional Quantum Critical points*

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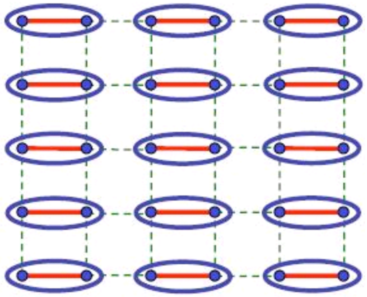


## *Conventional and Unconventional Quantum Critical points*

# Outline:

1, Conventional QCP within GL formalism

Basics of scaling, dynamical scaling, QCP with Fermi liquid, QCP with superfluid, scaling and hyperscaling, classical phase transition dipole interaction.

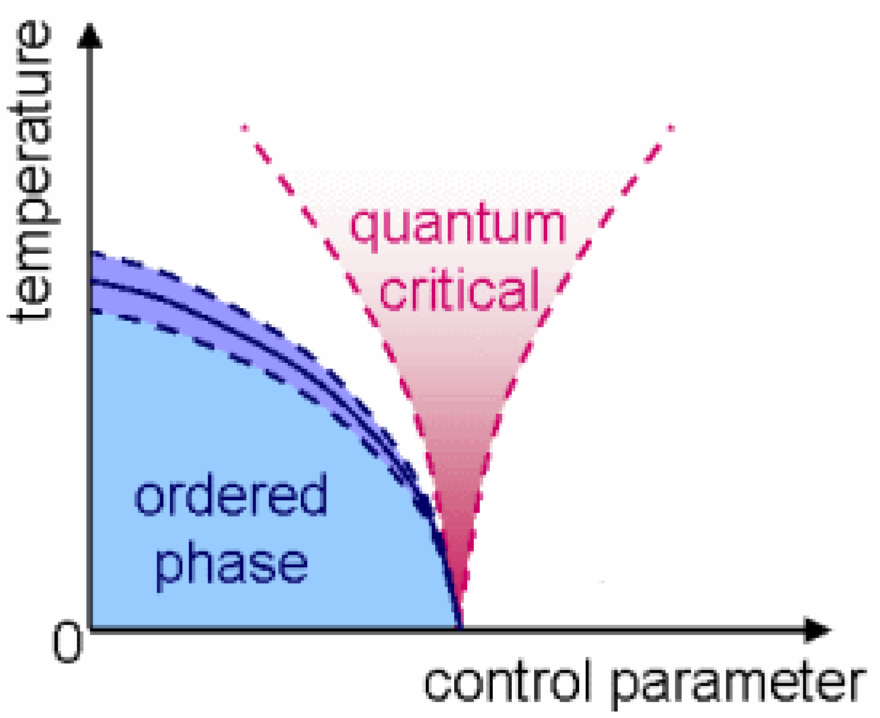
2, Unconventional QCP,

confine-deconfine transition, Z2, U(1), deconfined criticality and beyond, QCP between spin liquid and metal

## *Conventional and Unconventional Quantum Critical points*

# Quantum Phase Transitions:

http://en.wikipedia.org/wiki/Quantum_critical_point



## *Conventional and Unconventional Quantum Critical points*

### basics about scaling and RG

$$L = (\partial_\tau \phi)^2 + (\partial_i \phi)^2 + r\phi^2 + g\phi^4$$

Usually scaling laws are characterized by the following equations:

$$\langle \phi(0)\phi(l) \rangle \sim \frac{1}{l^{d+z-2+\eta}}, \quad \xi \sim r^{-\nu}$$

$$[\phi] = (d + z - 2 + \eta)/2 \qquad\qquad [\phi^2] = d + z - 1/\nu$$

For a large class of critical points, all the scaling exponents ONLY depend on two numbers, $\eta$ and $\nu$, these systems satisfy "**hyperscaling**".

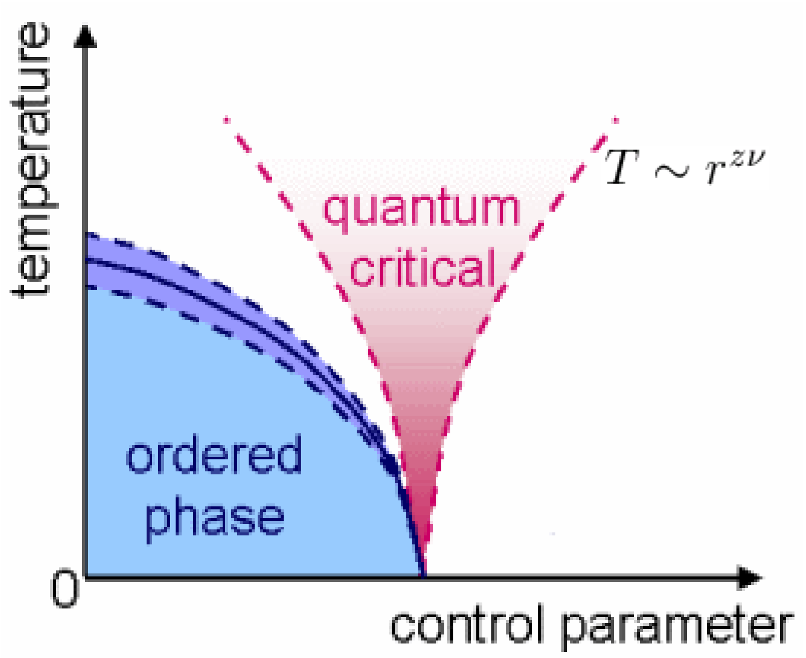
## *Conventional and Unconventional Quantum Critical points*

$$L = (\partial_\tau \phi)^2 + (\partial_i \phi)^2 + r\phi^2 + g\phi^4$$

If there are two relevant perturbations, say *T* and *r*, then these two relevant perturbations compete when $\xi_T \sim \xi_r$

This implies that the competition occurs at

$T \sim r^{z\nu}$



## *Conventional and Unconventional Quantum Critical points*

### Scaling and hyperscaling:

For a large class of critical point, all the scaling functions ONLY depend on two numbers, $\eta$ and $\nu$, these systems satisfy “**hyperscaling**”.

$$[\phi] = (d + z - 2 + \eta)/2 \qquad [r] = 1/\nu$$

$$\langle \phi \rangle \sim r^{\nu(d+z-2+\eta)/2} \qquad \chi = \langle \frac{d\phi}{dh} \rangle \sim r^{-\nu(2-\eta)}$$

Equations with equal scaling dimensions on both sides.

We can always do this when $d+z < 4$, however, when $d+z > 4$, we lose this kind of beauty.

# *Conventional* and *Unconventional Quantum Critical points*

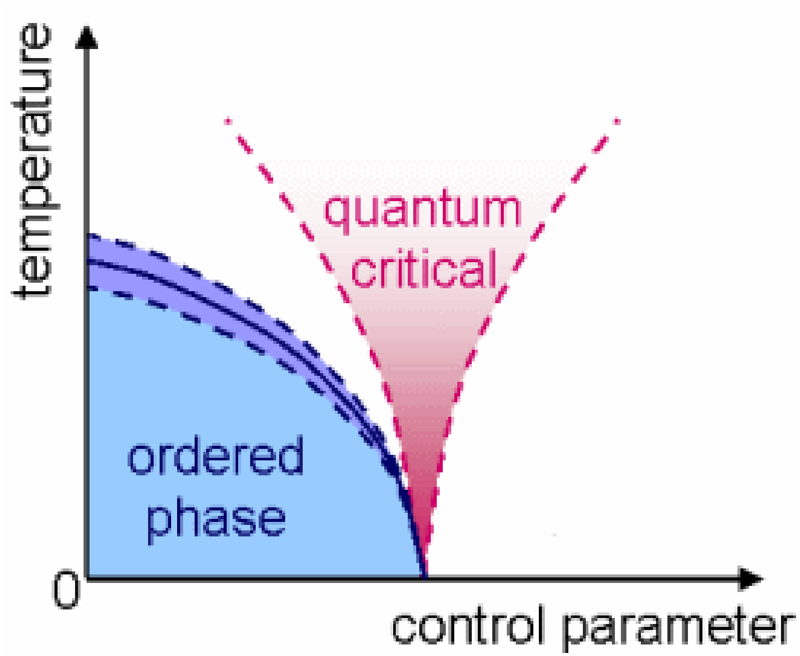
$$L = (\partial_\tau \phi)^2 + (\partial_i \phi)^2 + r\phi^2 + g\phi^4$$

We want to consider the scaling between $T_c$ and $r$.

Without hyperscaling, irrelevant parameter $g$ should enter scaling:

$$r_{eff} = r + gT^{\frac{d+z-2}{z}}$$

$$T_c \sim r^{\frac{z}{d+z-2}}$$



## *Conventional and Unconventional Quantum Critical points*

### Quantum Wilson-Fisher fixed point

Now let us take *d=2, z=1.* g is relevant, and the system is no longer described by the Gaussian theory. Instead, it is described by a Wilson-Fisher fixed point.

$$L = \sum_{a=1}^{N} (\partial_\tau \phi^a)^2 + (\partial_i \phi^a)^2 + r(\phi^a)^2 + g|\sum_a (\phi^a)^2|^2$$

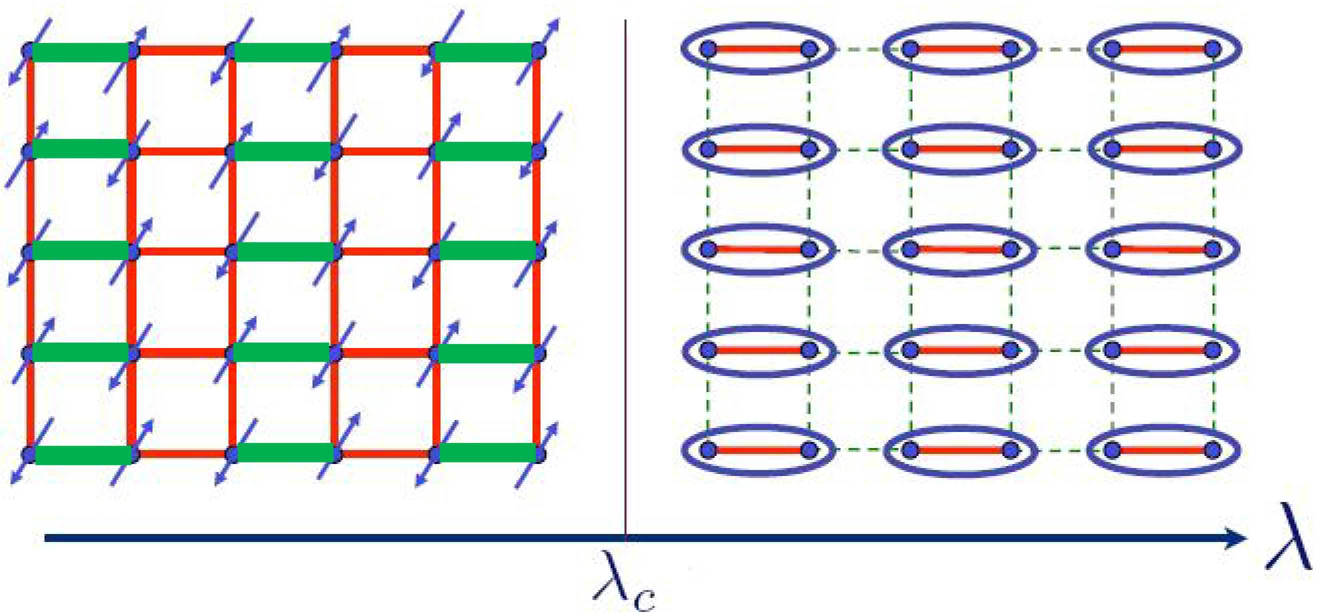
$N = \infty, \nu = 1, \eta = 0$

| $N$ | | $\nu$ | $\eta$ |
|---|---|---|---|
| 1 | FD exp [13] | 0.6303(8) | 0.0335(25) |
| | $\epsilon$ exp [13] | 0.6305(25) | 0.0365(50) |
| | HT exp [14] | 0.63012(16) | 0.03639(15) |
| | MC [15] | 0.6297(5) | 0.0362(8) |
| 2 | FD exp [13] | 0.6703(15) | 0.0354(25) |
| | $\epsilon$ exp [13] | 0.6680(35) | 0.0380(50) |
| | HT exp [16] | 0.67155(27) | 0.0380(4) |
| | MC [16] | 0.6716(5) | 0.0380(5) |
| 3 | FD exp [13] | 0.7073(35) | 0.0355(25) |
| | $\epsilon$ exp [13] | 0.7045(55) | 0.0375(45) |
| | HT exp [17] | 0.7112(5) | 0.0375(5) |
| | MC [17] | 0.7113(11) | 0.0378(6) |

Calabrese *et.al.*
Condmat/0306273

# *Conventional and Unconventional Quantum Critical points*

Realize O(3) Wilson-Fisher fixed point:




$$H = \sum_{i \in \mathrm{odd}} \lambda J \vec{S}_i \cdot \vec{S}_{i+x} + \sum_{<i,j>} J \vec{S}_i \cdot \vec{S}_j$$

Staggered VBS, not O(3) Wilson-Fisher fixed point.

***Conventional and Unconventional Quantum Critical points***

## Dynamical scaling exponent

Quantum critical points are described by quantum field theories. In condensed matter, no guarantee to have relativistic field theories.

$$L = (\partial_\tau \phi)^2 + (\partial_i \phi)^2 + r\phi^2 + g\phi^4$$

Quantum Ising transition, z=1. $\phi \partial_\tau \phi$ vanishes, but….

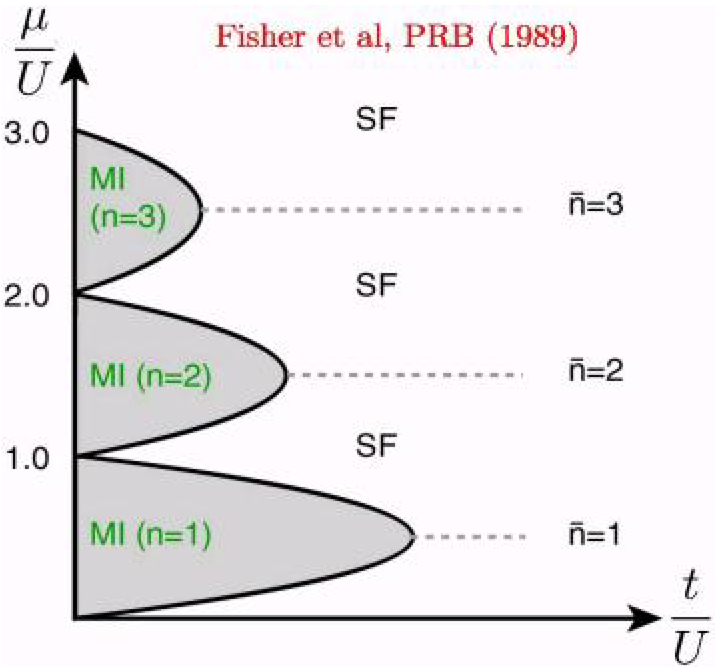
Transverse field quantum Ising model:

$$H_{Q,\ d} = \sum_{\ll i,j \gg} K\sigma_i^z \sigma_j^z + h\sigma_i^x, \qquad H_{C,\ d+1} = \sum_{\ll i,j \gg} K\sigma_i^z \sigma_j^z.$$

# *Conventional* and *Unconventional Quantum Critical points*

$$L = a(\psi^* \partial_\tau \psi) + |\partial_\tau \psi|^2 + r|\psi|^2 + g|\psi|^4 + \cdots$$

z = 2, Bose Hubbard model away from the tip of the MI lobe:




$$H = -t\sum_{\langle i,j\rangle}\left(b_i^\dagger b_j + b_j^\dagger b_i\right) + \frac{U}{2}\sum_i \hat{n}_i\left(\hat{n}_i - 1\right) - \mu\sum_i \hat{n}_i$$

Emergent PH symmetry at low energy at tip. No $\omega$ term, but maybe $\omega^3$ term.

$$\psi \rightarrow \psi^*$$

## *Conventional and Unconventional Quantum Critical points*

$$L = \psi^* \partial_\tau \psi + |\partial_i \psi|^2 + g|\psi|^4$$

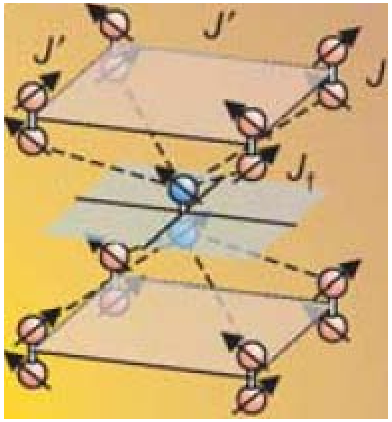
↓

$$L = \vec{\phi} \times \partial_\tau \vec{\phi} + |\partial_i \vec{\phi}|^2 + r|\vec{\phi}|^2 + g|\vec{\phi}|^4$$

Describes Larmor precession of XY spin in magnetic field.

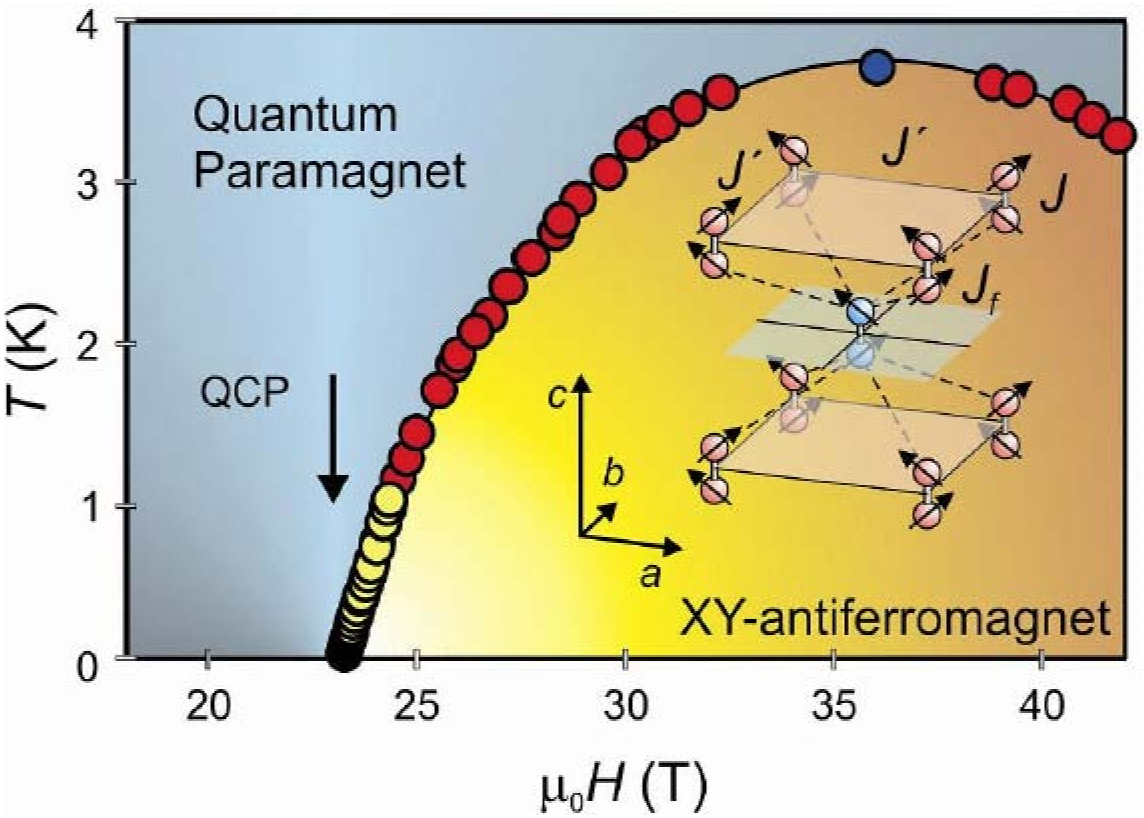
Example, quantum critical point in $BaCuSi_2O_6$

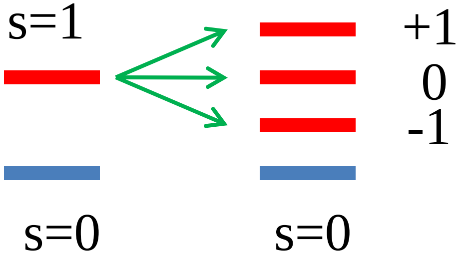
Nature, 441, 617




# *Conventional* and *Unconventional Quantum Critical points*

$$L = \vec{\phi} \times \partial_\tau \vec{\phi} + |\partial_i \vec{\phi}|^2 + r|\vec{\phi}|^2 + g|\vec{\phi}|^4$$



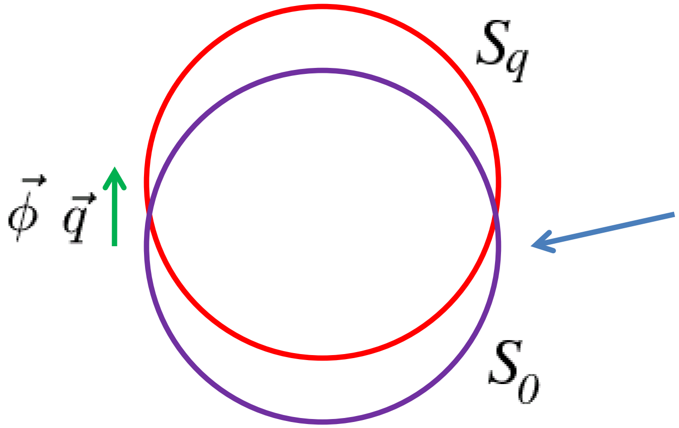
## *Conventional and Unconventional Quantum Critical points*

Hertz-Millis theory of QCP in fermi liquid

Example: FM transition in fermi liquid

$$L = |\partial_\tau \vec{\phi}|^2 + |\nabla_\mu \vec{\phi}|^2 + r|\vec{\phi}|^2 + g|\vec{\phi}|^4 + \vec{\phi} \cdot \psi^\dagger \vec{\sigma} \psi$$

FM order parameter can decay into particle-hole pairs




Couples most strongly to the intersection, $\vec{k}_f \perp \vec{q}$ where the particle-hole excitation is softest.

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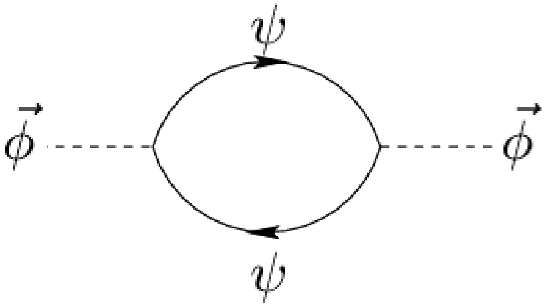
Example: FM transition in fermi liquid

$$L = |\partial_\tau \vec{\phi}|^2 + |\nabla_\mu \vec{\phi}|^2 + r|\vec{\phi}|^2 + g|\vec{\phi}|^4 + \vec{\phi}\cdot\psi^\dagger\vec{\sigma}\psi$$

FM order parameter can decay into particle-hole pairs, *i.e.* the imaginary part of the self-energy.

$$\begin{aligned}\mathrm{Im}[\Sigma_\phi(\omega, q)] &\sim \int \frac{d^2k}{(2\pi)^2}[f(\epsilon_{k+q}) - f(\epsilon_k)] \\ &\times \delta(|\omega| - \epsilon_{k+q} + \epsilon_k)|\langle k|\psi_k^\dagger\vec{\sigma}\psi_{k+q}|k+q\rangle|^2 \\ &\sim \frac{|\omega|}{v_f q}.\end{aligned}$$

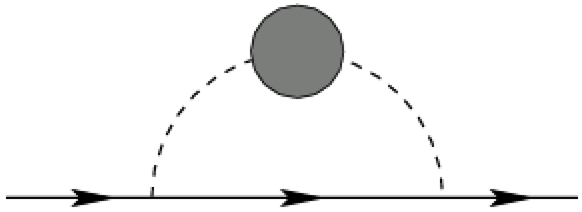


# *Conventional and Unconventional Quantum Critical points*

$$L = (\frac{|\omega|}{q} + q^2 + r)|\vec{\phi}_q|^2 + g\vec{\phi}^4 \qquad [g] = -1$$

Conclusion: FM transition in ordinary fermi liquid has $z = 3$. Perturbations of $g$ does not lead to divergence, *i.e.* it is a mean field transition.

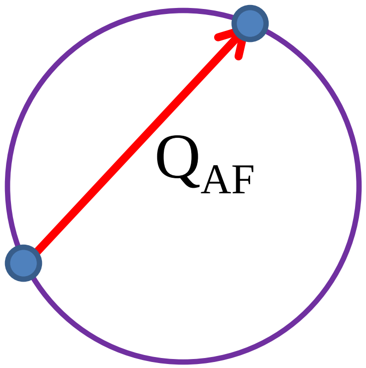
$$\Sigma_\psi \sim \omega^{2/3}\mathrm{sgn}[\omega] > \omega$$

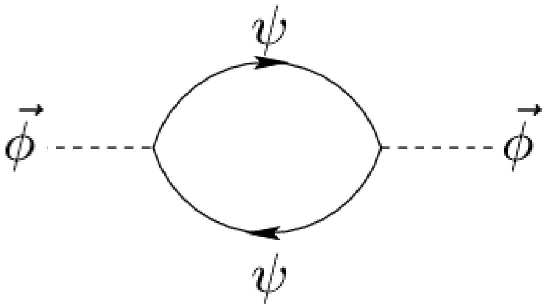


The self-energy dominates the fermi liquid behavior, at the quantum critical point there is no well-defined QP.

Same analysis for all other orders at zero momentum.

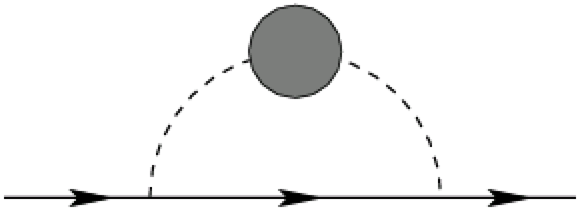
# *Conventional and Unconventional Quantum Critical points*




AF transition, z =2. as long as the wave-vector connects two separate points of the fermi surface. $L = (|\omega| + q^2 + r)|\vec{\phi}_{\omega,\vec{q}}|^2 \quad z = 2$

$$\Sigma_\psi \sim |\omega|^{1/2}\mathrm{sgn}[\omega]$$



## *Conventional and Unconventional Quantum Critical points*

# Scaling and Harris criterion

Couple the Lagrangian to random potential *V(r)*.

$$L = L_0 + hV(r)\phi(r)^2, \quad \langle V(r)V(r')\rangle \sim \delta^d(r - r')$$

$$[h] = d + z - (3d/2 + z - 1/\nu) = 1/\nu - d/2$$

This analysis implies that, when $\nu > 2/d$, the random potential is irrelevant, when $\nu < 2/d$, the random potential is relevant. This is called the Harris criterion.

2d quantum critical point is always instable against disorder.
3d classical critical point, depends on $\nu$.

## Conventional and Unconventional Quantum Critical points

### Phase transition with fermi surface

$$L = L_0 + h\phi^2\psi^\dagger\psi$$

We integrate out the fermions, obtain the following term:

$$L \sim L_0 + J(\phi^2)_{-k,-\omega}\frac{|\omega|}{k}(\phi^2)_{k,\omega}$$

$\frac{|\omega|}{k}$ is a singular momentum/frequency dependence, so it does not renormalize. All the renormalizations come from $\phi^2$

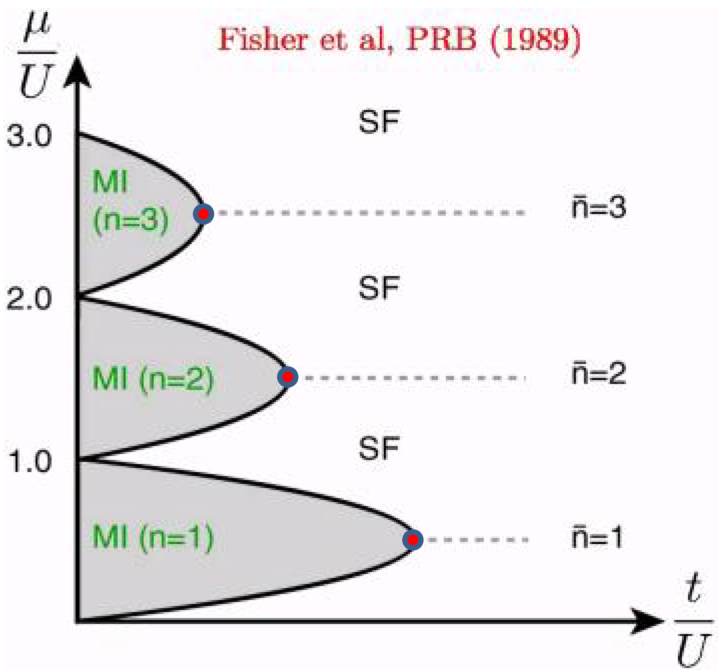
$$[J] = d + z - 2(d + z - 1/\nu) = 2/\nu - d - z$$

For $d = 2$, $z = 1$, for $N > 2$, $J$ is irrelevant.

## *Conventional and Unconventional Quantum Critical points*

# SF-MI transition inside a fermi liquid:

$$L = a(\psi^* \partial_\tau \psi) + |\partial_\tau \psi|^2 + r|\psi|^2 + g|\psi|^4 + u|\psi|^2 f^\dagger f$$




$$|\psi|^2_{\omega,k} \frac{|\omega|}{k} |\psi|^2_{-\omega,-k}$$

For 3d XY transition, ν > 2/3, thus
for z = 1 points, irrelevant,
for z = 2 points, obviously irrelevant

## *Conventional and Unconventional Quantum Critical points*

### Phase transition with superfluid

$$L = L_0 + h\phi^2 \delta n \sim L_0 + h\phi^2 \partial_\tau \theta$$

We integrate out the fermions, obtain the following term:

$$L \sim L_0 + J(\phi^2)_{-k,-\omega} \frac{\omega^2}{\omega^2 + c^2 k^2} (\phi^2)_{k,\omega}$$

Vertex still has a singular momentum/frequency dependence, so it does not renormalize. All the renormalizations come from $\phi^2$

$$[J] = d + z - 2(d + z - 1/\nu) = 2/\nu - d - z$$

For $d = 2$, $z = 1$, for $N > 2$, $J$ is irrelevant.
(Frey, Balents, PRB, 55, 1050)

# *Conventional* *and Unconventional Quantum Critical points*

## Classical phase transitions with "dynamical" scaling

Consider a classical Ising transition with Dipole interaction.

$$\int d^3r\, d^3r' \phi(r)\phi(r') \frac{3(r-r')_z^2 - (r-r')^2}{(r-r')^5}$$

After Fourier transformation:

$$F = (\frac{k_z^2}{k^2} + k^2)|\phi_k|^2 + \cdots \qquad [k_z] = 2,\ [k_x] = 1,\ [k_y] = 1$$

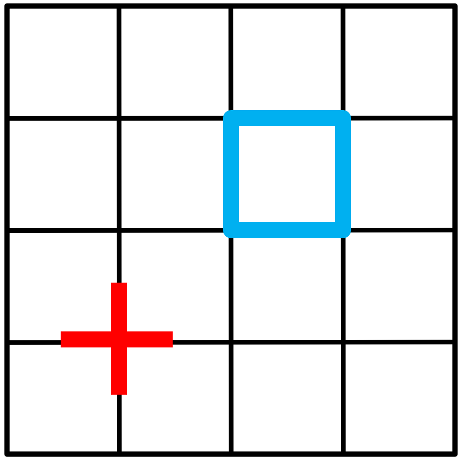
Now simplified as:

$$F = (\frac{k_z^2}{k_x^2 + k_y^2} + k_x^2 + k_y^2)|\phi_k|^2 + \cdots$$

Fisher, Aharony, late 70s, series of papers.

## *Conventional and Unconventional Quantum Critical points*

### Confine-deconfine transition:



Simplest example: Z2 gauge theory. Define Ising variables on the links of the lattice.

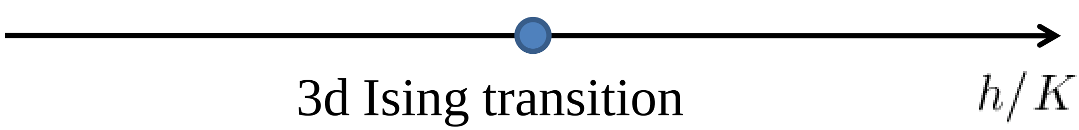
Constraint: $$\prod_{+} \sigma^x = 1$$

$$H = \sum_i -K\sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+y,x}\sigma^z_{i+x,y} - \sum_{i,\mu} h\sigma^x_{i,\mu}$$

$$\sigma^z_{i,\mu} \rightarrow \eta_i \sigma^z_{i,\mu} \eta_{i+\mu}$$

$$\mathcal{W} \sim \exp(-\mathcal{L})$$

$$\mathcal{W} = \langle \prod_{\mathcal{C}} \sigma^z \rangle \sim \exp(-\mathcal{A})$$

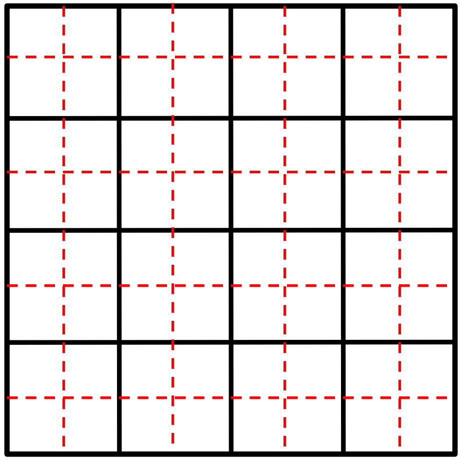


# *Conventional and Unconventional Quantum Critical points*

$\mathcal{W} \sim \exp(-\mathcal{L})$ $\qquad$ $\mathcal{W} = \langle \prod_{\mathcal{C}} \sigma^z \rangle \sim \exp(-\mathcal{A})$

3d Ising transition $\qquad h/K$

Go to the dual picture, solve the constraint:



$$\sigma^x_{i,\mu} = \tau^z_{i+\frac{\mu}{2}-\frac{\nu}{2}} \tau^z_{i+\frac{\mu}{2}+\frac{\nu}{2}}$$

$$\sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+y,x}\sigma^z_{i+x,y} = \tau^x_{i+x+y}$$

$$H = \sum_i -K\sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+y,x}\sigma^z_{i+x,y} - \sum_{i,\mu} h\sigma^x_{i,\mu}$$

$\Updownarrow$

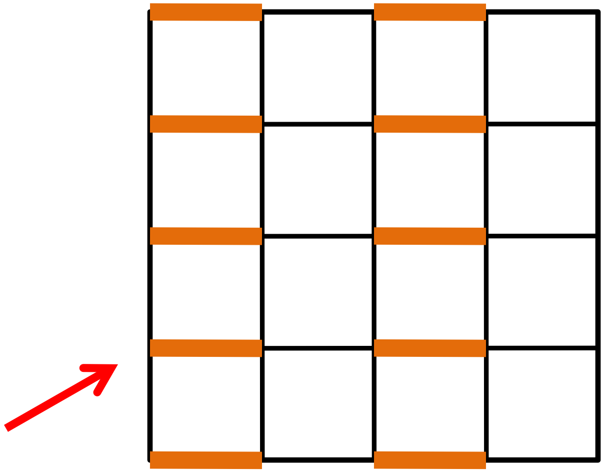
$$H_{\text{dual}} = \sum_{\bar{i}} -K\tau^x_{\bar{i}} - \sum_{\bar{i},\mu} h\tau^z_{\bar{i}}\tau^z_{\bar{i}+\mu}.$$

## *Conventional and Unconventional Quantum Critical points*

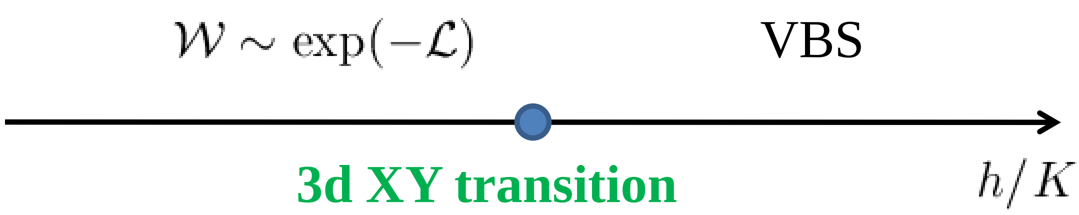
Modifications of the Z2 gauge theory:

$$\prod_{+} \sigma^x = -1$$

$$H = \sum_i -K\sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+y,x}\sigma^z_{i+x,y} - \sum_{i,\mu} h\sigma^x_{i,\mu}$$



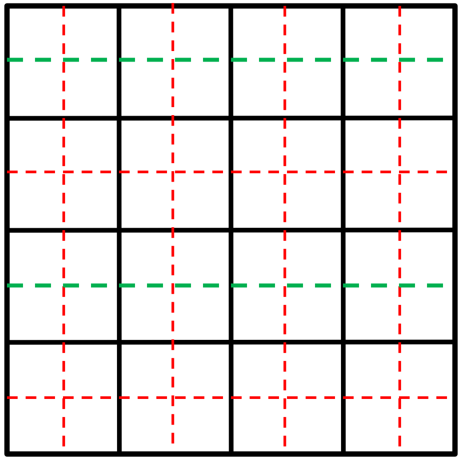
Hamiltonian dual to frustrated Ising model



## *Conventional and Unconventional Quantum Critical points*

Go to the dual picture, becomes the fully-frustrate-Quantum-Ising model

Solve the band structure of the FF-Ising model, the Ising spin will condense at two different points in the BZ.



$$\tau^z \sim \varphi e^{iQ\cdot r} + \varphi^* e^{-iQ\cdot r}$$

$$\mathcal{L}_{dual} = |\partial_\mu \varphi|^2 + r|\varphi|^2 + g|\varphi|^4 + u(\varphi^4 + \varphi^{*4})$$

3d XY transition with Z4 anisotropy, corresponding to four fold VBS pattern.

Z4 anisotropy is irrelevant at the 3d XY universality class.

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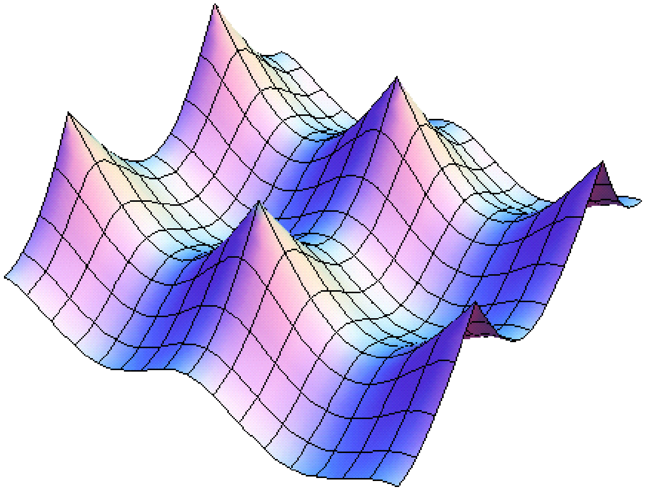

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3d XY transition with Z4 anisotropy, corresponding to four fold VBS pattern.

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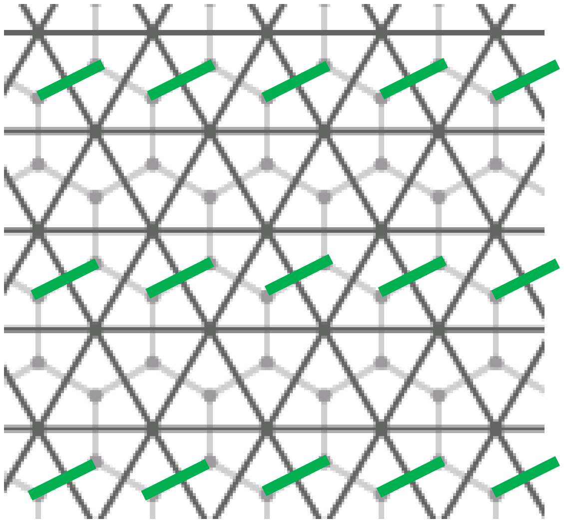
## *Conventional and Unconventional Quantum Critical points*

Define odd Z2 gauge field on the triangular lattice, dual to a FF-Ising model on the honeycomb lattice.



Four minima in the BZ.

The confine-deconfine transition of odd Z2 gauge field on the triangular lattice is an O(4) transition. (Moessner, Sondhi, 2001)

Application: spin liquid – VBS transition of the triangular lattice quantum dimer model is O(4) transition.

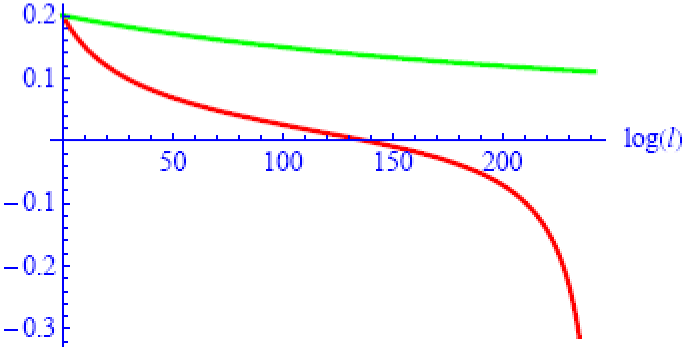
## Phase transitions of compact U(1) gauge field (sketch)

Compact U(1) gauge field, physically means introducing magnetic monopoles or magnetic charges.

In 3+1d, EM field is self-dual. The condensation of magnetic monopole is the confined phase for electric charge; while the condensation of electric charge is a confined phase for magnetic monopoles.

$$\mathcal{L} = |(\partial_\mu - ia_\mu)\varphi|^2 + r|\varphi|^2 + g|\varphi|^4 + \cdots$$

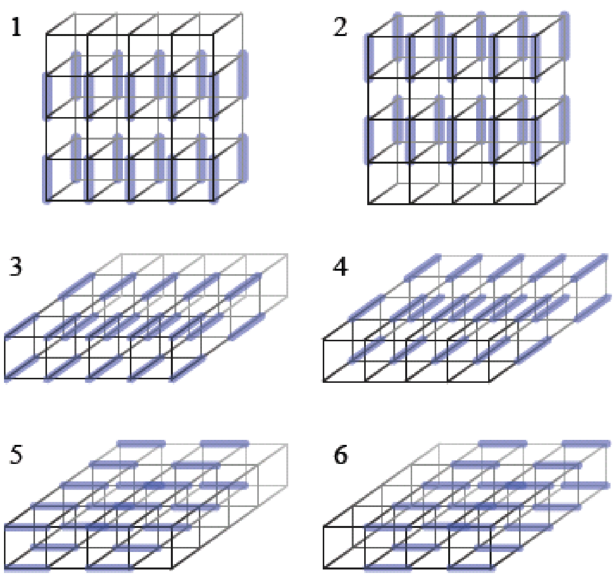
Coupled RG calculation for *g* and *e* yields a "extremely" weak first order transition, almost mean field.



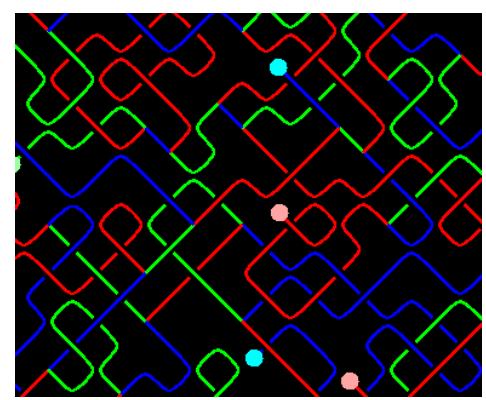
# *Conventional and Unconventional Quantum Critical points*

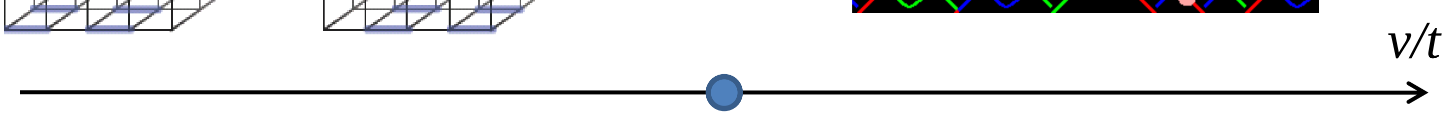
Application: Coulomb (photon) phase to VBS transition for 3d quantum dimer model:

$$H = -t\left(|=\rangle\langle \|\,| + h.c.\right) + v\left(|=\rangle\langle=| + |\|\rangle\langle\|\,|\right)$$




*Photon phase*





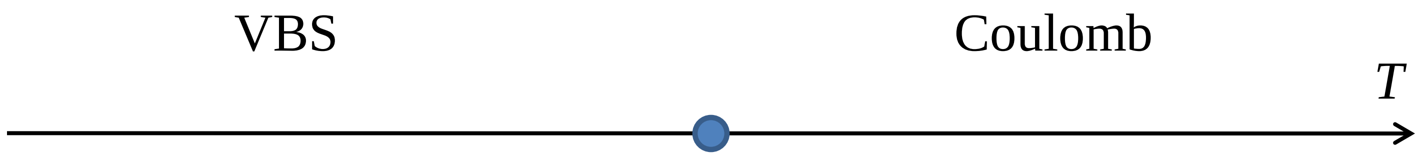

$$\mathcal{L} = \sum_a |(\partial_\mu - i a_\mu)\varphi_a|^2 + r|\varphi_a|^2 + g(\sum_a |\varphi_a|^2)^2 + \cdots$$

## *Conventional and Unconventional Quantum Critical points*

# Coulomb to VBS transition of classical dimer model

$$\mathcal{H}_{\text{6-GS}} = -\sum_{\square}(n_{=} + n_{//} + n_{||})$$

$$Z = \sum_{\mathcal{C}} \exp(-\frac{\mathcal{H}}{T})$$



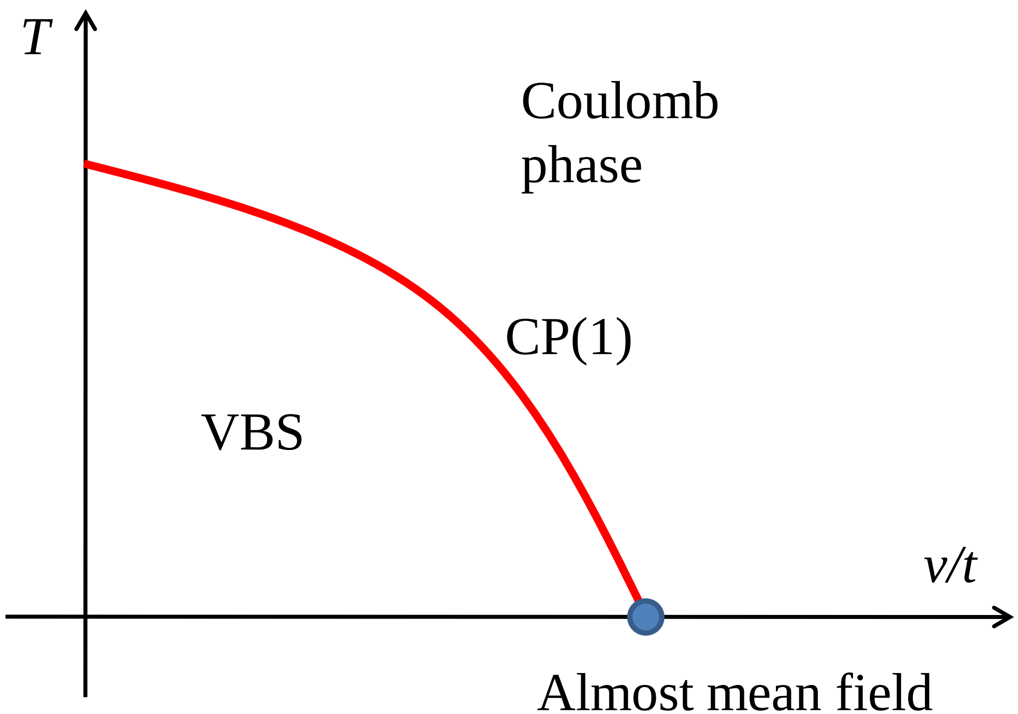

$$\mathcal{L} = \sum_{a} |(\partial_\mu - i a_\mu)\varphi_a|^2 + r|\varphi_a|^2 + g(\sum_{a} |\varphi_a|^2)^2 + \cdots$$

3d CP(1) model, (Gang Chen et.al. PRB, 80, 045112)

## *Conventional and Unconventional Quantum Critical points*

Global phase diagram of 3d quantum dimer model

$$H = -t\left(|\,=\,\rangle\langle\,\|\,| + h.c.\right) + v\left(|\,=\,\rangle\langle\,=\,| + |\,\|\,\rangle\langle\,\|\,|\right)$$



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Electric-magnetic duality:

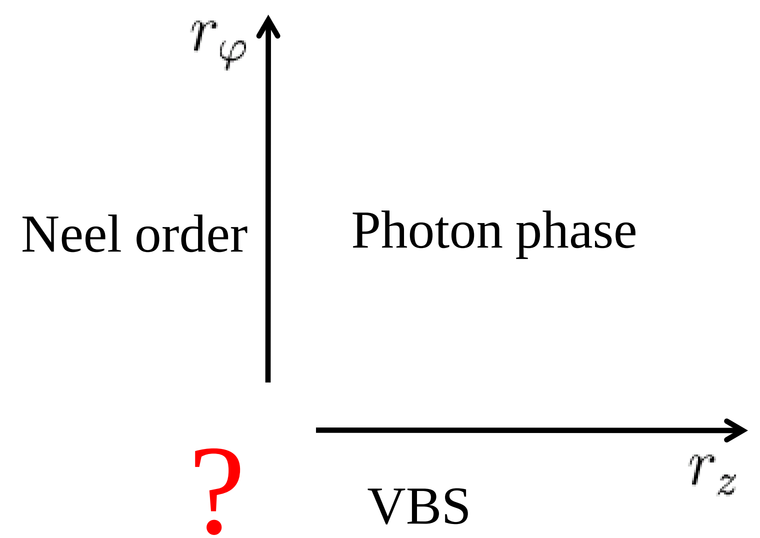
$$\mathcal{L} = \sum_a |(\partial_\mu - iA_\mu) z_a|^2 + r_z|z_a|^2 + g(\sum_a |z_a|^2)^2 + \cdots \qquad z^\dagger \vec{\sigma} z, \quad \text{Neel}$$

$$\mathcal{L} = \sum_a |(\partial_\mu - ia_\mu)\varphi_a|^2 + r_\varphi|\varphi_a|^2 + g(\sum_a |\varphi_a|^2)^2 + \cdots \qquad \varphi^\dagger \vec{\sigma} \varphi, \quad \text{VBS}$$

Spinon condense, Neel order,
Monopole condense, VBS order,
Both gapped, photon phase

Both condense?
Direct transition?

Motrunich, Senthil 2004



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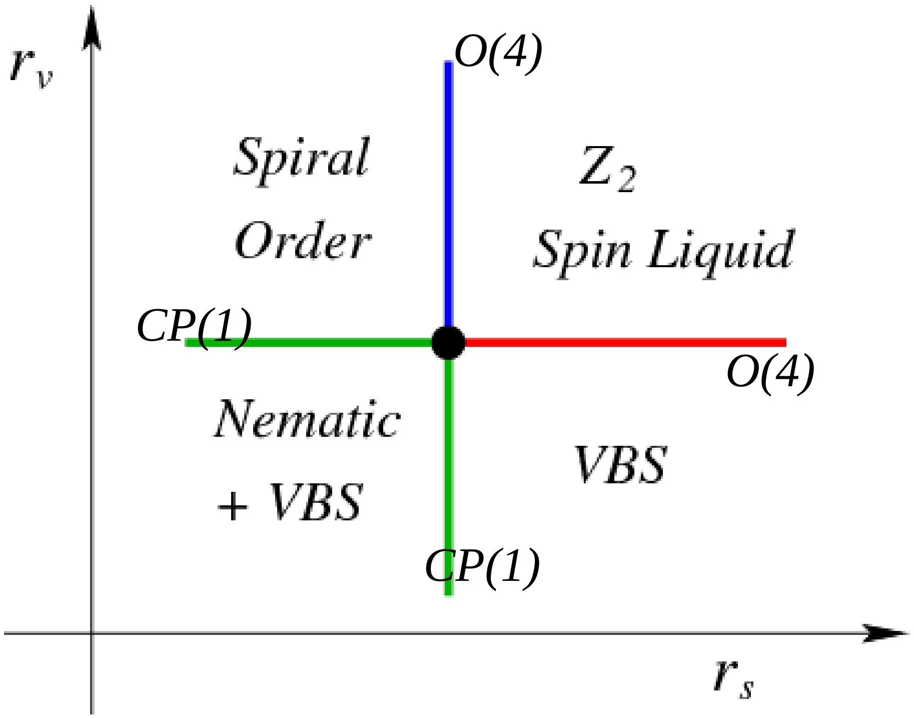
2d triangular lattice, spinon and vison also dual to each other:

$$L = \sum_{\alpha=1}^{2} |(\partial_\mu - i a_\mu) z_\alpha|^2 + |(\partial_\mu - i b_\mu) v_\alpha|^2 + \frac{ik}{4\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu b_\rho + r_s |z_\alpha|^2 + r_v |v_\alpha|^2 + \cdots$$




Xu, Sachdev, 2008

# *To Continue ……*