UNIVERSAL OPERATOR GROWTH HYPOTHESIS FOR HAMILTONIAN DYNAMICS

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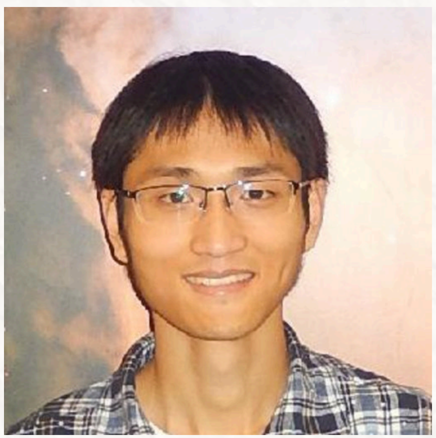
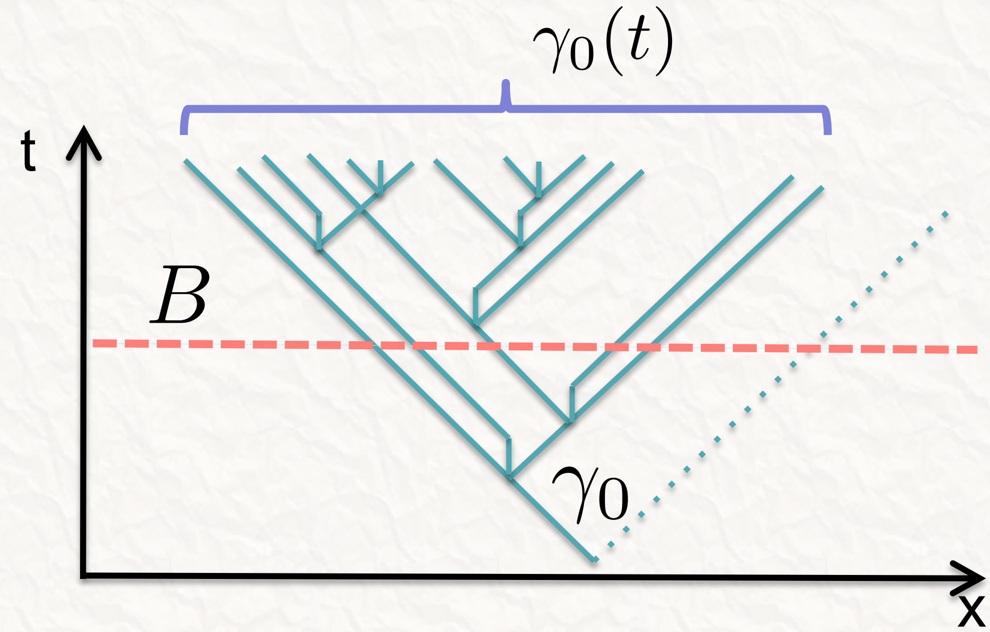


Operator growth

- Operator size; Lyapunov exponents (measured by OTOC)
- Operator complexity
- Emergent Hydrodynamics (Diffusion constants)

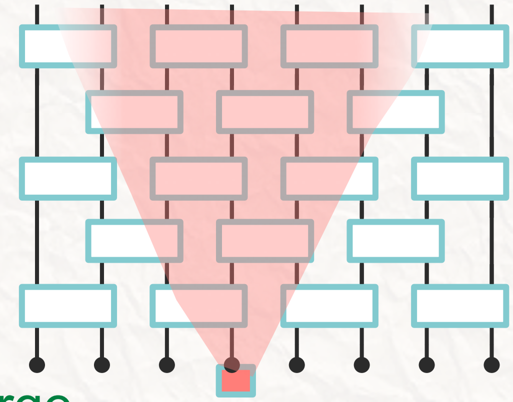


Is there a universal structure that governs and relates these quantities in generic systems?

Can we utilize such a structure to enable computation of dynamics? (e.g. compute transport coefficients of strongly coupled systems.)

Recent progress from special models

Random unitary networks:

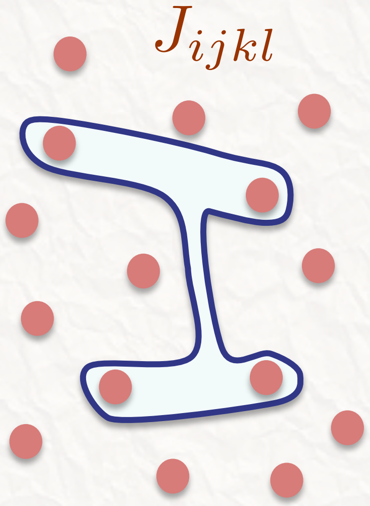


Nahum (2018), von Keyserlingk (2018) Khemani (2018)

✔ Local model and finite N per site
Universal operator front propagation.
Emergent dissipation / diffusion of conserved charge

✗ Non hamiltonian dynamics.
No energy conservation or notion of temperature.
Lyapunov exponents not well defined

SYK model



Sachdev, Kitaev, Stanford-Maldacena, …

✔ Hamiltonian dynamics, Lyapunov exponents;
Some connections to energy transport.

✗ Non generic features: Large-N / non locality

This talk

- **Preliminaries: operator dynamics, recursion methods**

- **A hypothesis for universal operator growth**

- **Evidence for the hypothesis:**
  (i) Numerical (Spin chains)
  (ii) Analytical (SYK models)
  (iii) Physical arguments (generalized RMT for extended system)

- **Application: generalized notion of quantum chaos**

- **Application: an accurate computational approach**
  Transport coefficients in strongly coupled systems

Operator dynamics: basics

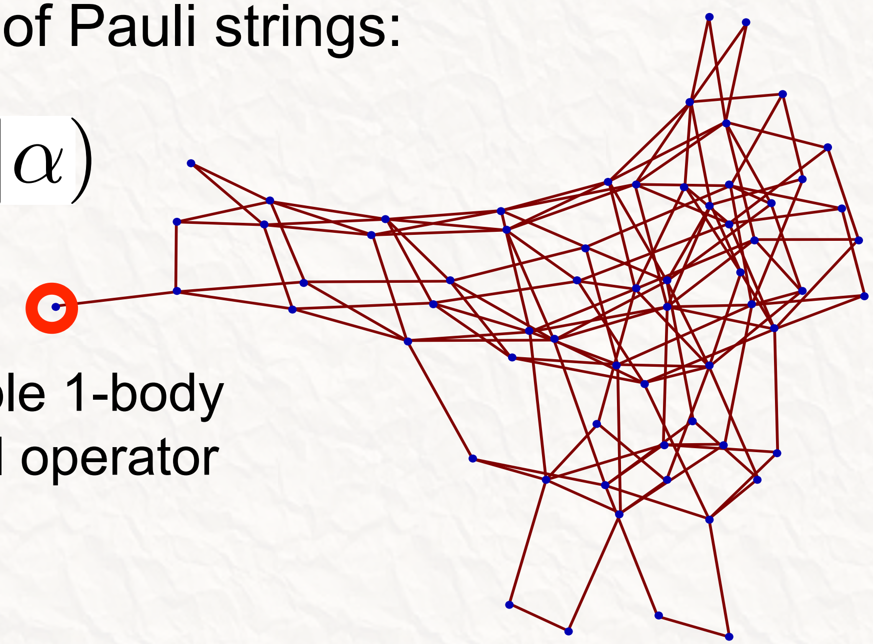
$$-i\frac{d\hat{A}}{dt} = [H, \hat{A}] \quad \Leftrightarrow \quad -i\frac{d|A)}{dt} = \mathcal{L}|A)$$

$$(A|B) = \mathrm{tr}(AB) \qquad |O(t)) = e^{-i\mathcal{L}t}|O)$$

For spin-1/2 problems use basis of Pauli strings:

$$\sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \ldots \otimes \sigma^{\alpha_N} \equiv |\alpha)$$

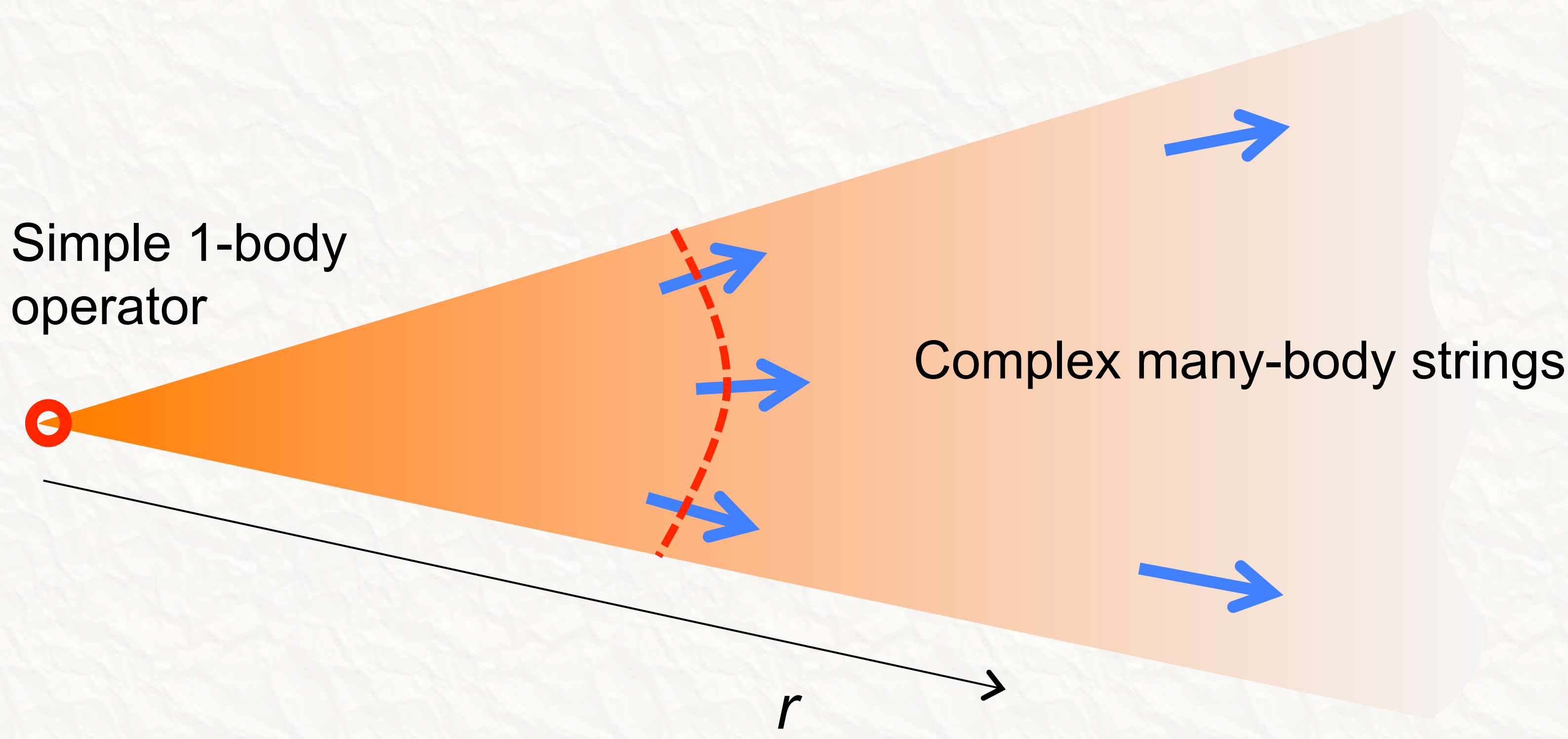
$$\alpha_i = 0, 1, 2, 3$$



Simple 1-body initial operator

$\mathcal{L}$ can be viewed as a Hamiltonian of a single particle hopping between Pauli strings on this graph. Due to hermiticity: No diagonal terms.

The basic idea



Operators flow from simple to more complex

When an operator becomes sufficiently complex its dynamics should be governed by a universal statistical description.
Our goal now is to formulate this universal description

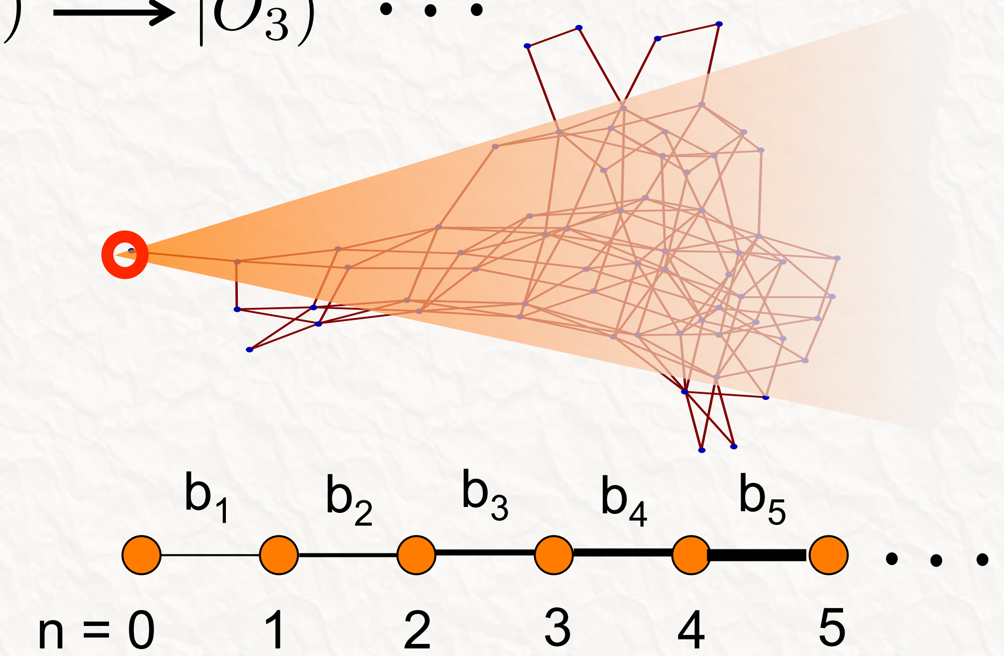
Krylov basis: folding the graph on a line

Generate orthonormal basis from successive application of $\mathcal{L}$

$$|O) \xrightarrow{\mathcal{L}} |O_1) \xrightarrow{\mathcal{L}} |O_2) \xrightarrow{\mathcal{L}} |O_3) \cdots$$

$$(\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

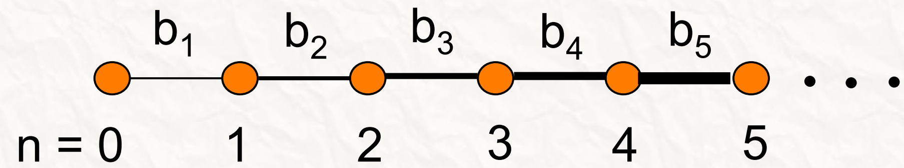
"***Lanczos Coefficients***"



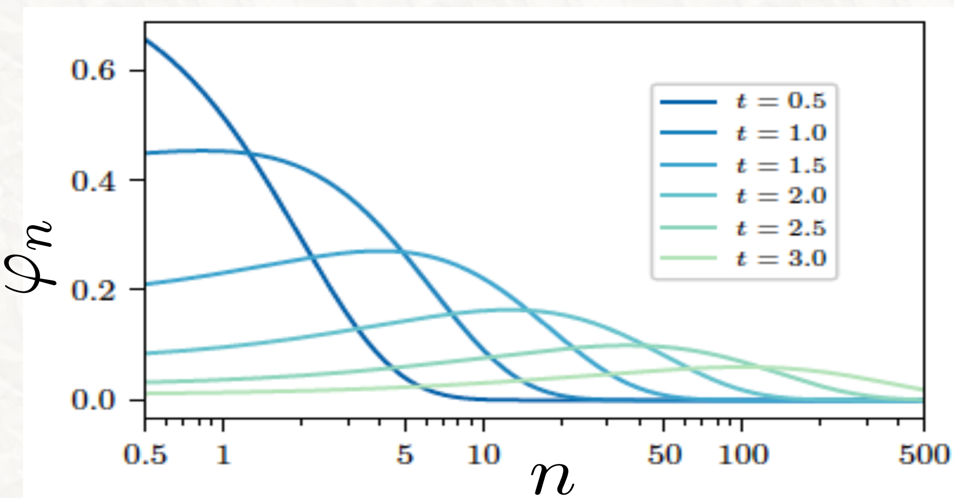
- Problem mapped to single-particle hopping on a semi-infinite chain !
- Krylov index ~ operator complexity

"Operator wavefunction" in Krylov space

$$\varphi_n(t) = (\mathcal{O}_n|\mathcal{O}(t))$$



$$\partial_t \varphi_n = -b_{n+1}\varphi_{n+1} + b_n \varphi_{n-1}, \qquad \varphi_n(0) = \delta_{n0}$$



The autocorrelation function:

$$C(t) = \mathrm{tr}\left[\mathcal{O}(t)\mathcal{O}\right] = \varphi_0(t)$$

Operator complexity:

$$\langle n(t) \rangle = \sum_{n=0}^{\infty} |\varphi_n(t)|^2 n$$

The hypothesis

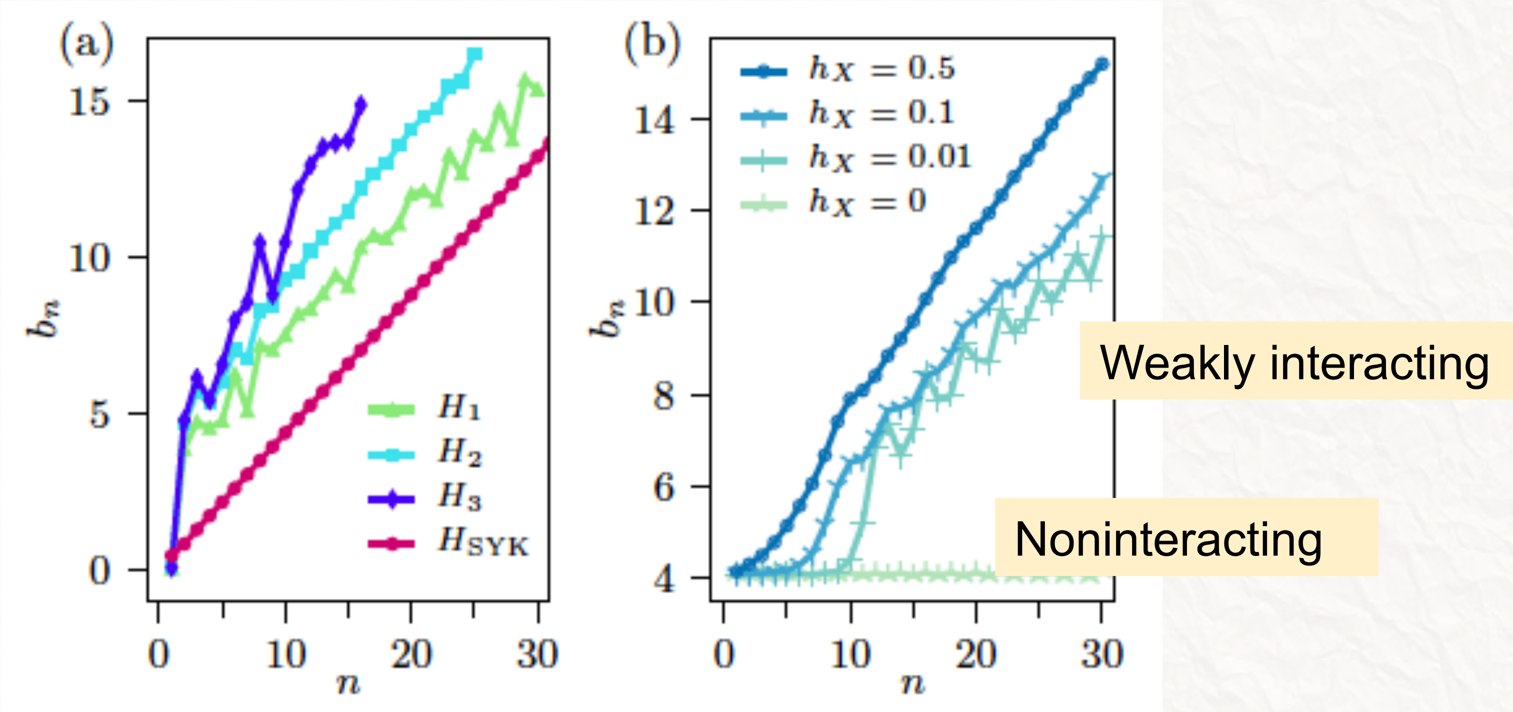
In an infinite non-integrable many-body system the Lanczos coefficients of a generic local operator are asymptotically linear:

$$b_n = \alpha n + \beta + o(1), \quad n \to \infty$$

We term the slope α, the “growth rate” of the operator for reasons that will become clear.

The evidence

<u>Numerical:</u> Many distinct nonintegrable spin chains, SYK model



<u>Analytical:</u> SYK model in the limit of large q

$$b_n \to \sqrt{q\, n(n-1)/2} \qquad n \geq 2$$

Physical origin of linear Lanczos coefficients in models with short range couplings

Use relation between b_n and moments:

$$b_n \sim n \quad \Longleftrightarrow \quad \mu_{2n} \equiv (\mathcal{O}|\mathcal{L}^{2n}|\mathcal{O}) \sim n^{2n}$$

H is local then $\mathcal{O}_n$ is at most of size n. To compute μ_{2n} we can use $H_{(n)}$ (H restricted to a subsystem that covers the support of $\mathcal{O}_n$)

$$\mathcal{L}^n_{(n)}|\mathcal{O}) = \sum_{\alpha,\beta}(E_\alpha - E_\beta)^n|\mathcal{O}_{\alpha\beta})(\mathcal{O}_{\alpha\beta}|\mathcal{O}) \qquad \mathcal{O}_{\alpha\beta} = D^{-1/2}|E_\alpha\rangle\langle E_\beta|$$

$$(\mathcal{O}_{\alpha\beta}|O) = D^{-\frac{1}{2}}\langle E_\beta|\mathcal{O}|E\alpha\rangle = f(E_\alpha - E_\beta)D^{-1}R_{\alpha\beta}$$ (ETH)

At frequency above local bandwidth we assume: $f(\omega) \sim e^{-|\omega|/\omega_0}$

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At frequency above local bandwidth we assume: $f(\omega) \sim e^{-|\omega|/2\omega_0}$

$$\mu_{2n} = (\mathcal{O}|\mathcal{L}^{2n}_{(n)}|\mathcal{O}) = \sum_{\alpha,\beta=1}^{D} f(E,\omega)^2 D^{-2}(E_\beta - E_\alpha)^{2n}$$

$$\sim \int_{\epsilon,\epsilon'} e^{ns(\epsilon)+ns(\epsilon')} e^{-n\frac{|\epsilon-\epsilon'|}{\omega_0}} \underbrace{n^{2n}(\epsilon-\epsilon')^{2n}}_{E=n\epsilon} \sim \boxed{n^{2n}e^{-S_{sp}n}}$$

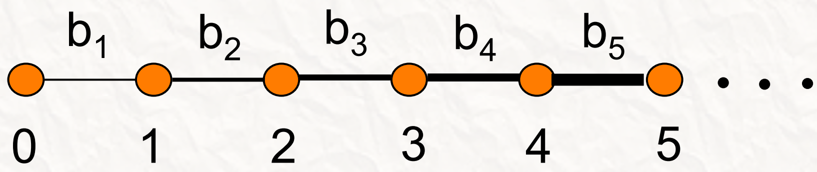
More precise relation to spectral function

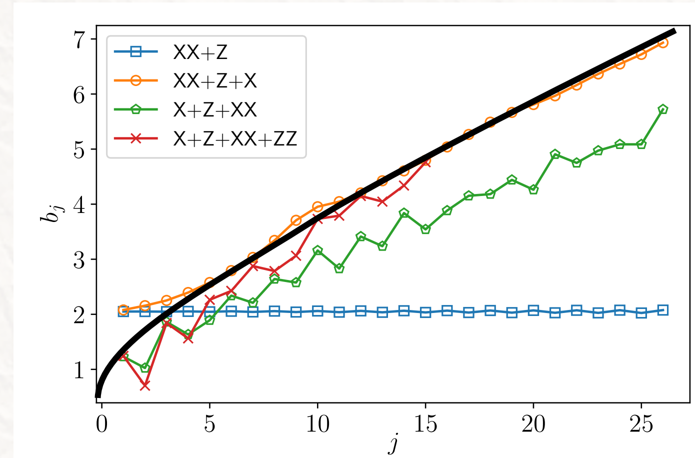
$$\Phi(\omega) = \int_{-\infty}^{\infty} dt C(t) e^{-i\omega t} = \int_{-\infty}^{\infty} dt \, \mathrm{tr}\left[\mathcal{O}(t)\mathcal{O}\right] e^{-i\omega t}$$

$$b_n = \alpha \, n + O(1) \Longleftrightarrow \Phi(\omega) \sim e^{-\pi \frac{|\omega|}{2\alpha}}$$

The operator "growth rate" α, is directly related to the decay of the spectral function

Phenomenology of the semi infinite chain

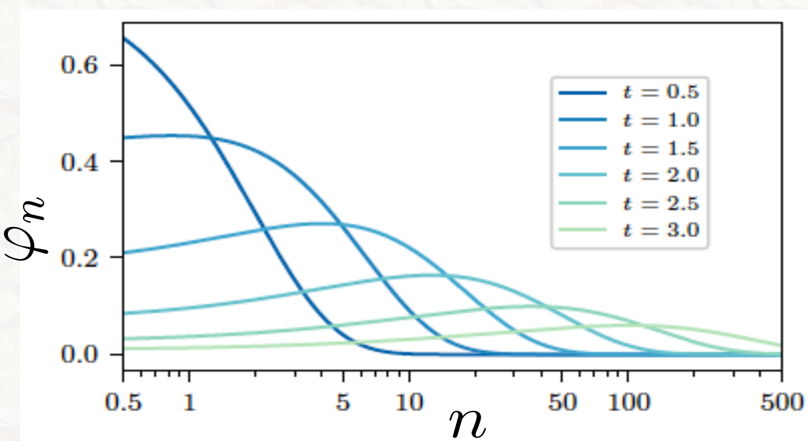




Exactly solvable "universal" model:

$$\tilde{b}_n = \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \beta$$

$$\langle n(t)\rangle = \eta \sinh(\alpha t)^2 \sim \eta e^{2\alpha t}$$



Exponential growth of complexity.

What is the relation to chaos?

$$C(t) = \varphi_0(t) \sim e^{-2\alpha t}$$

Can we utilize this to compute dynamical correlations, transport in real models?

Relation to chaos I : SYK model

Compare growth rate α to Lyapunov exp. λ_L in the SYK model

1. Infinite temperature:

$$C(t) := \frac{1}{2}([O(t), A] \,|\, [O(t), A]) \sim e^{2\lambda_L t}$$

q	2	3	4	7	10	∞
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/\mathcal{J}$	0	0.454	0.620	0.799	0.863	1

Relation to chaos I : SYK model

Compare growth rate α to Lyapunov exp. λ_L in the SYK model

2. Low temperature limit

Modify inner product: $$(A|B) := \mathrm{tr}[\rho A^\dagger B(i\beta/2)]$$

➡ $$C(t) \sim \mathrm{tr}[\rho\gamma_1\gamma_1(i\beta/2+t)] \sim \mathrm{sech}\,(t\pi T)^{2/q}$$

➡ $$b_n = \pi T\sqrt{n(n-1+2/q)}$$

➡ $$\boxed{\alpha = \pi T = \lambda_L}$$

Relation to chaos II: classical limit

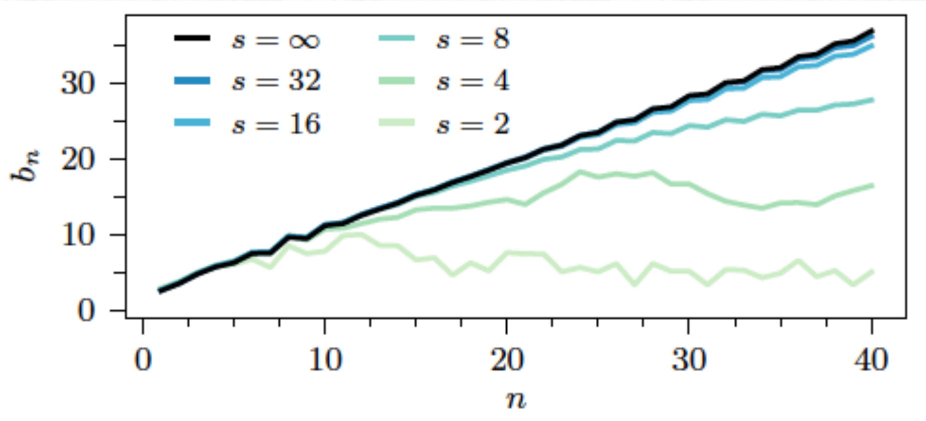
The framework carries over for classical dynamics with:

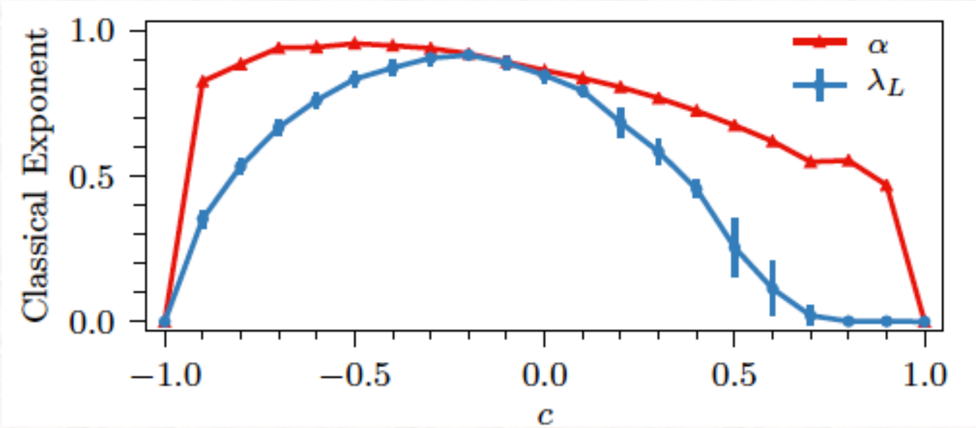
Liouvillian ⟶ $\mathcal{L} = i\{\mathcal{H}, \cdot\}$

Operators ⟶ Functions on the classical phase space

Compare α to λ_L Peres- Feingold model:

$$H_{\mathrm{FP}} = (1+c)\left[S_1^z + S_2^z\right] + 4s^{-1}(1-c)S_1^x S_2^x$$





The two exponents coincide where the model is most chaotic. Otherwise α appears to be an upper bound on λ_L

Relation to chaos: summary

- We conjecture that $\alpha \geq \lambda_L$ and that the two exponents coincide in maximally chaotic systems.

- The complexity growth rate α gives a measure of chaos even in systems where the Lyapunov exponent is not well defined. e.g. generic, non-semiclassical systems.

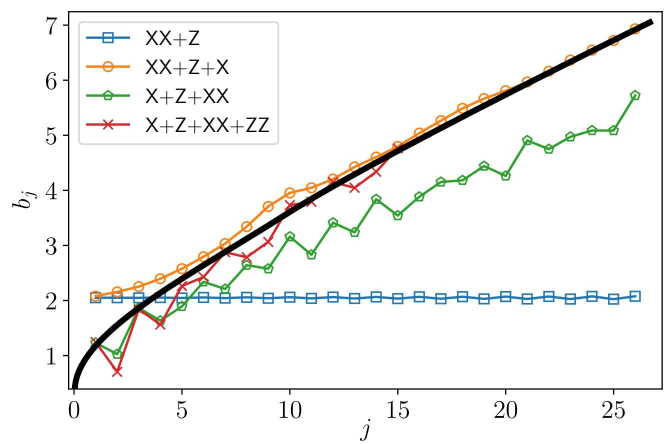
- α is measureable with a standard local probe through the high frequency limit of the spectral function.

Application: computing operator decay

The basic idea:

1. Compute the first m Lanczos coefficient numerically.

2. Complete with the fitted “universal” model at larger values of m



$$\widetilde{b}_n = \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \beta$$

3. Stich the small n and large m wavefunctions to get an approximation of the decay of C(t).
   In practice we utilize it to get an approximation for:

$$G(z) = i\int_0^\infty C(t)e^{-izt}dt = \langle\mathcal{O}|\frac{1}{z-\mathcal{L}}|\mathcal{O}\rangle$$

Meromorphic approximation for the Green's function

Continued fraction expansion:

$$G(z) = \cfrac{1}{z - \cfrac{|b_1|^2}{z - \cfrac{|b_2|^2}{z - |b_3|^2 G^{(3)}(z)}}} = M_1 \circ M_2 \circ \ldots M_n G^{(n)}(z)$$

Suppose we can obtain $b_1,\ldots,b_n$ numerically, from which we can already extract the parameters of the universal model.

Then we can substitute $G^{(n)}(z) \to \tilde{G}^{(n)}(z)$

$$G(z) \approx M_1 \circ M_2 \circ \ldots M_n \tilde{G}^{(n)}(z;\alpha,\eta)$$

$$\approx M_1 \circ \ldots M_n \tilde{M}_n^{-1} \circ \ldots \tilde{M}_n^{-1} \tilde{G}(z;\alpha,\eta)$$

Example: computing diffusion coefficient

<u>Model:</u>

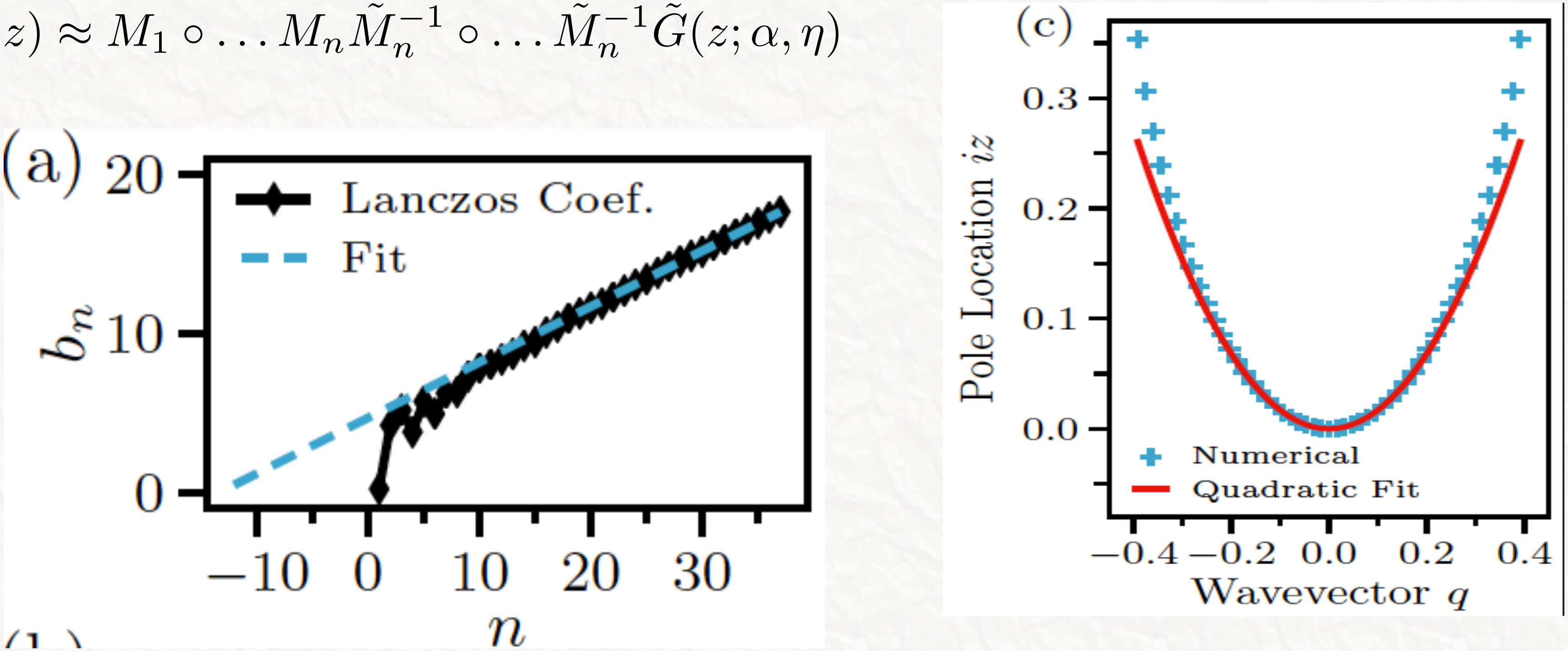
$$H = \sum_i h_i = \sum_i X_i X_{i+1} - 1.05 Z_i + X_i$$

<u>Operator of interest:</u>

$$H_q = \sum_i e^{iqx_i} h_i$$

1. Obtain first 35 Lanczos coefficients numerically to fit α and η
2. Find the smallest imaginary pole of the approximate Green's function:

$$G_q(z) \approx M_1 \circ \ldots M_n \tilde{M}_n^{-1} \circ \ldots \tilde{M}_n^{-1} \tilde{G}(z; \alpha, \eta)$$



Summary

- Hypothesis for universal operator dynamics supported by extensive evidence. Linear growth of Lanczos coefficients

$$b_n = \alpha n + \beta + o(1), \quad n \to \infty$$

- Implies exponential growth in operator complexity with a exponent α

- The complexity growth offers a generalized notion of chaos even where the Lyapunov exponent is ill defined. Conjecture: $\alpha \geq \lambda_L$, coincide for maximally chaotic.

- The hypothesis enables a new numerical scheme to compute dynamical correlations and transport coefficients.

Outlook

- Rigorous proofs of the hypothesis and the conjecture concerning quantum chaos?
  Perhaps within a generalized random matrix description for infinite systems with local interactions?

- Generalization to finite temperature

- Develop computational scheme for strongly correlated models at finite T. Use QMC to compute moments ?