The Quantum Mechanics of JT gravity

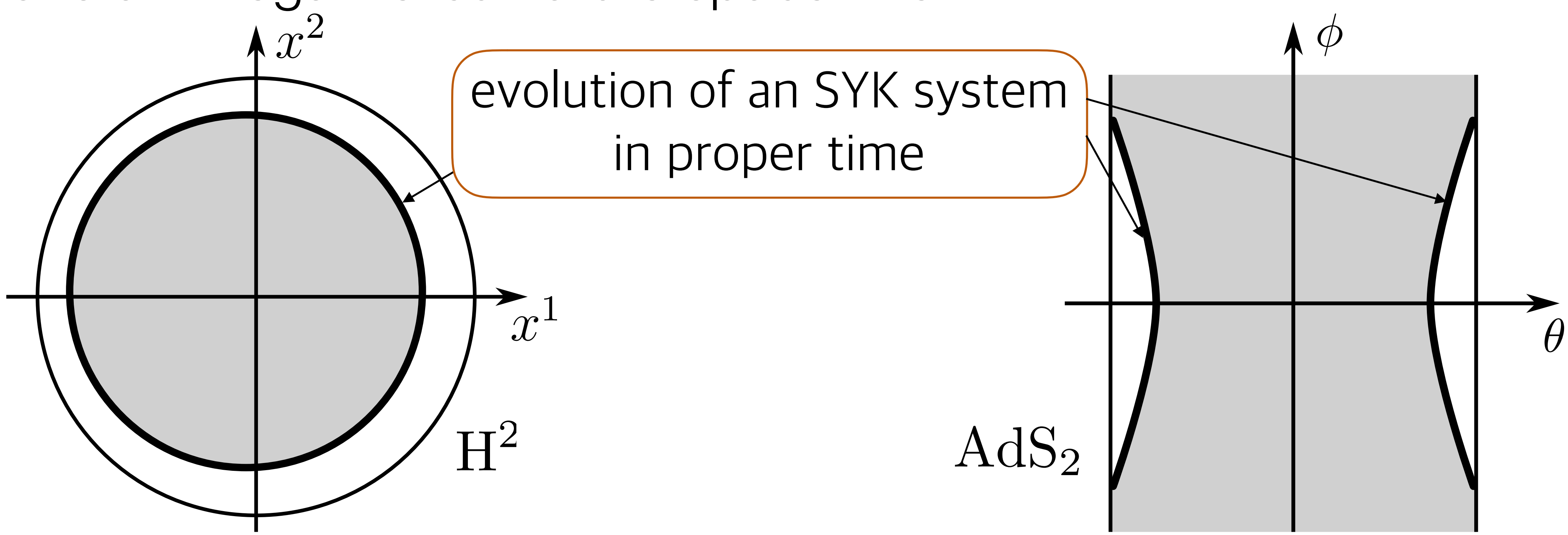
Josephine Suh

arXiv:1808.07032 + work in progress, w/ A. Kitaev

Order from Chaos conference, KITP
Dec. 11, 2018

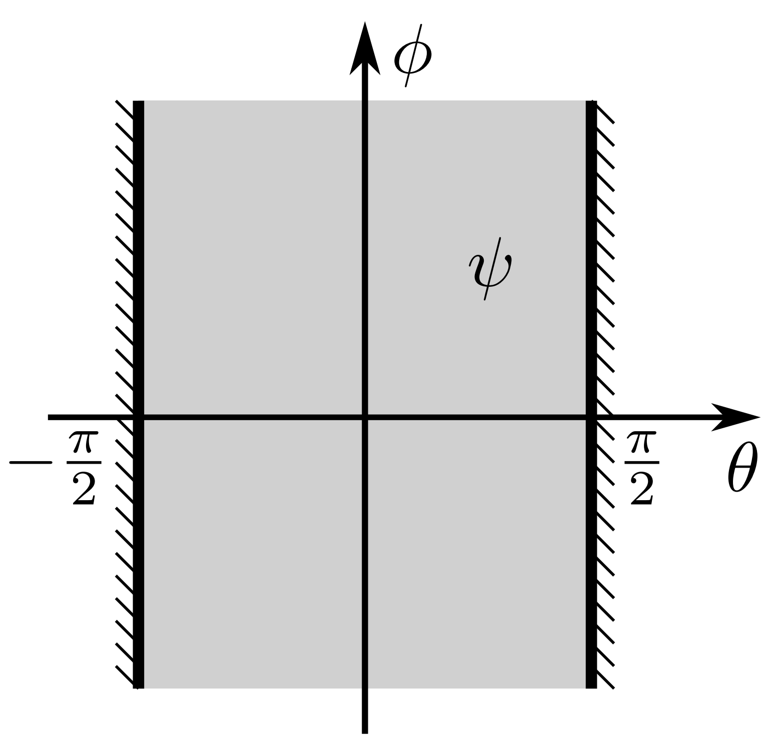
Goal

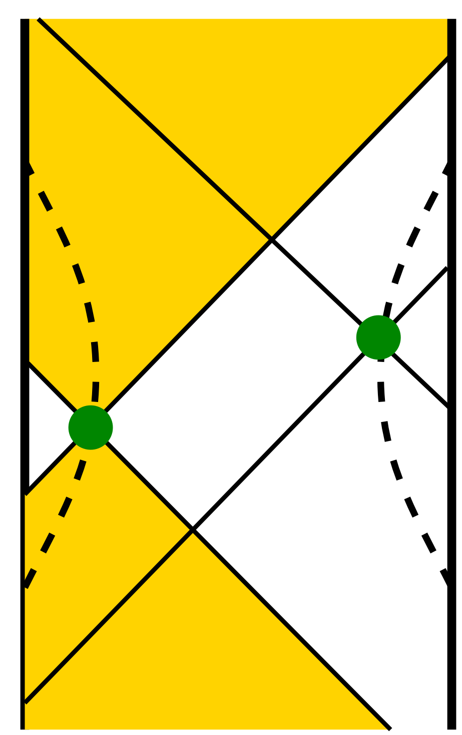
- Map the low-temperature dynamics of the SYK model to a consistent quantum mechanical theory defined on the emergent two-dimensional spacetime.
  - The relevant two-dimensional theory is a simple dilaton gravity, Jackiw-Teitelboim gravity, which is topological. [Jensen; Maldacena, Stanford, Yang; Engelsoy, Mertens, Verlinde]
  - Quantize the motion of boundaries of a rigid, two-dimensional, constant-negative-curvature spacetime.



Overview

- Quantum theory of JT gravity
  - Hilbert space (for a single boundary) consists of wavefunctions living on the universal cover of Lorentzian AdS_2.
  - We define an inner product and a regularized trace on this Hilbert space.
  - We can construct two-sided wavefunctions for an eternal black hole with fixed energy and exact causal disconnection between boundaries.

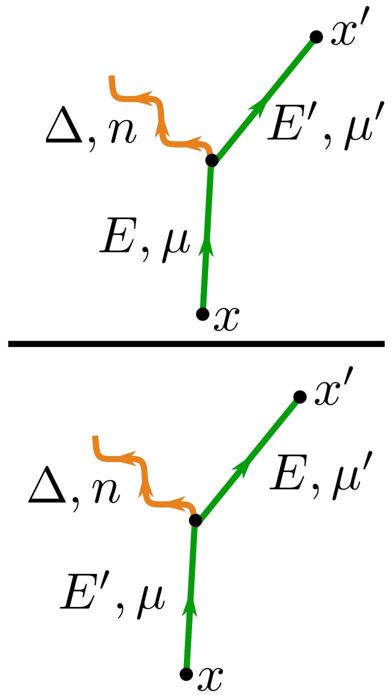




Overview (continued)

- Using the wavefunctions and trace, we can build density matrices and compute observables for an AdS_2 black hole, using a slight generalization of the rules used in discrete quantum systems.

- Wavefunctions in our Hilbert space seem to reproduce properties of the microscopic SYK model in a precise manner:
  - analytic continuation relations satisfied by observables (correlation functions of external operators on boundary)
  - density of states

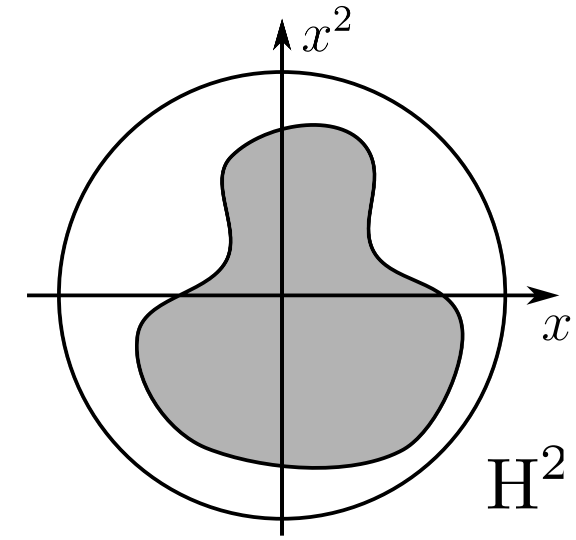


$$\sim \frac{\rho(E')}{\rho(E)}$$

JT gravity: particle with imaginary spin

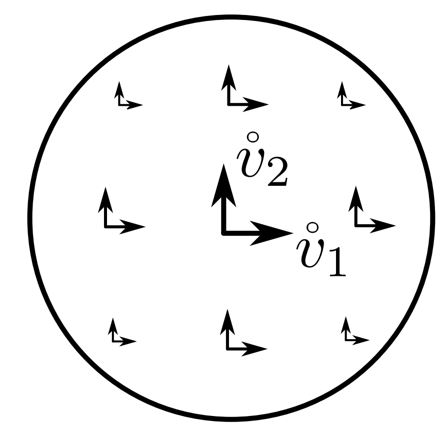
$$I_{\rm JT}[g,\Phi] = -\frac{1}{4\pi}\int_D d^2x\sqrt{g}\,\Phi(R+2) - \frac{\Phi_*}{2\pi}\int_{\partial D} dl\, K$$

$$\to I_{\rm pt}[X] = -\gamma(2\pi + {\rm area}[X]), \qquad \gamma = \frac{\Phi_*}{2\pi},\ \ L = {\rm length}[X]$$

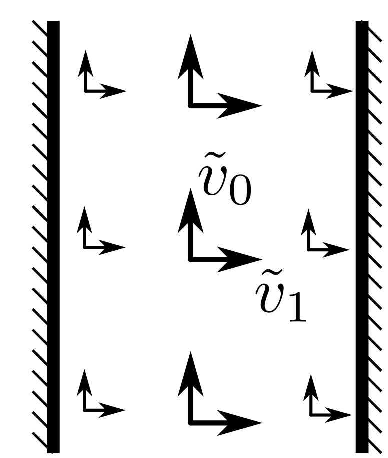


Spin connection: analogue of gauge field for group of rotations at each point of space(time)

$${\rm area}[X] = \int dX^\mu \omega_\mu$$

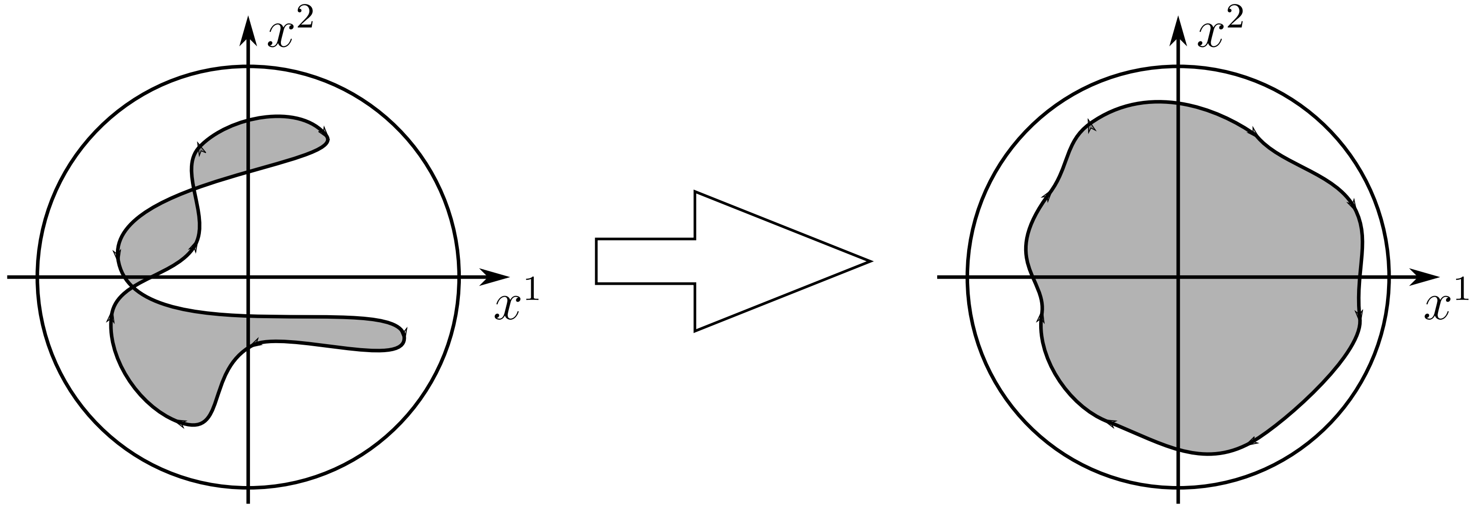


- Particle has spin $\nu = -i\gamma$.
- Generator for group of rotations at each point act as multiplication by $-\nu$ on its wavefunction.
- On $\widetilde{\rm AdS}_2$, ν–spinors are functions that transform in the same way under the generator of the analogous group $\widetilde{\rm SL}(2,\mathbb{R})/\widetilde{\rm AdS}_2$.



Recovery of the Schwarzian

- Take the limit $\gamma, L \gg 1$.



$$ds^2 = 4\left(\frac{d\varphi^2 + r^2 d\varphi^2}{(1-r^2)^2}\right)$$

$$K - 1 \approx \mathrm{Sch}(e^{i\varphi(\ell)}, \ell), \quad \mathrm{Sch}(f(x), x) = \frac{f^{(3)}}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

$$I_{\mathrm{pt}}[X] \approx I_{\mathrm{Sch}}[\varphi] + \mathrm{const.},$$

$$I_{\mathrm{Sch}}[\varphi] = -\gamma \int d\ell \, \mathrm{Sch}(e^{i\varphi(\ell)}, \ell) = -\frac{2\pi\gamma}{L} \int d\theta \, \mathrm{Sch}\left(e^{i\varphi(\theta)}, \theta\right)$$

- In the SYK model: $\gamma \sim \frac{N}{\epsilon}, \quad L \sim \frac{\beta J}{\epsilon}, \quad \frac{\gamma}{L} \sim \frac{N}{\beta J}$

$$G(\theta_1, \theta_2) = G_c(\varphi(\theta_1), \varphi(\theta_2))\varphi'(\theta_1)^{\Delta}\varphi'(\theta_2)^{\Delta}$$

[Kitaev; Kitaev, JS;…]

Outline

- Single-particle wavefunctions
- Two-sided black hole wavefunctions.
- General prescription for trace, observables.
- Correlation functions of external matter fields, kinematically interacting with boundary particle (checks of consistency with microscopic SYK).

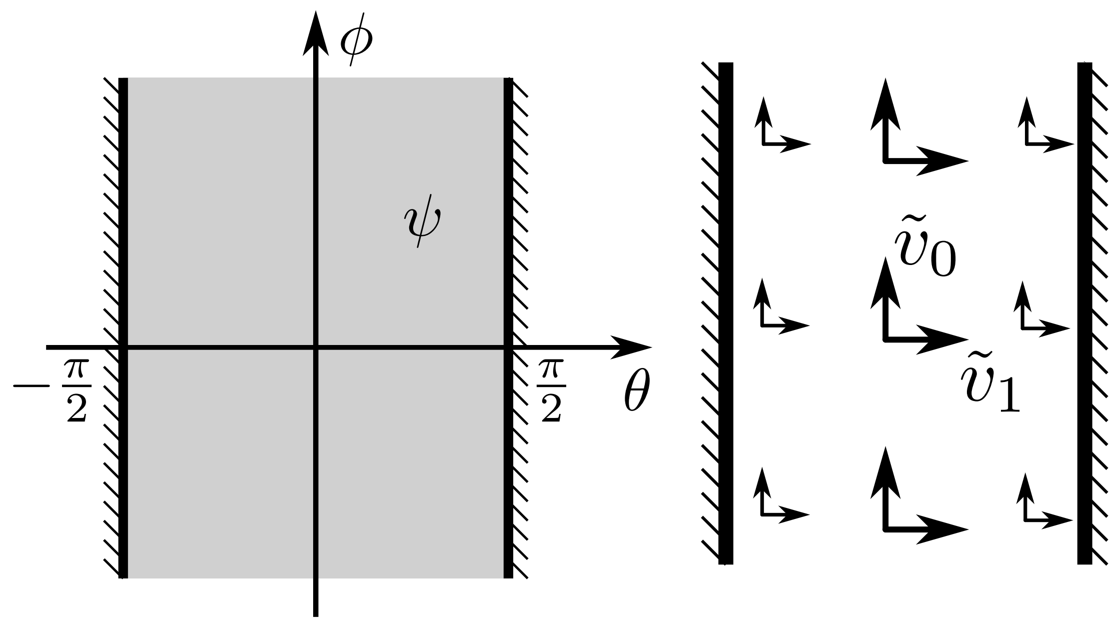
Single-particle wavefunctions

- Regularized particle action

$$I_{\text{pt,reg.}} = \int dT \left(\frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + \gamma \omega_\mu \dot{X}^\mu \right), \qquad \nu = -i\gamma$$

$$-\frac{1}{2}\nabla^2 \psi = E\psi \quad \Leftrightarrow \quad Q\psi = \lambda(1-\lambda)\psi \qquad -\nabla^2 = Q + \nu^2, \quad E = \frac{1}{2}(q - \gamma^2)$$

$$ds^2 = \frac{(-d\phi^2 + d\theta^2)}{\cos^2\theta}$$



- Inner product

$$\psi(\phi, \theta) = f(\theta) e^{im\phi}, \quad \langle \psi_1 | \psi_2 \rangle = \int_{\widetilde{\text{AdS}}_2} d^2x \sqrt{-g}\, \psi_1^*(x) \psi_2(x)$$

Possible $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ irreps in Hilbert space

- Our Hilbert space will be spanned by some ν–spinors, specifically Casimir eigenfunctions, organized into irreps of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

- In general, an irrep of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is labeled by two numbers, the Casimir eigenvalue and central element:

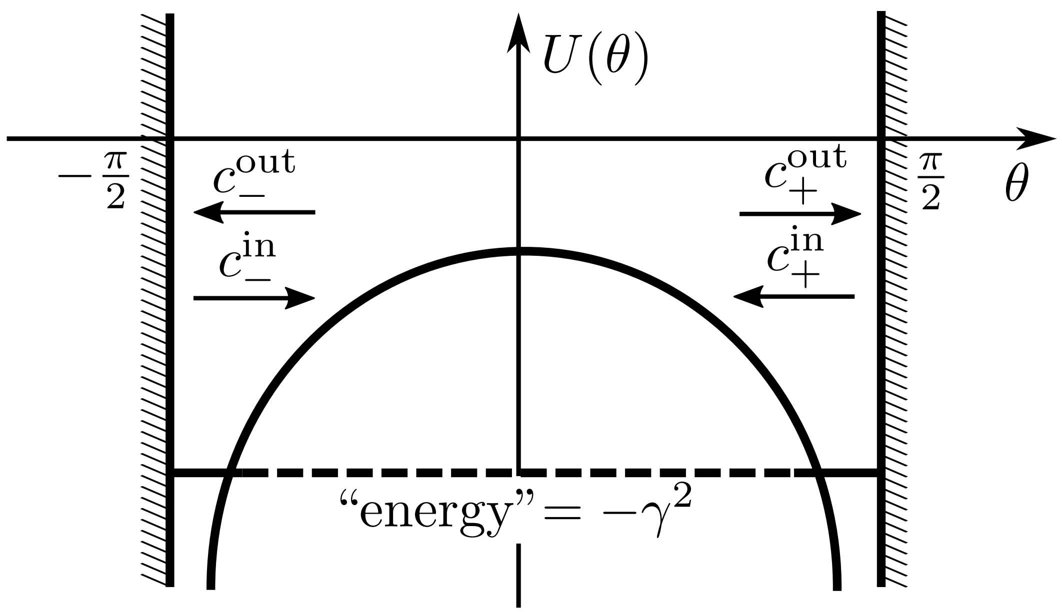
$$Q = \lambda(1-\lambda), \qquad e^{2\pi i L_0} = e^{2\pi i \mu}, \quad \mu \in \mathbb{R}/\mathbb{Z}$$

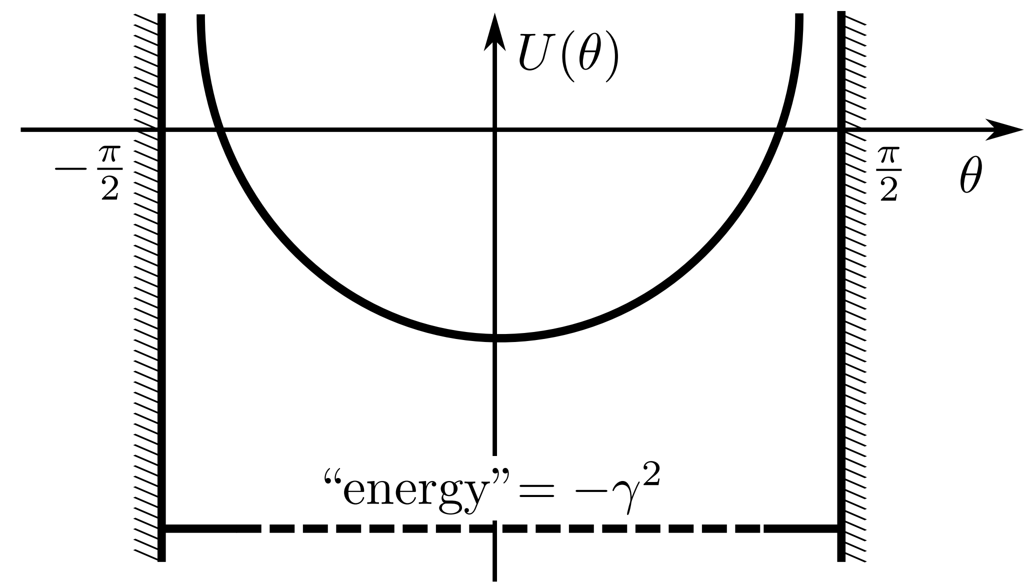
  - Within each irrep, states are labeled by $L_0 = -m,\ m \in \mu + \mathbb{Z}$.
  - An irrep falls into one of three categories:
    - Principal series: $\lambda = \frac{1}{2} + is,\ s \in \mathbb{R}, \quad \mu \text{ arbitrary}$
    - Discrete series: $\lambda > 0, \quad \mu = \pm\lambda, \quad \pm m = \lambda, \lambda+1, \lambda+2, \ldots$
    - Complementary series:…

- ν–spinors in principal and discrete series are normalizable under our inner product.

Physical distinction between wavefunctions

- The spatial part of a spin-ν Casimir eigenfunction sees a potential in a time-independent Schrodinger equation.





- Tunneling probability computed with wavefunctions in the principal series reproduces the density of states of the system.

$$\begin{pmatrix} c_+^{\text{out}} \\ c_-^{\text{out}} \end{pmatrix} = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \begin{pmatrix} c_+^{\text{in}} \\ c_-^{\text{in}} \end{pmatrix}, \quad p(s,\mu) = |S_{-+}|^2$$

$$\int d\mu\, p(s,\mu) = (2\pi)^2 \rho(E), \quad \rho(E) = (2\pi)^{-2} \frac{\sinh(2\pi s)}{\cosh(2\pi\gamma) + \cosh(2\pi s)}$$

Operators on particle Hilbert space

- A general $\widetilde{\mathrm{SL}}(2,\mathbb{R})$-invariant operator takes the form

$$\Psi^{\nu}_{\lambda,\mu}[R] = \sum_{\alpha,\beta} R_{\alpha\beta}(\lambda,\mu) \sum_{m\in\mu+\mathbb{Z}} \big|\, (\psi_\alpha)^{\nu}_{\lambda,m} \big\rangle\big\langle\, (\psi_\beta)^{\nu}_{\lambda,m} \big|, \quad \alpha = 1,2$$

$(\psi(x) = \langle x|\psi\rangle)$

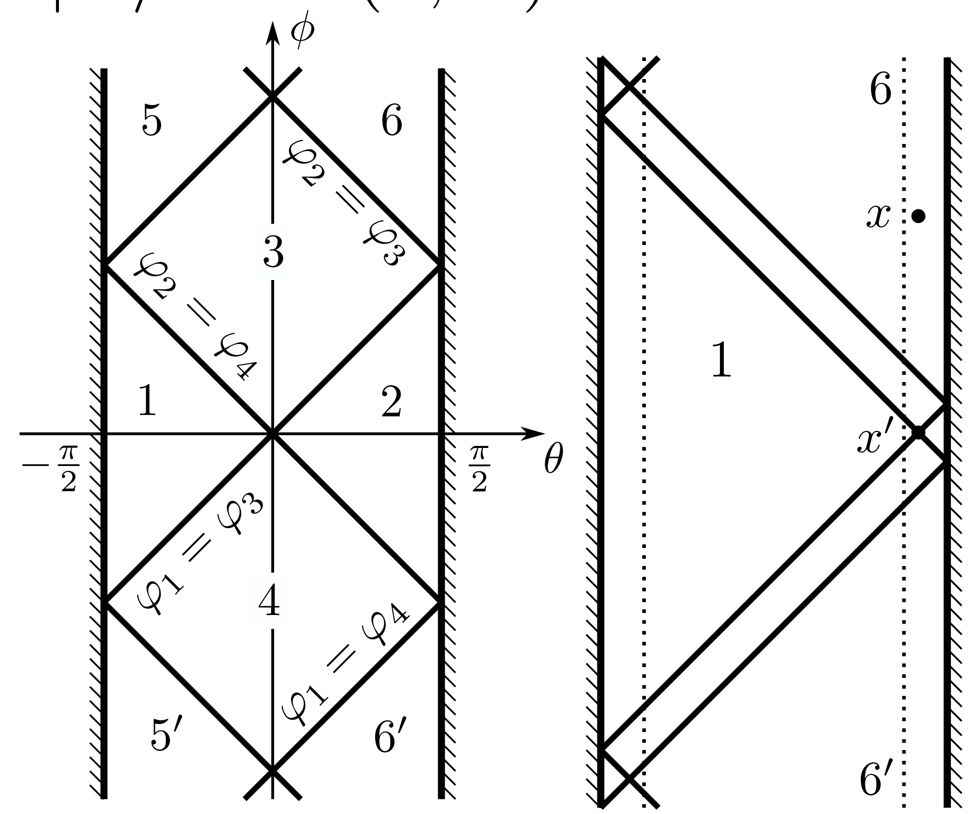
$$\Psi^{\nu}[R] = \int dE \underbrace{\int d\mu\, \rho_{\mathrm{Pl}}(E,\mu)\Psi^{\nu}_{1/2+is,\mu}[R(s,\mu)]}_{\Psi^{\nu}_{E}[R]}$$

$$\left(E = \tfrac{1}{2}\left(\tfrac{1}{4} + s^2 - \gamma^2\right), \quad \rho_{\mathrm{Pl}}(E,\mu) = (2\pi)^{-2}\frac{\sinh(2\pi s)}{\cosh(2\pi s)+\cos(2\pi\mu)}\right)$$

- Two-point functions of a ν–particle $\langle x|\Psi^{\nu}|x'\rangle = \Psi^{\nu}(x;x')$ have the structure on $\widetilde{\mathrm{AdS}}_2$:

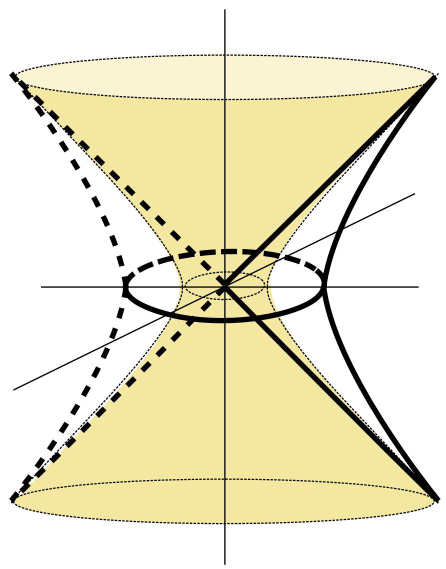
$$\Psi^{\nu}(x;x') = \left|\frac{\varphi_{23}}{\varphi_{14}}\right|^{\nu} f_j(u), \quad u = \frac{\varphi_{13}\varphi_{24}}{\varphi_{14}\varphi_{23}}$$

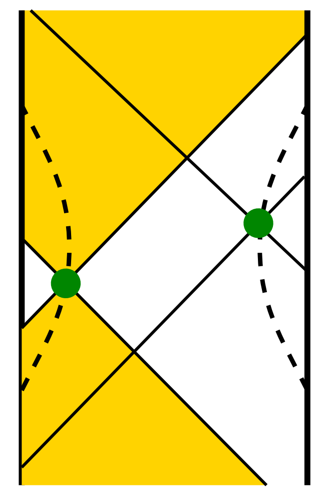
$$\varphi_{ij} = 2\sin\left(\frac{\varphi_i - \varphi_j}{2}\right)$$



Black hole wavefunction (tunneling operator)

- We can uniquely specify the wavefunction for a two-sided black hole at each energy $\Phi_E(x;x')$





$$\Phi_E(x;0) = \begin{cases} \pm 2\pi\rho(E) A_{\lambda,\nu,-\nu}(u) & (+ : \text{in region } 2, - : \text{in region } 1) \\ 0 & \text{in all other regions} \end{cases}$$

$$A_{\lambda,\nu,-\nu}(u) = (1-u)^{\lambda} \tilde{F}(\lambda+\nu, \lambda-\nu, 1; u)$$

- Due to its space-like support, Φ_E can also be interpreted as a tunneling operator.

Trace operation on Hilbert space

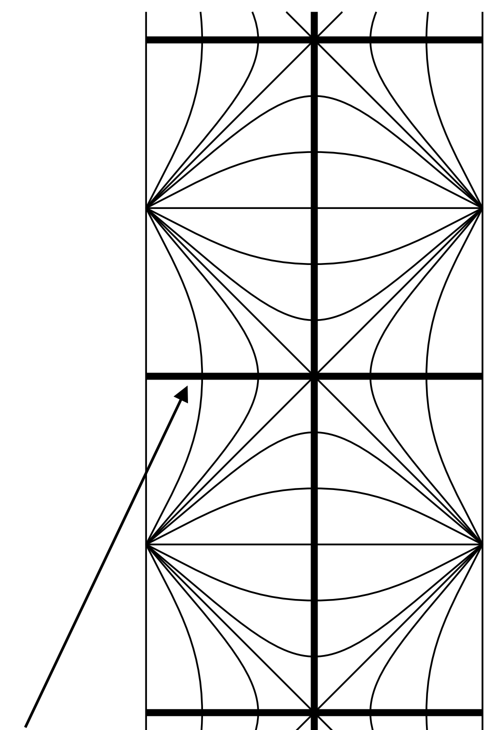
- We can define an algebraic trace operation on an arbitrary operator on our Hilbert space as

$$\mathrm{tr}(\Psi^{\nu}[R]) = \int dE \int d\mu\, \rho_{\mathrm{Pl}}(E,\mu)\mathrm{Tr}(R(s,\mu))$$

- The same trace, for a product of two operators, can also be evaluated over spacetime points, by factoring out the volume of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$:

$$F(x;x') = \left|\frac{\varphi_{23}}{\varphi_{14}}\right|^{\nu} f_j(u),\ \ G(x;x') = \left|\frac{\varphi_{23}}{\varphi_{14}}\right|^{\nu} g_j(u)$$

$$\mathrm{tr}(F^{\dagger}G) = \sum_j \frac{2du}{(1-u)^2} f_j^*(u) g_j(u),$$



$(\widetilde{\mathrm{AdS}}_2 \times \widetilde{\mathrm{AdS}}_2)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$

Prescription for black hole observables

- The thermal partition function for a one-sided black hole, with our construction of the quantum theory, takes the form

$$Z = \mathrm{tr}(e^{-\beta H}\mathrm{P}), \quad \mathrm{P} = \Phi^\dagger\Phi = -\Phi^2$$

- The square of the tunneling operator, P, encodes the density of states in the black hole system as $\mathrm{tr}(\mathrm{P}_E)/2 = \rho(E)$.

- In the absence of matter in the bulk, most general density matrix for a black hole is

$$\varrho = \Psi^\nu[fI]\cdot\mathrm{P} = \int dE\, f(E)\mathrm{P}_E$$

- To calculate general expectation values, we use the usual rules of quantum mechanics, employing our trace, except that density matrices are multiplied by an extra P. For quantum entropy,

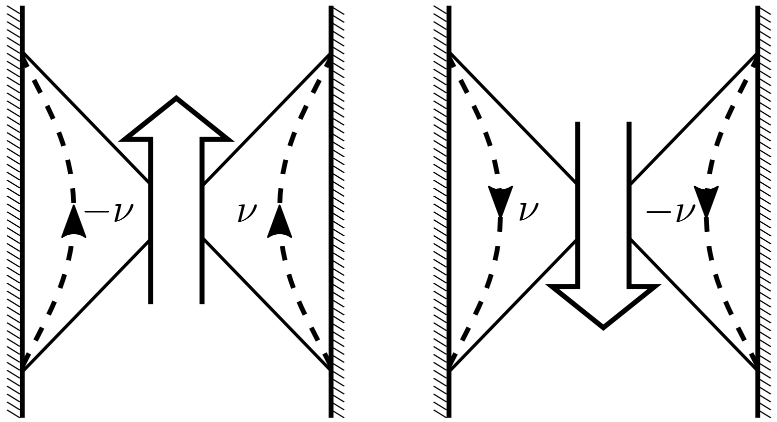
.

$$S = -\frac{1}{2}\mathrm{tr}(\varrho(\ln\varrho - \ln\mathrm{P})) = -\int dE\,\rho(E)f(E)\ln f(E)$$

Correlation functions of external operators

- Consider adding matter fields decoupled from the boundaries which are second-quantized on $\widetilde{\mathrm{AdS}_2}$

$$\mathcal{H} \subset (\mathcal{H}_{\text{fields}} \oplus \mathcal{H}^*_{\text{fields}}) \otimes \mathcal{H}^{\nu} \otimes \mathcal{H}^{-\nu}$$



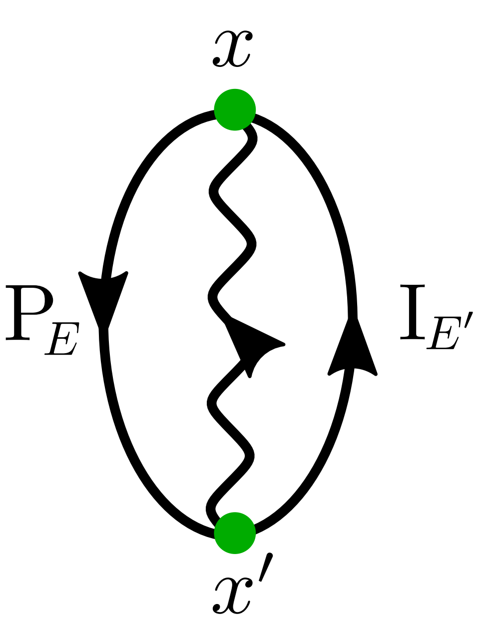
- Operator acting at position of ν–particle:

$$\hat{\mathcal{O}}^{\nu} = \int d^2x\sqrt{-g}\,\mathcal{O}(x) \otimes |x\rangle\langle x| \otimes \mathbf{1}, \quad \hat{O}^{\nu}(T) = e^{iH^{\nu}T}\hat{O}^{\nu}e^{-iH^{\nu}T}$$

- E.g. a one-sided correlator

$$\mathcal{F}^{\nu,\nu}_{X,Y}(T,0) = \left\langle \frac{1}{2}\mathrm{tr}\left(Z^{-1}e^{-\beta H}\mathrm{P}\,\hat{X}(T)\,\mathrm{I}\,\hat{Y}(0)\right)\right\rangle_{\text{fields}}$$

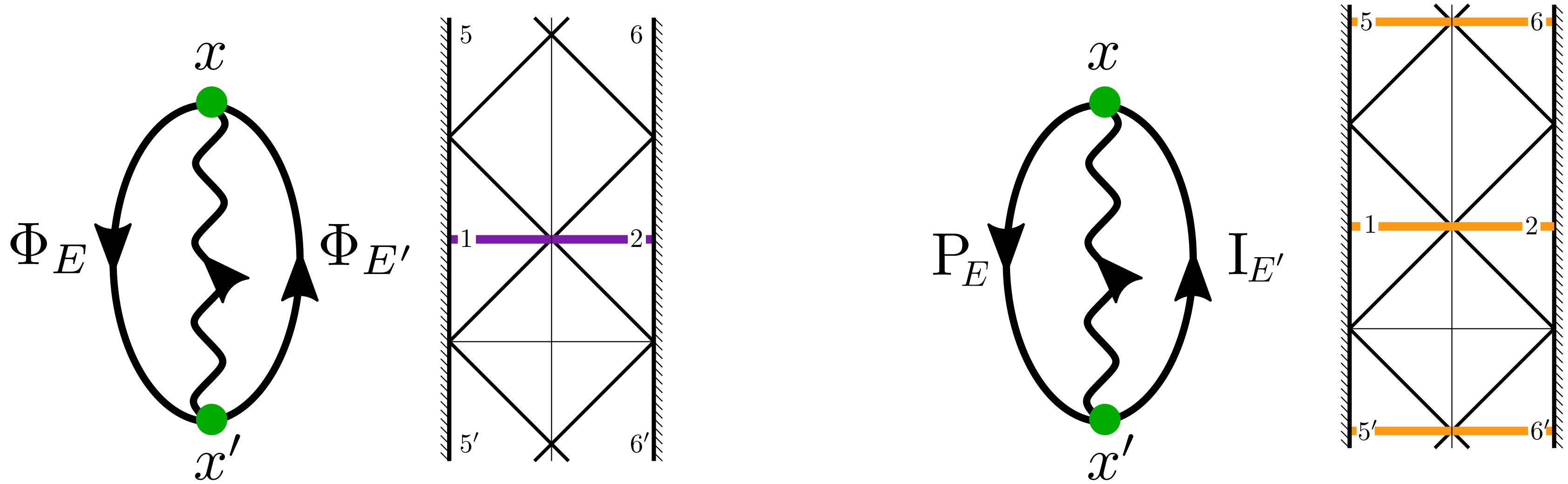
$$= Z^{-1}\int dEdE' e^{-\beta E}e^{i(E-E')T} \times$$



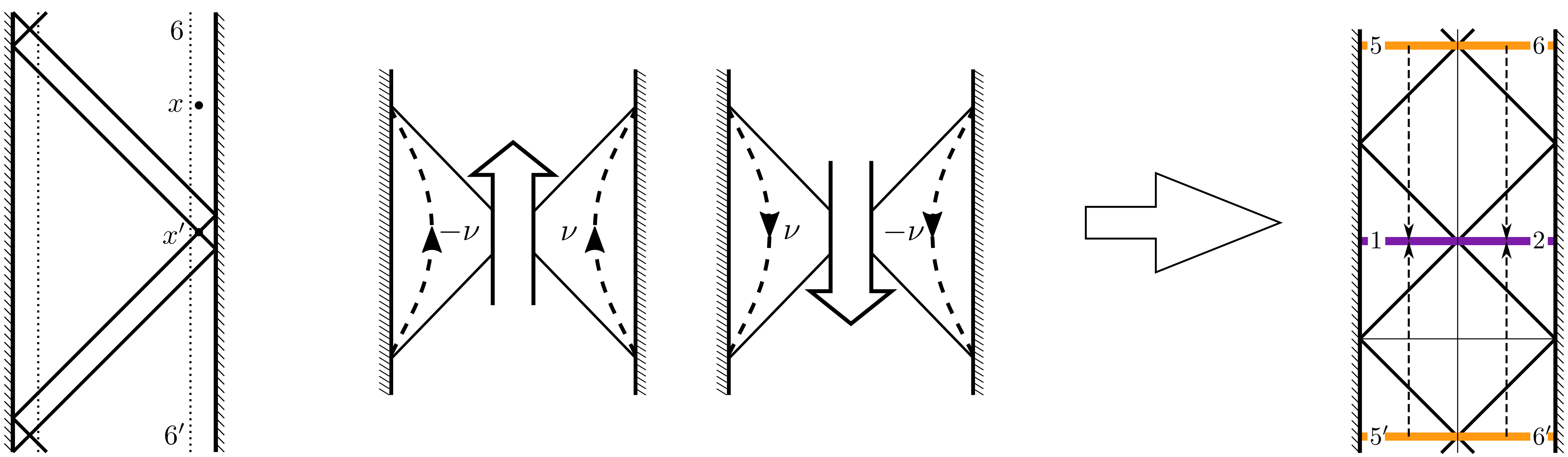
Constraints from analyticity of 2-point function

- Analogue of analyticity holding in a microscopic quantum system

$$\langle \mathrm{TFD} | \hat{X}^R(iT)\hat{Y}^L(0) | \mathrm{TFD} \rangle = \langle \mathrm{TFD} | \hat{X}^R(T)\hat{Y}^R(0) | \mathrm{TFD} \rangle$$



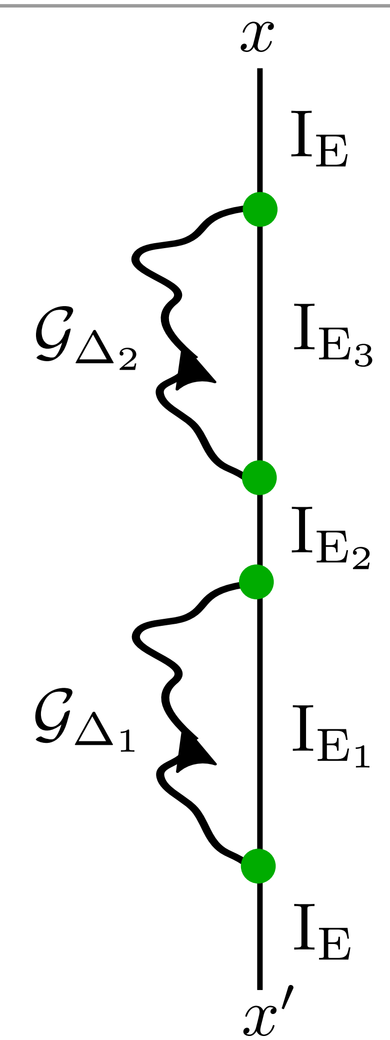
- This imposes non-trivial constraints on our quantum theory



$$\mathcal{H} = \left(\mathcal{H}_{\text{fields}} \otimes \mathcal{H}_{\mathrm{R}}^{\nu} \otimes \mathcal{H}_{\mathrm{L}}^{-\nu}\right) \oplus \left(\mathcal{H}_{\text{fields}}^{*} \otimes \mathcal{H}_{\mathrm{L}}^{\nu} \otimes \mathcal{H}_{\mathrm{R}}^{-\nu}\right)$$

Structure of higher-point correlation functions

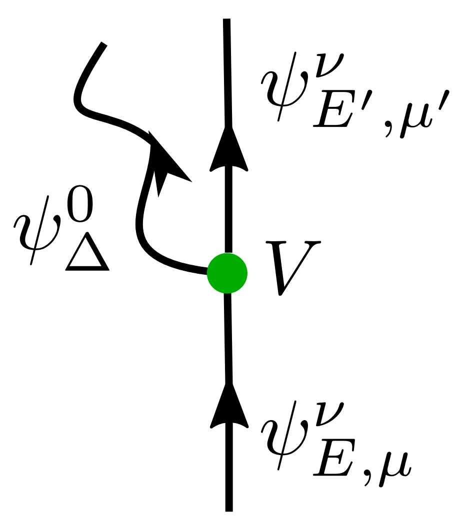
- There is group-theoretic structure in a general Lorentzian correlator.



$$\mathrm{I}_E = \int d\mu\, \rho_{\mathrm{Pl}}(E,\mu) \sum_{m\in\mu+\mathbb{Z}} \big|\psi^{\nu}_{E,m}\big\rangle\big\langle\psi^{\nu}_{E,m}$$

$$\mathcal{G}_\Delta = \sum_{n\in\mathbb{N}} \big|\psi^{0}_{\Delta,\Delta+n}\big\rangle\big\langle\psi^{0}_{\Delta,\Delta+n}\big|$$

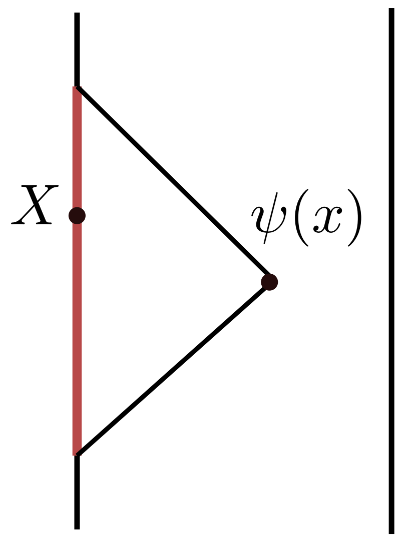
- Each vertex is an intertwiner, whose matrix elements are Clebsch-Gordon coefficients.



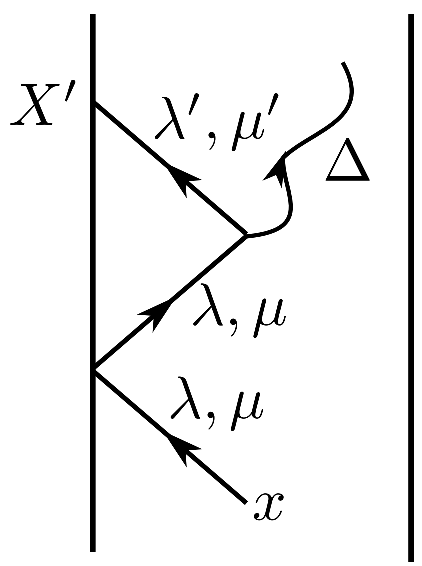
$$\big\langle X', n\big|V_{E',\mu',\Delta;E,\mu}\big|X\big\rangle =$$
$$\int d^2x\sqrt{-g}\,\psi^{\nu}_{E,\mu}(x|X)\psi^{\nu}_{E',\mu'}(x|X')^*\psi^{0}_{\Delta,n}(x)^*$$

Cancellation of μ-dependence

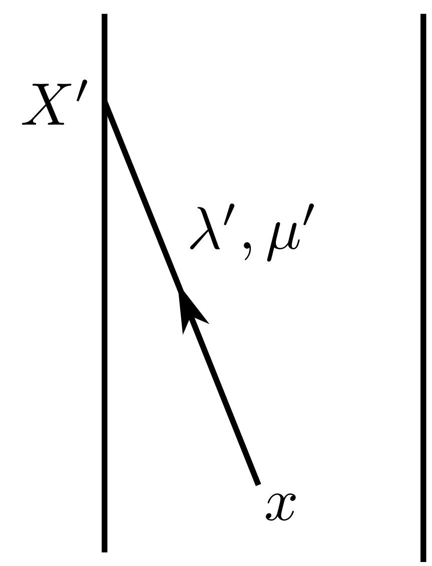
- Bulk-to-boundary propagators seem to constitute a nice basis, in which we can see cancellation of μ-dependence.



$$\Rightarrow \sum_{m\in\mu+\mathbb{Z}} \psi^{\nu}_{E,m}(x)\psi^{\nu}_{E,m}(x')^* = \int dX\, \psi^{\nu}_{E,\mu}(x|X)\psi^{\nu}_{E,\mu}(x'|X)^*$$



$$=$$



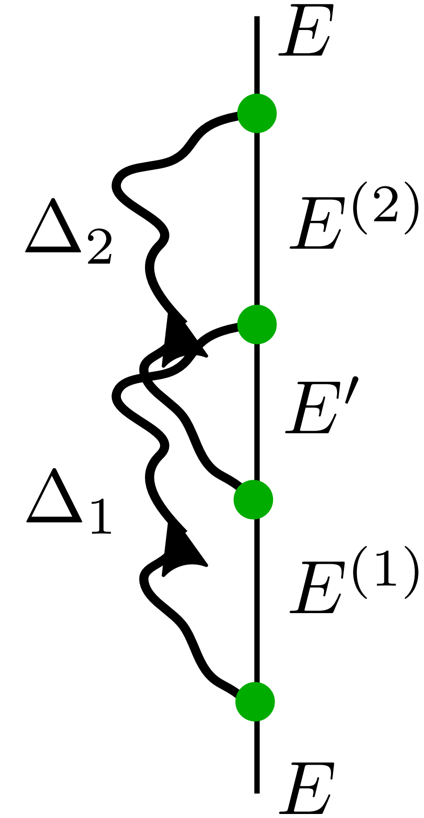
$$\times \mathcal{A}(\Delta, \lambda, \lambda')\delta(\mu - \mu' - \Delta)$$

(work in progress)

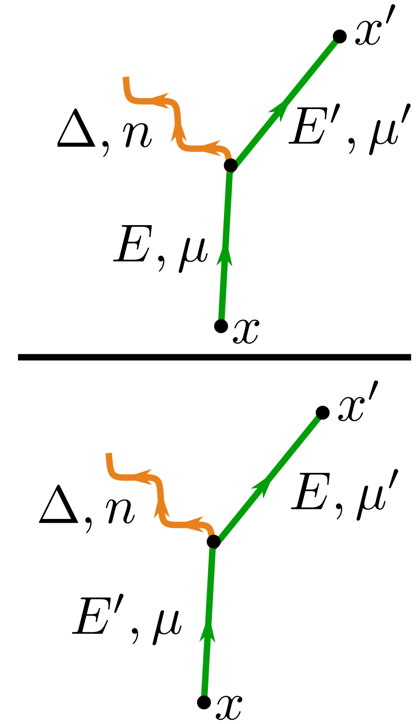
Outcomes

- Pending this cancellation, we can reproduce the classical 6j-symbol found in OTO 4-point functions using CFT bootstrap arguments in the Schwarzian theory.

  [Mertens, Turiaci, Verlinde]



- The density of states appears in the ratio between transition amplitudes:



$$\sim \frac{\rho(E')}{\rho(E)}$$

Summary

- We have built a quantum theory for the dynamics of the soft mode in the SYK model, which corresponds to a solvable, topological sector of a gravitational theory in two-dimensional anti-de Sitter space.
- Wavefunctions in our quantum theory pass some preliminary checks and can be viewed as resulting from coarse-graining of SYK microstates.
- More rigorous consistency checks on the Hilbert space remain, involving matrix elements of interaction with external matter.

Some open questions

- Can we couple additional degrees of freedom to JT gravity which reproduce the subleading-in-βJ exponent of the OTO 4-point function in the SYK model?
  - If so, can we quantize them?

- Can we quantize degrees of freedom that contribute at non-perturbative order (powers of e^{-N}) in JT gravity?