On the relation between the magnitude and exponent of OTOCs

(Based on the paper Yingfei Gu, AK [1812.00120])

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Outline

- Out-of-time-order correlators (OTOC)s:
  - Large N systems with all-to-all interactions
  - Single-mode ansatz for early-time OTOCs
- SYK model
- Kinetic equation for early-time OTOCs
- *Ladder identity* and *branching time*
- Some applications:
  - Near-maximal chaos at $\beta J \gg 1$
  - Maximal chaos in the butterfly wavefront
- Commutator OTOC and stringy states

Naturally ordered (Keldysh) correlators

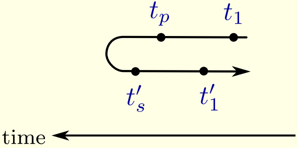
- Consider an abstract quantum experiment setup:
  - The initial state is $\rho = \rho_{\text{system}} \otimes \rho_{\text{probe}}$
  - The system and the probe interact and evolve *forward in time*:

$$H = H_{\text{system}} + H_{\text{probe}} - \sum_j \underset{\uparrow \text{ system}}{X_j} \underset{\nwarrow \text{ probe}}{Y_j}, \qquad U = \underset{\uparrow \text{ time ordering}}{\mathbf{T}} \exp\left(-i \int H(t)dt\right)$$

  - A measurement is performed

- The probability of a particular outcome, $P = \operatorname{Tr}\bigl(U^\dagger \Pi U \rho\bigr)$ expands into terms like this:

$$\boxed{\begin{gathered}\bigl\langle X_{j_1}(t'_1)^\dagger \cdots X_{j_s}(t'_s)^\dagger X_{k_p}(t_p) \cdots X_{k_1}(t_1)\bigr\rangle \\ t'_1 < \cdots < t'_s, \quad t_p > \cdots > t_1\end{gathered}}$$



where $\langle X\rangle := \operatorname{Tr}(X\rho_{\text{system}})$

Butterfly effect

- **"A flap of a butterfly's wing can change the weather."**

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  Is there a *direct* way to test this theory?

  Test 1: Run the time backward, introduce a butterfly, and run the time forward. Check if the weather is different.

  Test 2: Have two copies of the Earth, with and without the butterfly (but otherwise in the same state).

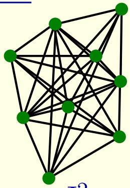
- Both tests are well-defined in the quantum setting. (Test 2 should be done on the thermofield double.)

- The butterfly effect can be characterized by out-of-time-order correlators like $\langle D(t)C(0)B(t)A(0)\rangle$, where A, B, C, D are some quantum observables.

Large N systems with all-to-all interactions

– Random Heisenberg model (N spins)

$$H = -\sum_{j<k}\sum_{\alpha} J_{jk} S_j^{\alpha} S_k^{\alpha}, \qquad \overline{J_{jk}^2} = \frac{J^2}{N}$$

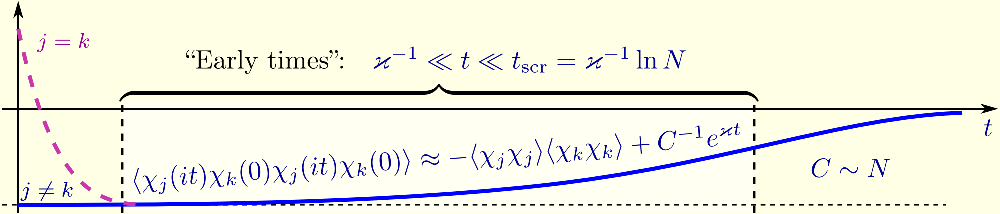


Random parameters

– SYK model (N Majorana modes)

$$H = -\frac{1}{4!}\sum_{j,k,l,m} J_{jklm}\,\chi_j\,\chi_k\,\chi_l\,\chi_m, \qquad \overline{J_{jklm}^2} = 3!\,\frac{J^2}{N^3}$$

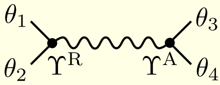
- OTOC as a function of time



- Chaos bound: $\varkappa \leqslant \frac{2\pi}{\beta}$ (Maldacena, Shenker, Stanford 2015)

Single-mode anzats for early-time OTOCs

$$\left\langle \chi_j(\theta_1)\chi_k(\theta_3)\chi_j(\theta_2)\chi_k(\theta_1)\right\rangle + \langle\chi_j\chi_j\rangle\langle\chi_k\chi_k\rangle$$
$$\approx C^{-1} e^{i\varkappa(\pi-\theta_1-\theta_2+\theta_3+\theta_4)/2}\, \Upsilon^{\mathrm{R}}(\theta_1-\theta_2)\, \Upsilon^{\mathrm{A}}(\theta_3-\theta_4)$$



- We use the complex time variable $\theta = it + \tau$ $\quad (0 < \tau < \beta)$
- For convenience, $\beta = 2\pi \quad \Rightarrow\ 0 \leqslant \varkappa \leqslant 1$

(Kitaev, Suh [arxiv:1711.08467])

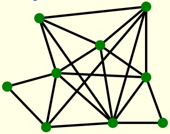
- Corollary (for $\mathrm{Re}\,\theta_1 = \mathrm{Re}\,\theta_3,\ \mathrm{Re}\,\theta_2 = \mathrm{Re}\,\theta_4$):

$$\left\langle \left\{\chi_j(\theta_1), \chi_k(\theta_3)\right\}\left\{\chi_j(\theta_1), \chi_k(\theta_3)\right\}\right\rangle$$
$$\approx \frac{2\cos(\varkappa\pi/2)}{C}\, e^{-i\varkappa(\theta_1+\theta_2-\theta_3-\theta_4)/2}\, \Upsilon^{\mathrm{R}}(\theta_1-\theta_2)\, \Upsilon^{\mathrm{A}}(\theta_3-\theta_4)$$

The SYK model

N Majorana operators χ_j

$$\dim \mathcal{H} = 2^{N/2}$$



antisymmetric

$$H = -\frac{1}{4!} \sum_{j,k,l,m} J_{jklm}\, \chi_j\, \chi_k\, \chi_l\, \chi_m$$

$$\overline{J_{jklm}} = 0, \qquad \overline{J^2_{jklm}} = 3! \frac{J^2}{N^3}$$

Operator algebra $= \mathrm{Cliff}(N)$:

$$\chi_j \chi_k + \chi_k \chi_j = \delta_{jk}$$

Hilbert space $\mathcal{H}$ is described by a Fock basis: built from $|0, \ldots, 0\rangle$ by the raising operators

$$a_n^\dagger = \frac{\chi_{2n-1} - i\chi_{2n}}{\sqrt{2}}, \quad n = 1, \ldots, \tfrac{N}{2}$$

– This is the $q = 4$ variant of the model.

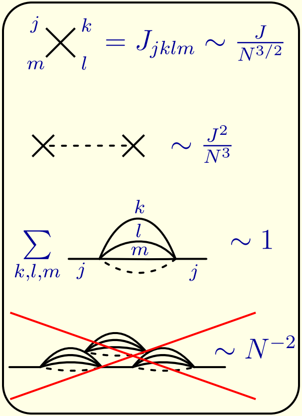
– More generally, one can consider interactions of order $q = 2, 4, 6, \ldots$, though the $q = 2$ case is degenerate.

(Sachdev, Ye (1992), Maldacena, Stanford (2016), Kitaev, Suh (2017))

SYK model: the Green function

Definition of the imaginary-time Green function:

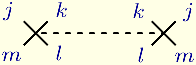
$$G(\tau_1, \tau_2) = -\langle \mathbf{T}\, \chi_j(\tau_1)\chi_j(\tau_2) \rangle$$



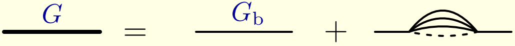
Bare Green function (for $H = 0$):

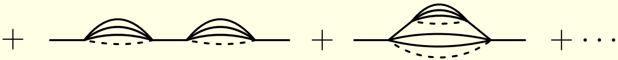
$$\hat{G}_\mathrm{b} = (-\partial_\tau)^{-1}, \quad G_\mathrm{b}(\tau_1, \tau_2) = -\tfrac{1}{2}\,\mathrm{sgn}(\tau_1 - \tau_2)$$

Disorder-averaged interaction:



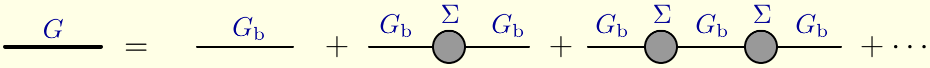
Taylor expansion in the interaction strength βJ:





SYK model: the Schwinger-Dyson equations

- General structure of the Green function:



- Schwinger-Dyson equations: $(-\partial_\tau - \hat{\Sigma})\hat{G} = 1$,

i.e. $(\Sigma G)(\tau_1, \tau_2) = \int d\tau\, \Sigma(\tau_1, \tau) G(\tau, \tau_2)$,

$$\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^{q-1}$$

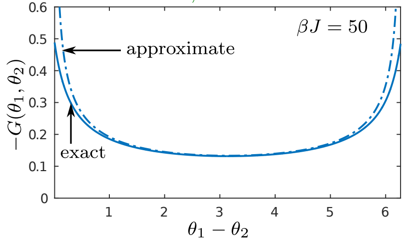
- Solution at $J \gg 1$: neglecting the ∂_τ term:

$$G(\theta_1, \theta_2) \approx -b^{\Delta} (J\theta_{12})^{-2\Delta}\, \mathrm{sgn}\, \theta_{12}$$

where $\boxed{\Delta = 1/q}$, $\quad \theta_{12} = 2 \sin \frac{\theta_1 - \theta_2}{2}$

(Parcollet, Georges 1998)

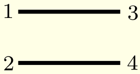
Maldacena, Stanford 2016

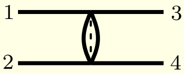


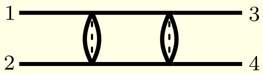
Connected four-point function $\mathcal{F}$

$$\left\langle \mathbf{T}\, \chi_j(\theta_1)\chi_j(\theta_2)\chi_k(\theta_3)\chi_k(\theta_4)\right\rangle = G(\theta_1,\theta_2)G(\theta_3,\theta_4) + \frac{1}{N}\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4)$$

- Diagrammatic expansion (up to subleading $1/N$ terms)

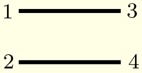
$\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4) = -$ 

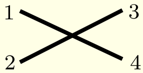
$-$ 

$-$ 

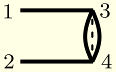
$- \ldots \quad + (3 \leftrightarrow 4)$

- Bethe-Salpeter equation: $\mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$

$\mathcal{F}_0(\theta_1,\theta_2,\theta_3,\theta_4) = -$ 

$+$ 

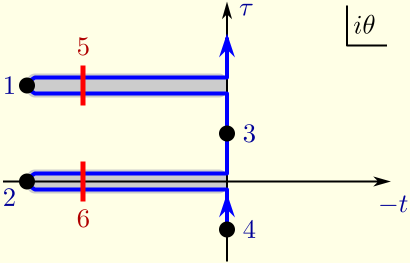
$\theta_{ab} := \theta_a - \theta_b$

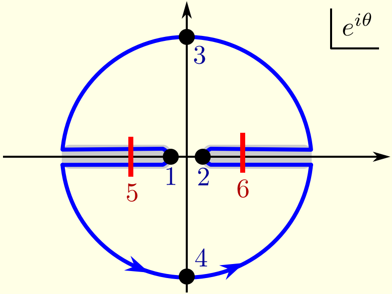
$K(\theta_1,\theta_2,\theta_3,\theta_4) = -$ 

$= -J^2(q-1)G(\theta_{13})G(\theta_{24})G(\theta_{34})^{q-2}$

Connected OTOC

$$\mathrm{OTOC}(t_1,t_2,t_3,t_4) = -N^{-1}\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4)$$
$$= \big\langle \chi_j(\theta_1)\chi_k(\theta_3)\chi_j(\theta_2)\chi_k(\theta_4)\big\rangle + \langle\chi_j\chi_j\rangle\langle\chi_k\chi_k\rangle$$
$$\theta_1 = it_1+\pi, \qquad \theta_2 = it_2, \qquad \theta_3 = it_3+\tfrac{\pi}{2}, \qquad \theta_4 = it_4-\tfrac{\pi}{2}$$



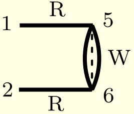


$$\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4) \approx \int_{\text{folds}} d\theta_5\, d\theta_6\, K(\theta_1,\theta_2,\theta_5,\theta_6)\, \mathcal{F}(\theta_5,\theta_6,\theta_3,\theta_4)$$

Kinetic equation and retarded kernel

- Let $F(t_1, t_2) = \mathrm{OTOC}(t_1, t_2, t_3, t_4)$. Then

$$F(t_1, t_2) = \int_{\mathbb{R}} dt_5\, dt_6\, K^{\mathrm{R}}(t_1, t_2, t_5, t_6)\, F(t_5, t_6)$$

$K^{\mathrm{R}}(t_1, t_2, t_5, t_6) = -$ 

$= -J^2(q-1) G^{\mathrm{R}}(t_{15}) G^{\mathrm{R}}(t_{26}) G^{\mathrm{W}}(t_{56})^{q-2}$

- Eigenfunctions: $\widetilde{\Upsilon}_\alpha(t_1, t_2) = e^{-\alpha(t_1+t_2)/2} \Upsilon(t_1 - t_2)$

$$K^{\mathrm{R}} \widetilde{\Upsilon}_\alpha = k_{\mathrm{R}}(\alpha) \widetilde{\Upsilon}_\alpha \quad \Leftrightarrow \quad K^{\mathrm{R}}_\alpha \Upsilon_\alpha = k_{\mathrm{R}}(\alpha) \Upsilon_\alpha \,,$$

where $K^{\mathrm{R}}_\alpha(t, t') = \int K^{\mathrm{R}}\bigl(s + \frac{t}{2}, s - \frac{t}{2}, \frac{t'}{2}, -\frac{t}{2}\bigr)\, e^{\alpha s}\, ds$

- Solving for the Lyapunov exponent: $k_{\mathrm{R}}(-\varkappa) = 1$

Example: SYK model in the conformal limit

- The model is maximally chaotic: $\varkappa \approx 1$.
- The eigenfunctions $\widetilde{\Upsilon}^{\mathrm{R}}_{-\varkappa}$ and $\widetilde{\Upsilon}^{\mathrm{A}}_{-\varkappa}$ are generated by the action of

$$L_{-1} = e^{t}(\partial_t + \Delta), \qquad L_1 = e^{-t}(\partial_t - \Delta)$$

on the first variable of the Wightman function

$$G^{\mathrm{W}}(t_1, t_2) = G(it_1 + \pi, it_2)\,.$$

$$G^{\mathrm{W}}(t_1, t_2) = -\frac{b^{\Delta}}{\left(2J\cosh\frac{t_{12}}{2}\right)^{2\Delta}} \quad \Rightarrow \quad \Upsilon^{\mathrm{R}}(t) = \Upsilon^{\mathrm{A}}(t) = -\frac{2\Delta b^{\Delta} J^{-2\Delta}}{\left(2\cosh\frac{t_{12}}{2}\right)^{2\Delta+1}}$$

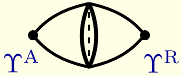
$$\mathrm{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{e^{(t_1+t_2-t_3-t_4)/2}}{C}\Upsilon^{\mathrm{R}}(t_{12})\Upsilon^{\mathrm{A}}(t_{34}), \qquad C = \frac{2\alpha_S N}{J}$$

(C is obtained from the Schwarzian theory)

Main results

- Ladder identity:

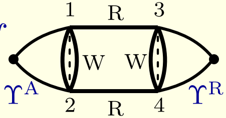
$$\boxed{N\frac{2\cos\frac{\varkappa\pi}{2}}{C}\,k_{\mathrm{R}}'(-\varkappa)\,(\Upsilon^{\mathrm{A}},\Upsilon^{\mathrm{R}})=1} \qquad (\Upsilon^{\mathrm{A}},\Upsilon^{\mathrm{R}}) =$$



  – Allows for the calculation of C from the retarded kernel;
  – Conversely, in the case of near-maximal chaos, one can calculate $\delta\varkappa = 1-\varkappa$ using $k_{\mathrm{R}}'(-1)$ from the conformal limit and C from the Schwarzian theory.

- *Branching time* $t_B = k_{\mathrm{R}}'(-\varkappa)$ is the average time separation s between adjacent rungs in a ladder diagram contributing to the OTOC:

$$t_B = \frac{1}{(\Upsilon^{\mathrm{A}},\Upsilon^{\mathrm{R}})}\int$$

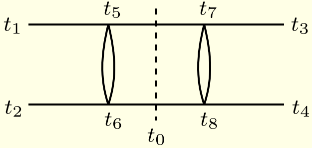


$$dt_{12}\,dt_{34}\,s\,ds, \qquad s=\frac{t_1+t_2}{2}-\frac{t_3+t_4}{2}$$

Derivation sketch

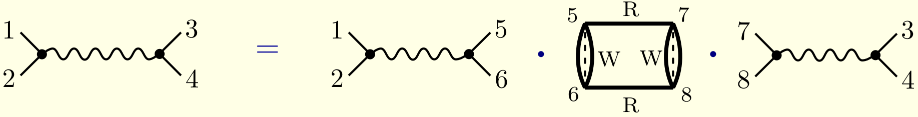
- Idea: cut a long ladder in half; find a consistency condition.

– Cuting the ladder: Fix t_0; find adjacent rungs such that



$$\frac{t_5 + t_6}{2} < t_0 < \frac{t_7 + t_8}{2}$$

– Consistency condition:



– The factor $2\cos\frac{\varkappa\pi}{2} = e^{i\varkappa\pi/2} + e^{-i\varkappa\pi/2}$ arises because there are two different ways to put θ_5, θ_6 on the double Keldysh contour.

Near-maximal chaos $(J \to \infty, \quad \varkappa \to 1)$

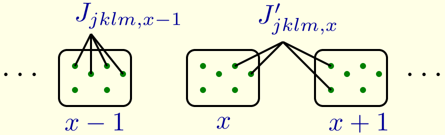
- The prefactor $r = \dfrac{2\cos(\varkappa\pi/2)}{C}$ in the commutator OTOC has a finite limit:

$$r = \Big(k_{\mathrm{R}}'(-1)\,\big(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{A}}\big)\Big)^{-1} N^{-1}$$

- The correction to the Lyapunov exponent is

$$1 - \varkappa \approx \frac{rC}{\pi} = \frac{2\alpha_S}{\pi k_{\mathrm{R}}'(-1)\,\big(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{A}}\big)}\, J^{-1}$$

Application to a 1D model



(Gu, Qi, Stanford 2016)

- $\mathrm{OTOC}_{x,0}(t_1,t_2,t_3,t_4) := \big\langle \chi_{j,x}(\theta_1)\chi_{k,0}(\theta_3)\chi_{j,x}(\theta_2)\chi_{k,0}(\theta_4)\big\rangle + \langle\cdots\rangle\langle\cdots\rangle$

- Fourier transform: $\displaystyle\int \frac{dp}{2\pi} e^{ipx} \underbrace{\mathrm{OTOC}_p(t_1,t_2,t_3,t_4)}_{\propto C(p)^{-1} e^{\varkappa(p)t}}$ $\quad t = \frac{t_1+t_2}{2} - \frac{t_3+t_4}{2}$

  $\varkappa(p) \approx \varkappa(0) - t_B a p^2$ is equal to 1 at some $p_1 = i|p_1|$,

  hence $C(p)^{-1} = \big(N\cdot 2\cos\frac{\varkappa(p)\pi}{2}\cdot t_B \cdot (\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}})\big)^{-1}$ has a pole.

- Result: The Lyapunov exponent *in the butterfly wavefront* is exactly 1 is J is above threshold.

Summary

- The ladder identity is very useful for calculating OTOCs.
- The inverse branching time t_B^{-1} characterizes the strength of "stringy" effects.
  - Challenge: construct a model with $t_B \gg 1$