Introduction to Tensor Models

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- Based in part on

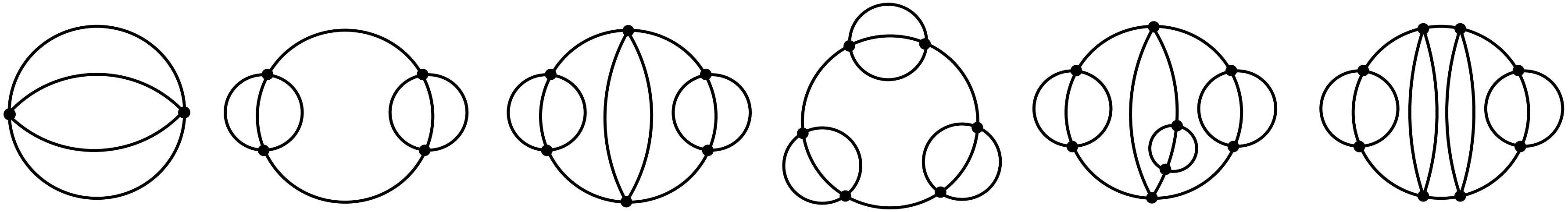
  IK, Fedor Popov, Grigory Tarnopolsky,

  "TASI Lectures on Large N Tensor Models,"

  arXiv: 1808.09434

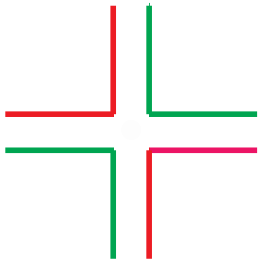
Three Large N Limits

- O(N) Vector: solvable because the bubble diagrams can be summed. 
- Matrix ('t Hooft) Limit: planar diagrams. Solvable only in special cases.
- Tensor of rank three and higher. When interactions are specially chosen, dominated by the "melonic" diagrams. Bonzom, Gurau, Riello, Rivasseau; Carrozza, Tanasa; Witten; IK, Tarnopolsky

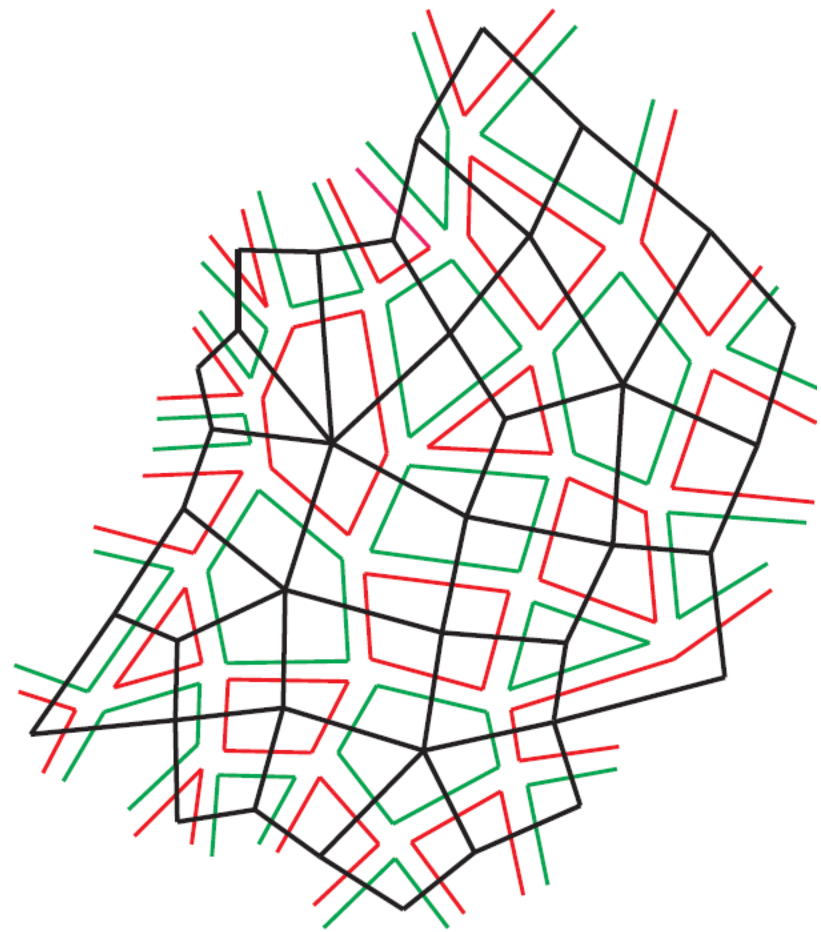


O(N) x O(N) Matrix Model

- Theory of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a \times O(N)_b$ symmetry.
- The interaction is at least quartic: $g \, \mathrm{tr}\, \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is
- In d=0 or 1 special limits describe two-dimensional quantum gravity.



- In the large N limit where gN is held fixed we find planar Feynman graphs, and each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

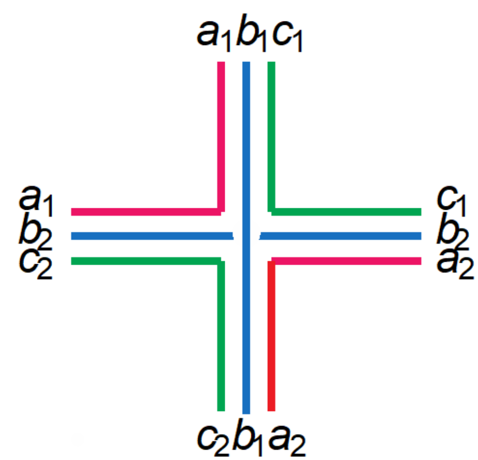
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

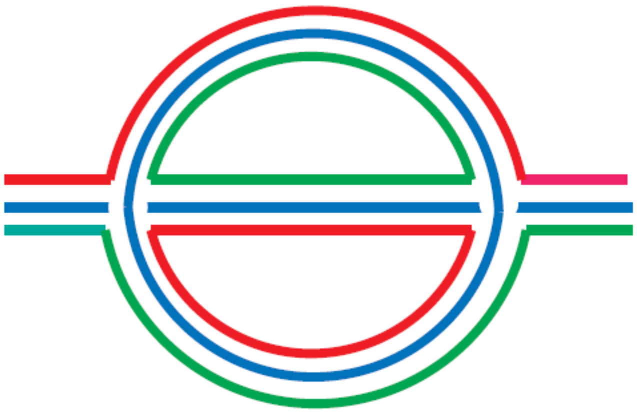
- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

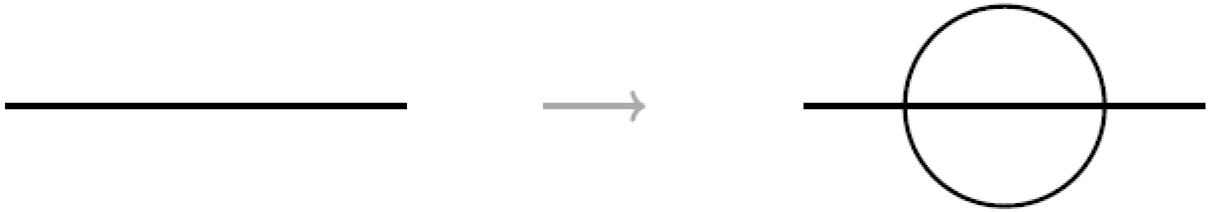
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



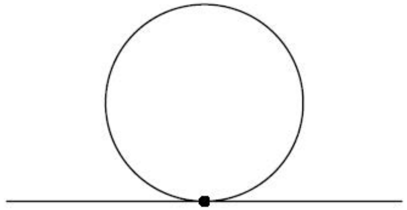
- Leading correction to the propagator has 3 index loops

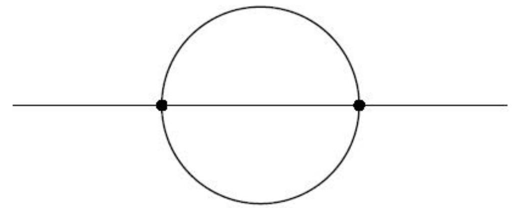


- Requiring that this "melon" insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



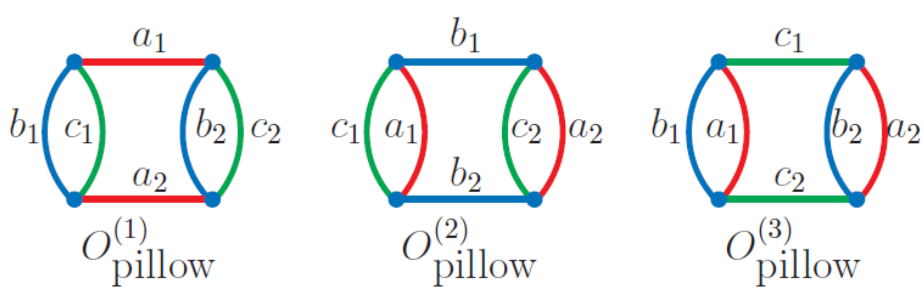
Snails vs. Melons



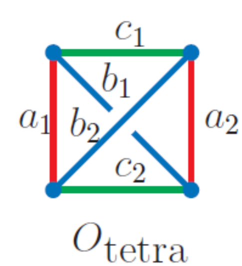


- In large N vector models snail diagrams dominate.
- In matrix models both contribute.
- In tensor models with tetrahedral interactions the melons dominate.

- The snail insertion scales as $gN \sim \frac{\lambda}{\sqrt{N}}$
- The melon insertion as $g^2 N^3 \sim \lambda^2$
- The melonic dominance would not hold if we adopted the "pillow interactions"

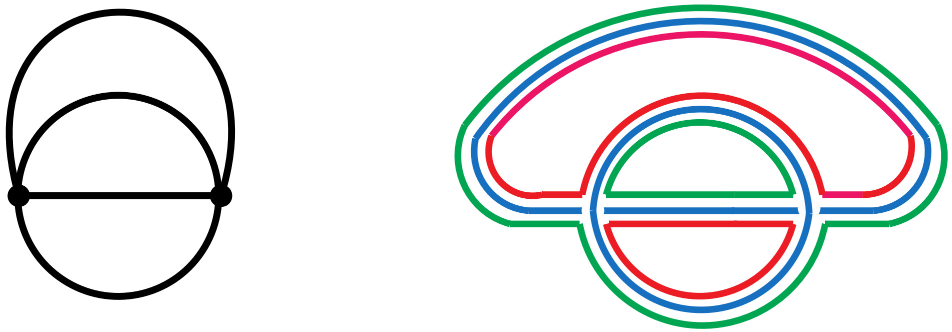


instead of the tetrahedral

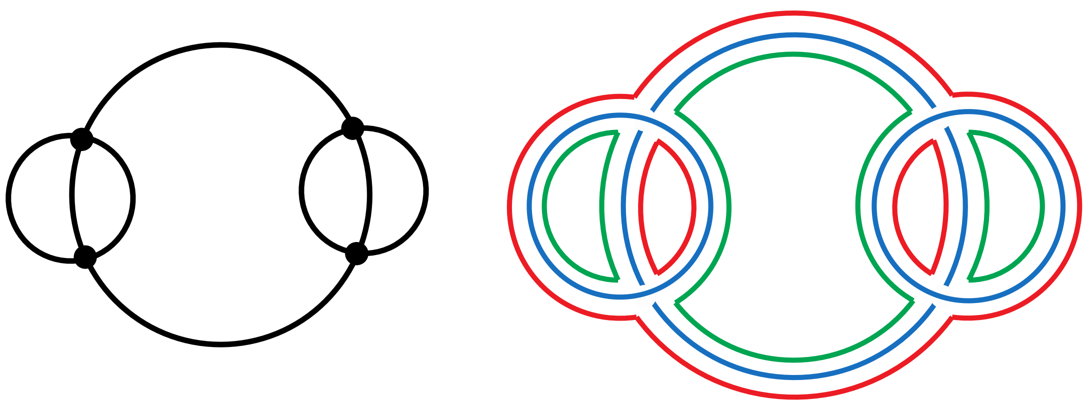


Cables and Wires

- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines)



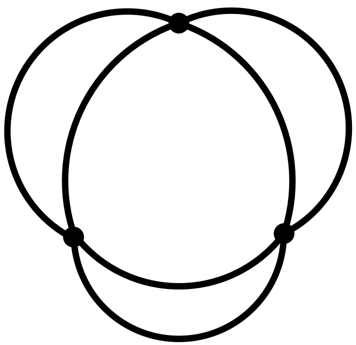
$$g^2 N^6 \sim N^3 \lambda^2$$

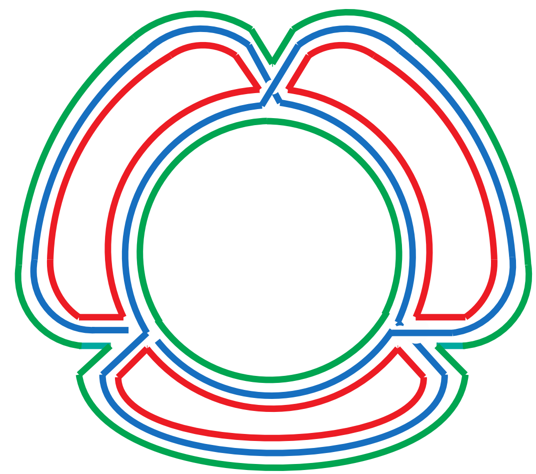


$$g^4 N^9 \sim N^3 \lambda^4$$

Non-Melonic Graphs

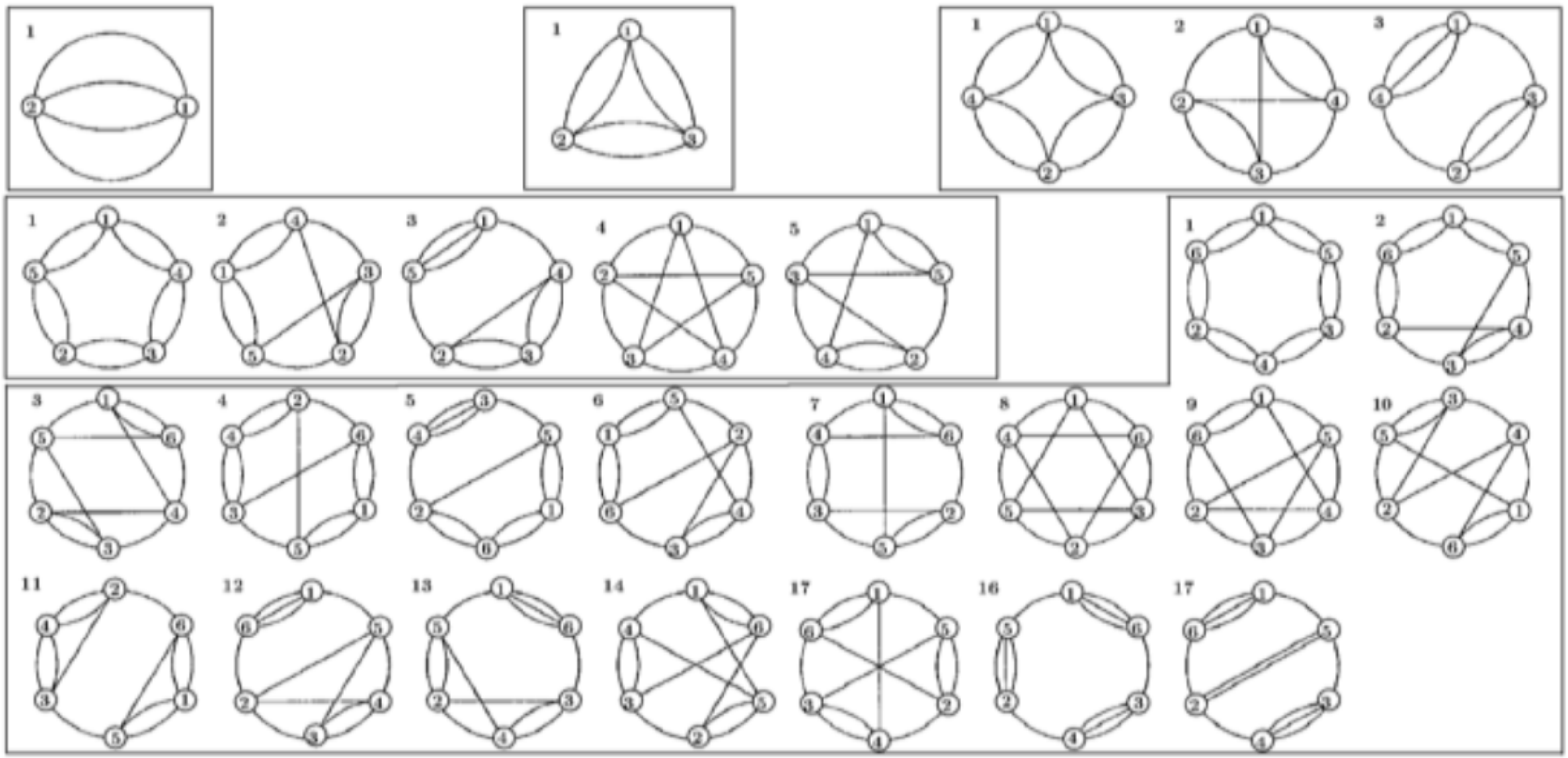
- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.





- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

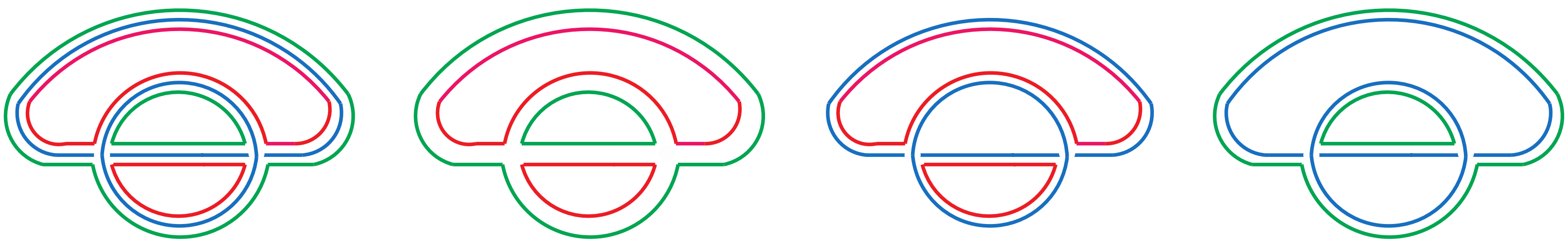
- Here is the list of snail-free vacuum graphs up to 6 vertices Kleinert, Schulte-Frohlinde



- Only 4 out of these 27 graphs are melonic.
- The number of melonic graphs with p vertices grows as C^p Bonzom, Gurau, Riello, Rivasseau

Large N Scaling

- ”Forgetting ” one color we get a double-line graph.



- The number of loops in a double-line graph is

$f = \chi + e - v$ where χ is the Euler characteristic, e is the number of edges, and v is the number of vertices, $e = 2v$

- If we erase the blue lines we get $f_{rg} = \chi_{rg} + v$

- Adding up such formulas, we find

$$f_{bg} + f_{rg} + f_{br} = 2(f_b + f_g + f_r) = \chi_{bg} + \chi_{br} + \chi_{rg} + 3v$$

- The total number of index loops is

$$f_{\text{total}} = f_b + f_g + f_r = \frac{3v}{2} + 3 - g_{bg} - g_{br} - g_{rg}$$

- The genus of a graph is $g = 1 - \chi/2$
- Since $g \geqslant 0$, for a "maximal graph" which dominates at large N all its subgraphs must have genus zero: $f_{\text{total}} = 3 + 3v/2$
- Scales as $N^3(gN^{3/2})^v$
- In the 3-tensor models $\lambda = gN^{3/2}$ must be held fixed in the large N limit.

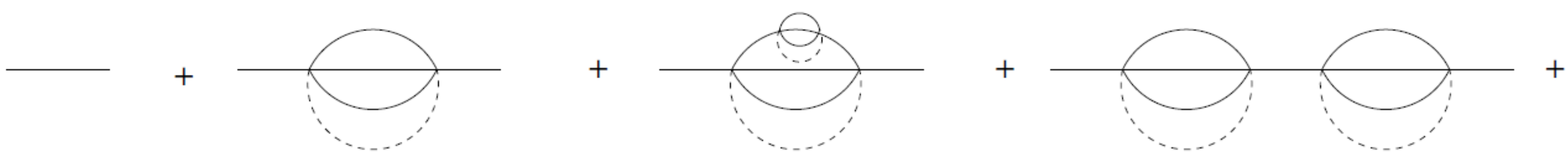
The Sachdev-Ye-Kitaev Model

- Quantum mechanics of a large number N_{SYK} of anti-commuting variables with action

$$I = \int \mathrm{d}t \left(\frac{\mathrm{i}}{2} \sum_i \psi_i \frac{\mathrm{d}}{\mathrm{d}t} \psi_i - \mathrm{i}^{q/2} j_{i_1 i_2 \dots i_q} \psi_{i_1} \psi_{i_2} \dots \psi_{i_q} \right)$$

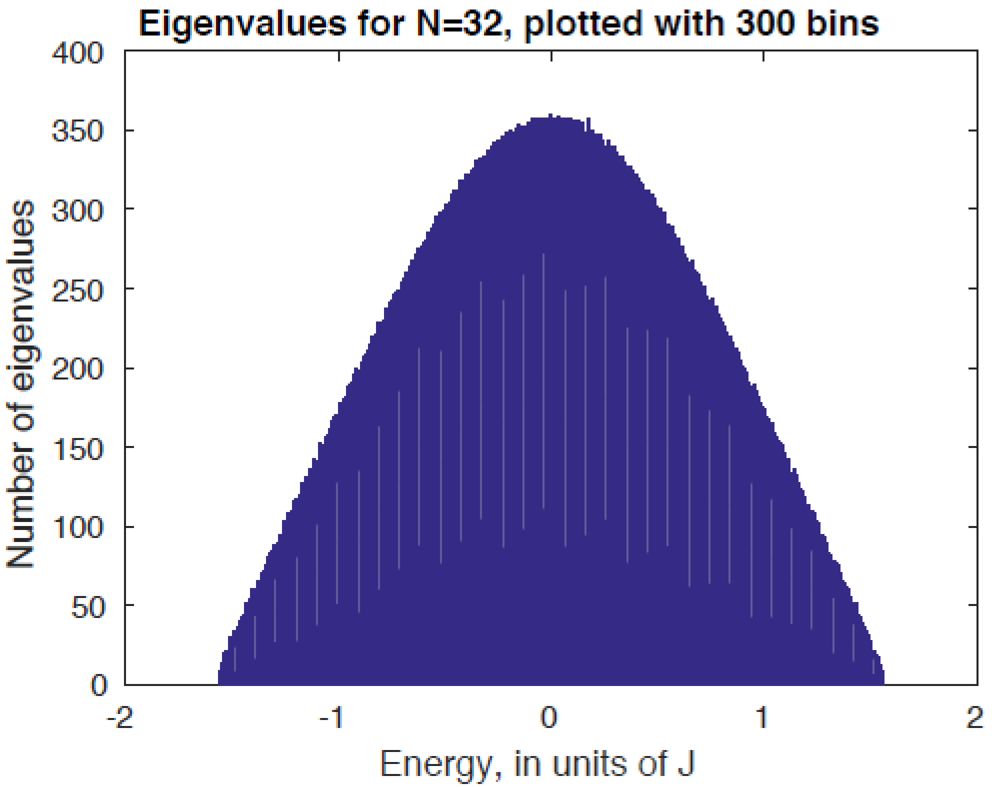
- Random couplings j have a Gaussian distribution with zero mean.
- The model flows to strong coupling and becomes nearly conformal. Georges, Parcollet, Sachdev; Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh

- The simplest interesting case is q=4.
- Exactly solvable in the large N_{SYK} limit because only the melon Feynman diagrams contribute



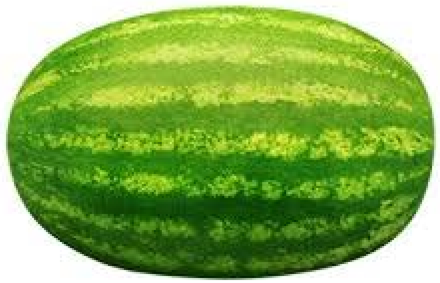
- Solid lines are fermion propagators, while dashed lines mean disorder average.
- The exact solution shows resemblance with physics of certain two-dimensional black holes. Kitaev; Almheiri, Polchinski; Sachdev; Maldacena, Stanford, Yang; Engelsoy, Merten, Verlinde; Jensen; Kitaev, Suh; …

- Spectrum for a single realization of N_{SYK}=32 model with q=4. Maldacena, Stanford
- No exact degeneracies, but the gaps are exponentially small. Large low T entropy.



SYK-Like Tensor Quantum Mechanics

- E. Witten, "An SYK-Like Model Without Disorder," arXiv: 1610.09758.
- Appeared on the evening of Halloween: October 31, 2016.


- It is sometimes tempting to change the term "melon diagrams" to "pumpkin diagrams."

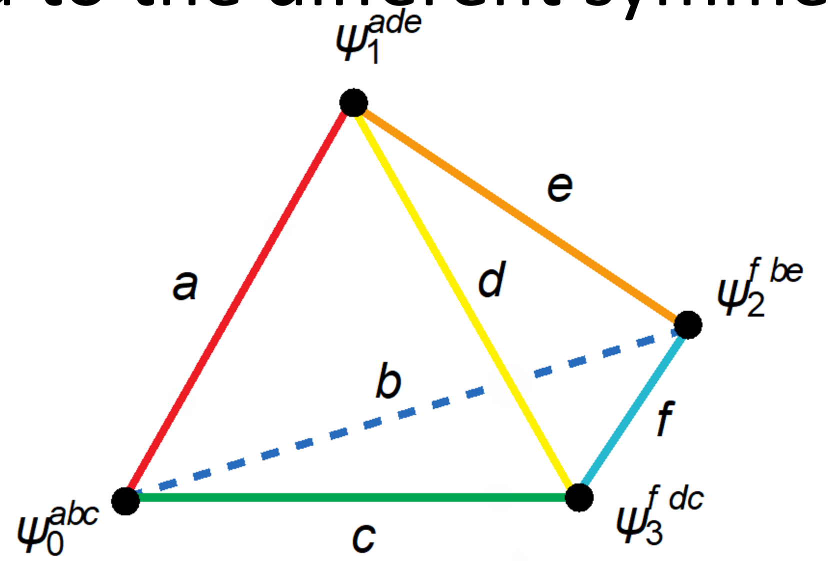
The Gurau-Witten Model

- This model is called "colored" in the random tensor literature because the anti-commuting 3-tensor fields ψ_A^{abc} carry a label A=0,1,2,3.

$$S_{\text{Gurau-Witten}} = \int dt \Big(\frac{i}{2} \psi_A^{abc} \partial_t \psi_A^{abc} + g \psi_0^{abc} \psi_1^{ade} \psi_2^{fbe} \psi_3^{fdc} \Big)$$

- Perhaps more natural to call it "flavored."
- The model has $O(N)^6$ symmetry with each tensor in a tri-fundamental under a different subset of the six symmetry groups.
- Contains $4N^3$ Majorana fermions.

- The 4 different fields may be associated with 4 vertices of a tetrahedron, and the 6 edges correspond to the different symmetry groups:



- As stressed by Witten, it may be advantageous to gauge the $SO(N)^6$ symmetry.
- This makes it a candidate gauge/gravity correspondence.

The $O(N)^3$ Model

- A pruned version: there are N^3 Majorana fermions IK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

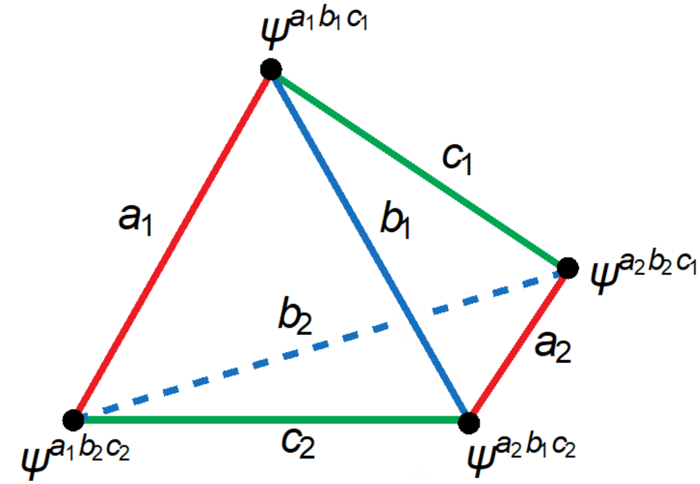
- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

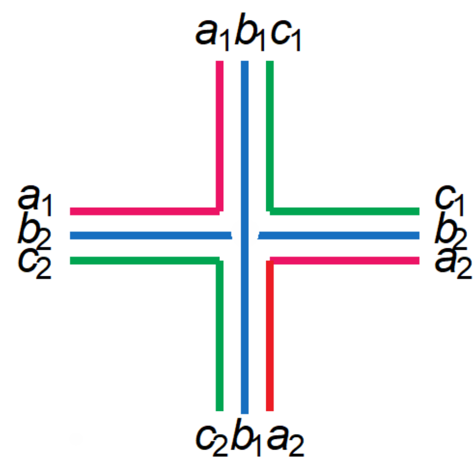
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}], \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}], \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.



- This is equivalent to



- The triple-line Feynman graphs are produced using the propagator

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

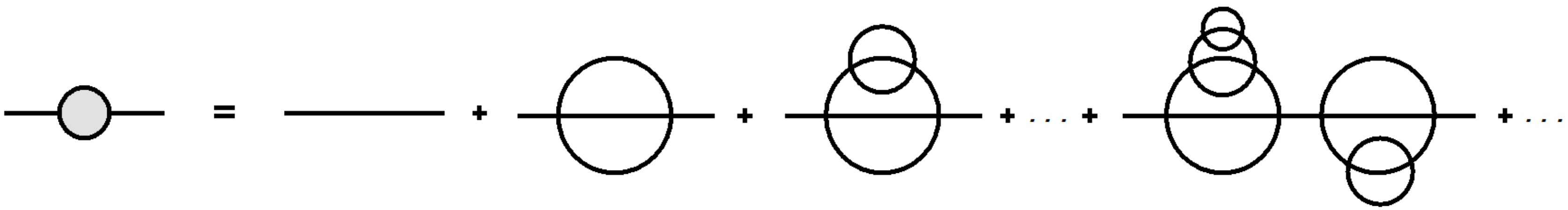
$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\text{SYK}} = N^3$

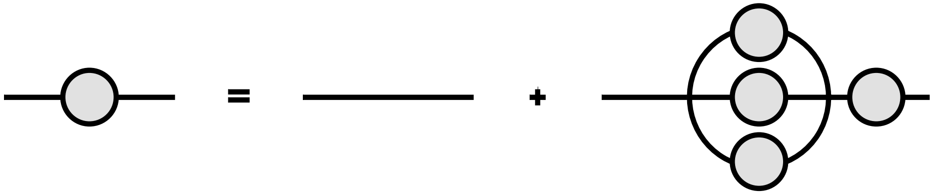
$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

Schwinger-Dyson Equations

- Some are the same as in the SYK model Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh



$$G(t_1 - t_2) = G_0(t_1 - t_2) + g^2 N^3 \int dt dt' G_0(t_1 - t) G(t - t')^3 G(t' - t_2)$$

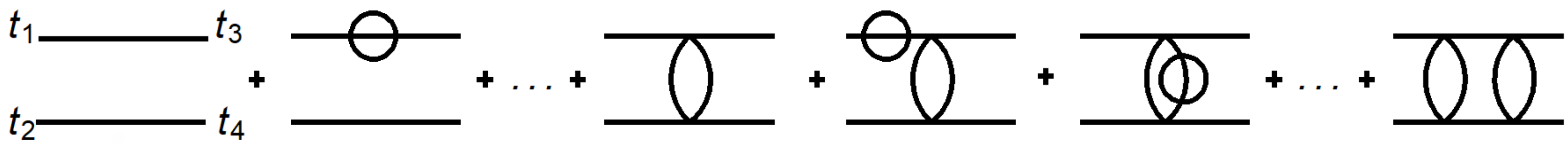


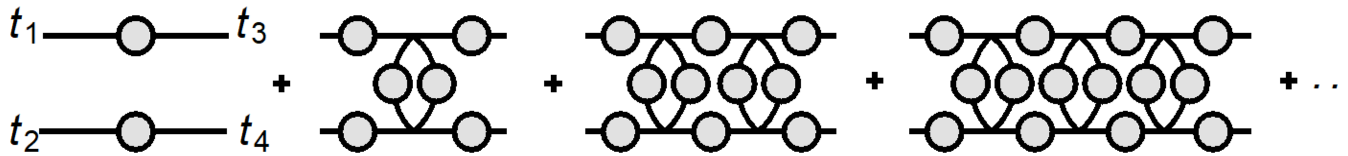
- Neglecting the left-hand side in IR we find

$$G(t_1 - t_2) = -\left(\frac{1}{4\pi g^2 N^3}\right)^{1/4} \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}}$$

- Four point function

$$\langle \psi^{a_1 b_1 c_1}(t_1)\psi^{a_1 b_1 c_1}(t_2)\psi^{a_2 b_2 c_2}(t_3)\psi^{a_2 b_2 c_2}(t_4)\rangle = N^6 G(t_{12})G(t_{34}) + \Gamma(t_1,\ldots,t_4)$$





- If we denote by Γ_n the ladder with n rungs

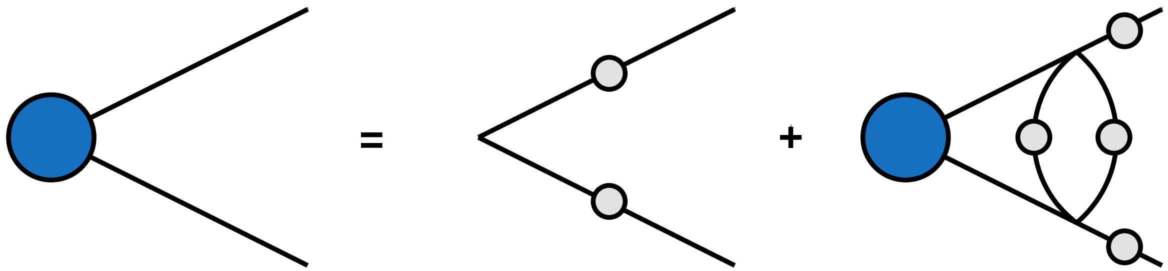
$$\Gamma = \sum_n \Gamma_n$$

$$\Gamma_{n+1}(t_1,\ldots,t_4) = \int dt dt' K(t_1,t_2;t,t')\Gamma_n(t,t',t_3,t_4)$$

$$K(t_1,t_2;t_3,t_4) = -3g^2 N^3 G(t_{13})G(t_{24})G(t_{34})^2$$

Spectrum of two-particle operators

- S-D equation for the three-point function Gross, Rosenhaus



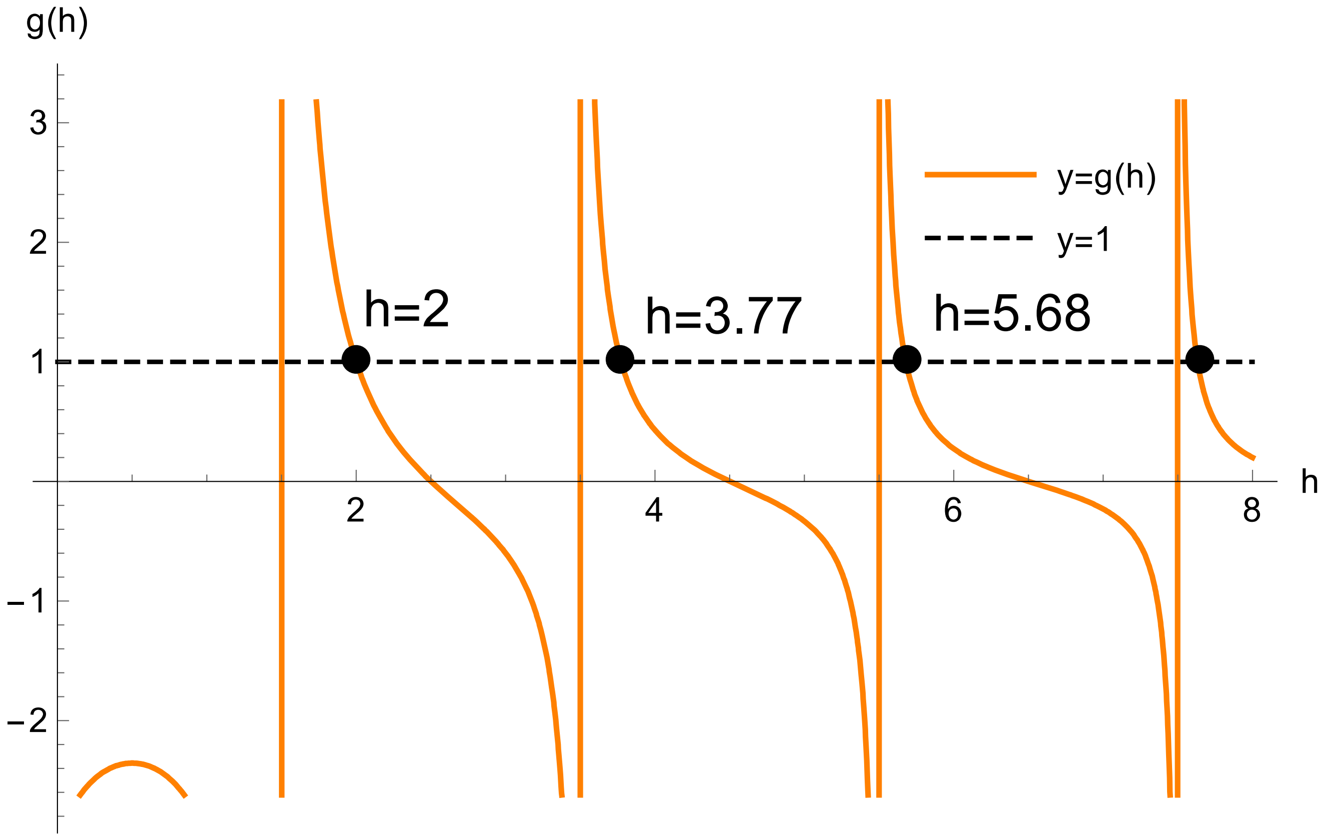
$$v(t_0,t_1,t_2)=g(h)\int dt_3dt_4K(t_1,t_2;t_3,t_4)v(t_0,t_3,t_4)$$

$$v(t_0,t_1,t_2)=\langle O_2^n(t_0)\psi^{abc}(t_1)\psi^{abc}(t_2)\rangle=\frac{\mathrm{sgn}(t_1-t_2)}{|t_0-t_1|^h|t_0-t_2|^h|t_1-t_2|^{1/2-h}}$$

- Scaling dimensions of operators $O_2^n=\psi^{abc}(D_t^n\psi)^{abc}$

$$g(h)=-\frac{3}{2}\frac{\tan(\frac{\pi}{2}(h-\frac{1}{2}))}{h-1/2}=1$$

- The first solution is h=2; dual to dilaton gravity.

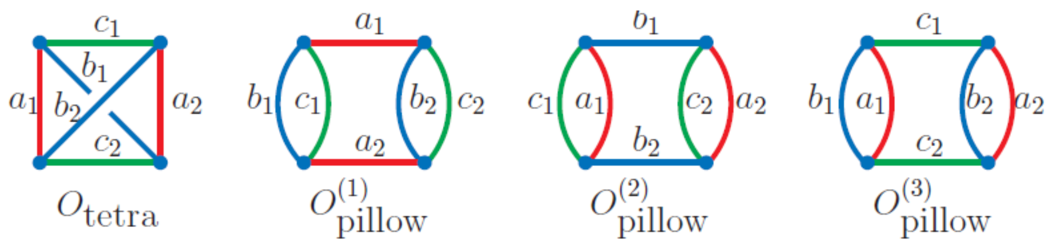


- The higher scaling dimensions are

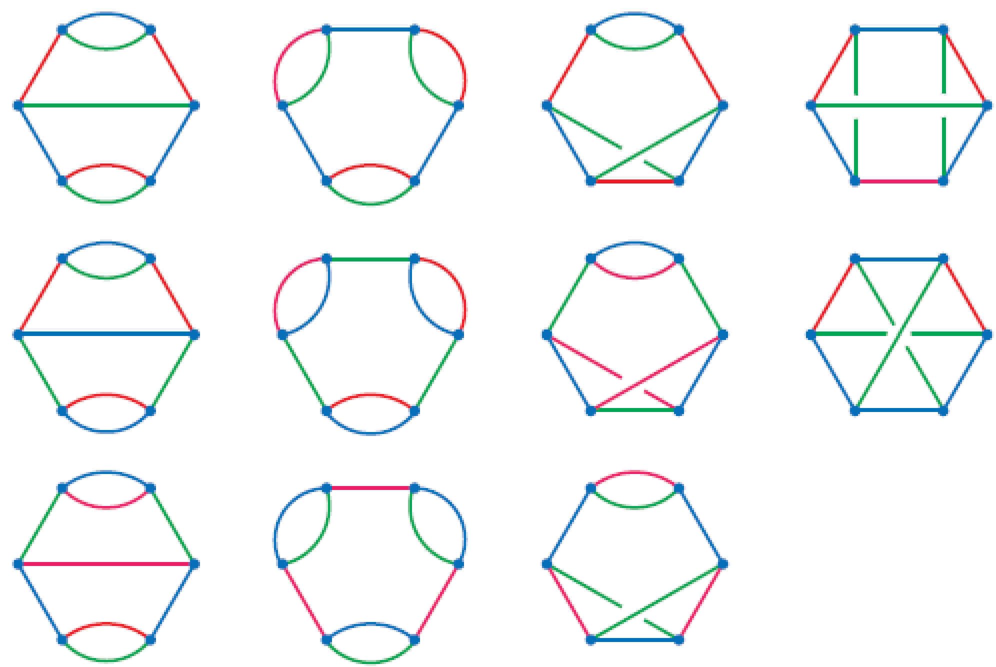
$h \approx 3.77,\ 5.68,\ 7.63,\ 9.60$ approaching $h_n \to n + \frac{1}{2}$

Gauge Invariant Operators

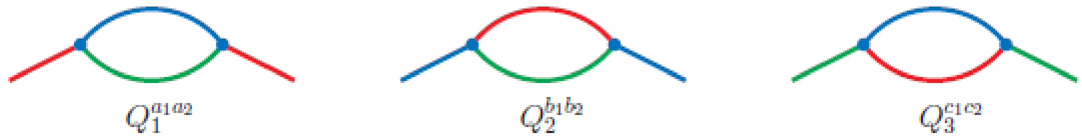
- Bilinear operators related by the EOM to some of the higher particle “single-sum” operators.



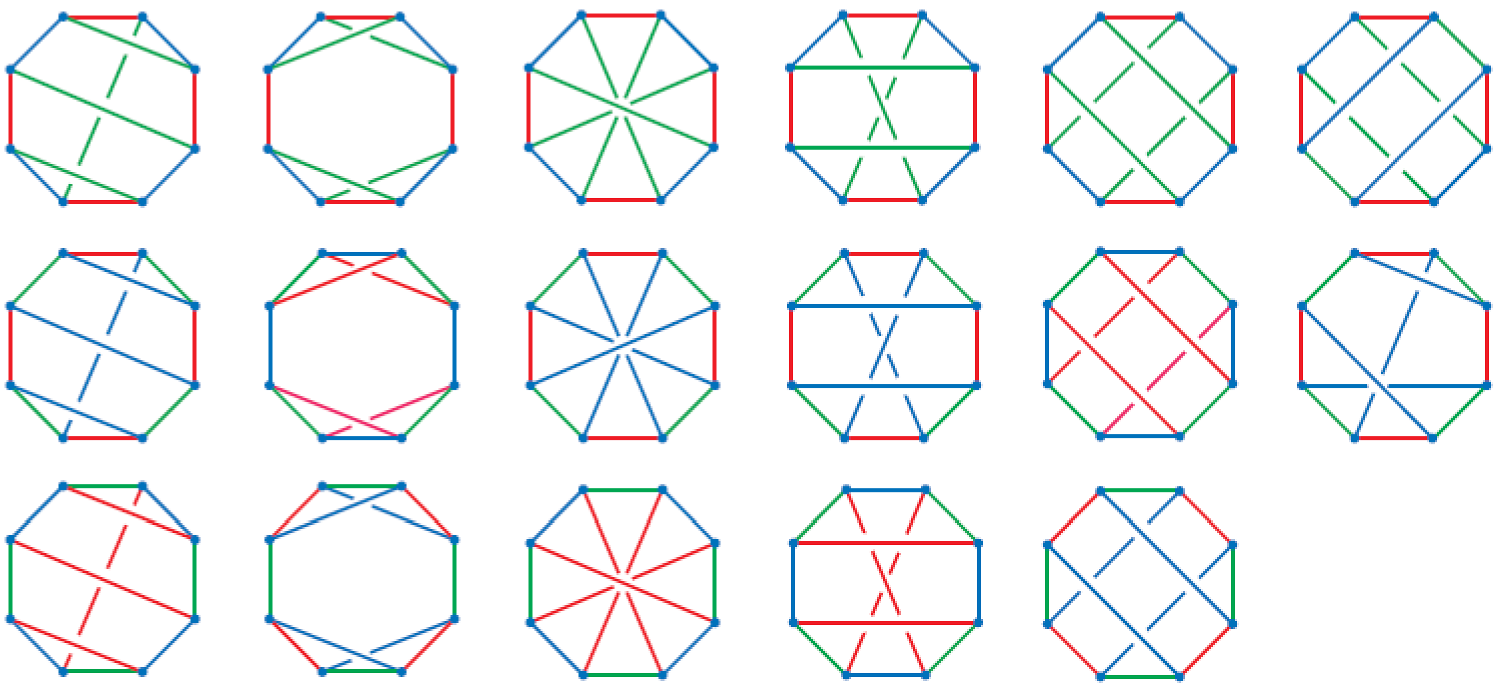
- All the 6-particle operators vanish by the Fermi statistics in the theory of one Majorana tensor



- The bubbles come from O(N) charges and vanish in the gauged model:



- The 17 single-sum 8-particle operators which do not include bubble insertions are



Factorial Growth

- There are 24 bubble-free 10-particle; 617 12-particle; 4887 14-particle; 82466 16-particle operators; etc.
- The number of (2k)-particle operators grows asymptotically as $k!\,2^k$. Bulycheva, IK, Milekhin, Tarnopolsky
- The Hagedorn temperature of the large N theory vanishes as 1/log N.
- The tensor models seem to lie "beyond string theory."
- Are they related to M-theory?

Spectra of Energy Eigenstates

- Generalize the Majorana tensor model to have $O(N_1) \times O(N_2) \times O(N_3)$ symmetry
- The traceless Hamiltonian is

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N_1N_2N_3(N_1 - N_2 + N_3)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$a = 1, \ldots, N_1;\ b = 1, \ldots, N_2;\ c = 1, \ldots, N_3$$

- The Hilbert space has dimension $2^{[N_1N_2N_3/2]}$
- Eigenstates of H form irreducible representations of the symmetry.

Complete Diagonalizations

- Generally possible only for small ranks. Krishnan, Pavan Kumar, Sanyal, Bala Subramanian, Rosa; Chaudhuri et al.; IK, Roberts, Stanford, Tarnopolsky
- For example IK, Milekhin, Popov, Tarnopolsky

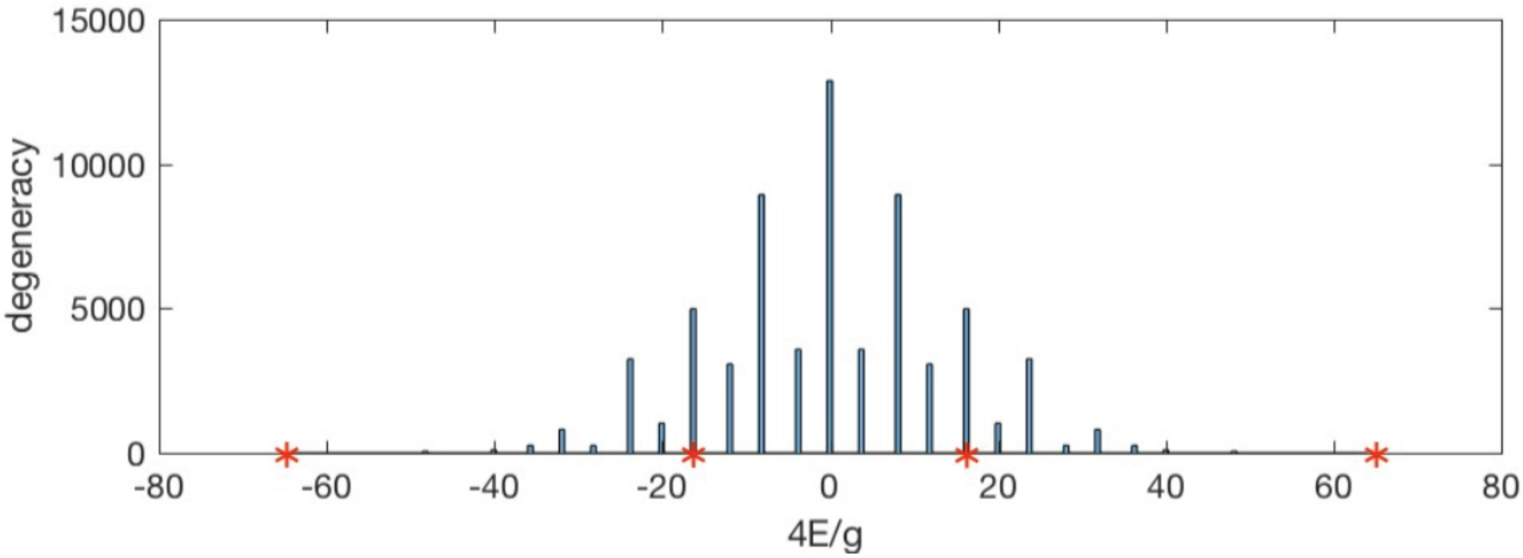


Figure 1: Spectrum of the $O(4)^2 \times O(2)$ model. There are four singlet states, and the stars mark their energies.

$\pm 16g$ and $\pm 4g$

- Spectra for N_3=2
- For the $O(2)^3$ model only two singlets at energies -2g and 2g.

(N_1, N_2)	(2,2)	(2,3)	(3,3)	(2,4)	(3,4)	(4,4)
$\frac{4}{g}E_{\text{degeneracy}}$	-8_1	-13_2	-20_6	-24_1	-34_6	-64_1
	0_{14}	-7_6	-16_{18}	-16_2	-28_{24}	-48_{55}
	8_1	-3_2	-12_{16}	-12_{16}	-24_8	-40_{106}
		-1_{22}	-8_{60}	-8_{23}	-22_{76}	-36_{256}
		1_{22}	-4_{42}	-4_{16}	-20_{40}	-32_{810}
		3_2	0_{228}	0_{140}	-18_{14}	-28_{256}
		7_6	4_{42}	4_{16}	-16_{152}	-24_{3250}
		13_2	8_{60}	8_{23}	-14_{168}	-20_{1024}
			12_{16}	12_{16}	-12_{40}	-16_{4985}
			16_{18}	16_2	-10_{170}	-12_{3072}
			20_6	24_1	-8_{240}	-8_{8932}
					-6_{194}	-4_{3584}
					-4_{384}	0_{12874}
					-2_{270}	4_{3584}
					0_{248}	8_{8932}
					2_{640}	12_{3072}
					4_{384}	16_{4985}
					6_{76}	20_{1024}
					8_{312}	24_{3250}
					10_{216}	28_{256}
					14_{32}	32_{810}
					16_{128}	36_{256}
					18_{168}	40_{106}
					20_{64}	48_{55}
					26_{10}	64_1
					28_{24}	
					30_6	
					38_2	

Energy Bounds

- The bound on the singlet ground state energy
  IK, Milekhin, Popov, Tarnopolsky

$$|E| \leq E_{bound} = \frac{g}{16} N^3 (N+2)\sqrt{N-1}$$

- In the melonic limit, this correctly scales as N^3.
- The gap to the lowest non-singlet state scales as 1/N.
- For unequal ranks the bound is

$$|E| \leq \frac{g}{16} N_1 N_2 N_3 (N_1 N_2 N_3 + N_1^2 + N_2^2 + N_3^2 - 4)^{1/2}$$

A Fermionic Matrix Model

- For N_3=2 the bound simplifies to

$$|E|_{N_3=2} \leq \frac{g}{8} N_1 N_2 (N_1 + N_2)$$

- Saturated by the ground state.
- This is a fermionic matrix model with symmetry

$$O(N_1) \times O(N_2) \times U(1)$$

$$\bar{\psi}_{ab} = \frac{1}{\sqrt{2}} \left(\psi^{ab1} + i\psi^{ab2} \right), \quad \psi_{ab} = \frac{1}{\sqrt{2}} \left(\psi^{ab1} - i\psi^{ab2} \right)$$

$$\{\bar{\psi}_{ab}, \bar{\psi}_{a'b'}\} = \{\psi_{ab}, \psi_{a'b'}\} = 0, \quad \{\bar{\psi}_{ab}, \psi_{a'b'}\} = \delta_{aa'} \delta_{bb'}$$

- The traceless Hamiltonian is

$$H = \frac{g}{2}(\bar{\psi}_{ab}\bar{\psi}_{ab'}\psi_{a'b}\psi_{a'b'} - \bar{\psi}_{ab}\bar{\psi}_{a'b}\psi_{ab'}\psi_{a'b'}) + \frac{g}{8}N_1N_2(N_2 - N_1)$$

- May be expressed in terms of quadratic Casimirs

$$-\frac{g}{2}\left(4C_2^{SU(N_1)} - C_2^{SO(N_1)} + C_2^{SO(N_2)} + \frac{2}{N_1}Q^2 + (N_2 - N_1)Q - \frac{1}{4}N_1N_2(N_1 + N_2)\right)$$

$SU(N_1) \times SU(N_2)$ is not a symmetry here but a spectrum generating algebra.

- For all N_1, N_2, the energy levels are integers in units of g/4.

Gauge Singlets

- To eliminate large degeneracies, focus on the states invariant under $SO(N_1) \times SO(N_2) \times SO(N_3)$
- Their number can be found by gauging the free theory $L = \psi^I \partial_t \psi^I + \psi^I A_{IJ} \psi^J$

$$A = A^1 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes A^2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes A^3$$

$$\#\text{singlet states} = \int d\lambda_G^N \prod_{a=1}^{M/2} 2\cos(\lambda_a/2)$$

$$d\lambda_{SO(2n)} = \prod_{i<j}^{n} \sin\left(\frac{x_i - x_j}{2}\right)^2 \sin\left(\frac{x_i + x_j}{2}\right)^2 dx_1 \ldots dx_n$$

Gauge Singlets in the $O(N)^3$ Model

- Their number vanishes for odd N due to a QM anomaly for odd numbers of flavors.
- Grows very rapidly for even N

N	# singlet states
2	2
4	36
6	595354780

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- The large low-temperature entropy suggests tiny gaps for singlet excitations ~ c^{-N^3}

Qubit Hamiltonian

- Convenient to introduce operator basis which breaks the third O(N) to U(N/2)

$$\bar{c}_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} + i\psi^{ab(2k+1)}\right), \quad c_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} - i\psi^{ab(2k+1)}\right),$$

$$\{c_{abk}, c_{a'b'k'}\} = \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0, \quad \{\bar{c}_{abk}, c_{a'b'k'}\} = \delta_{aa'}\delta_{bb'}\delta_{kk'},$$

$$a, b = 0, 1, \ldots, N-1, \text{ and } k = 0, \ldots, \tfrac{1}{2}N - 1$$

- Operators $c_{abk}, \bar{c}_{abk}$ correspond to qubit number $N^2 k + Nb + a$
- The Hamiltonian couples N/2 sets of N^2 qubits

$$H = 2\left(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'bk'}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'bk'}c_{ab'k'}c_{a'b'k}\right)$$

- The Cartan generators of U(N/2) are

$$Q_k = \sum_{a,b} \frac{1}{2}[\bar{c}_{abk}, c_{abk}] \ , \qquad k = 0, \ldots, \frac{1}{2}N - 1$$

- For the oscillator vaccuum

$$c_{abk} \left|\text{vac}\right\rangle = 0 \ , \qquad Q_k \left|\text{vac}\right\rangle = -\frac{N^2}{2} \left|\text{vac}\right\rangle$$

- The gauge singlet states appear in the sector where all these charges vanish: each set of N^2 qubits is at half filling.
- This reduces the number of states but it still grows rapidly. For N=4 there are 165636900, while for N=6 over 7.47 * 10^29

Spectrum of the Gauged N=4 Model

- Studied the system of 32=16+16 qubits

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- Needed to isolate the 36 states invariant under $SO(4)^3$ out of the 165080390 "half-half-filled" states.
- Diagonalize 4H/g + 100 C where C is the sum of three Casimir operators.
- A Lanczos type algorithm is well suited for this sparse operator.
- Find 15 distinct $SO(4)^3$ invariant energy levels: E=0 and 7 "mirror pairs" (E, -E).

Discrete Symmetries

- Act within the $SO(N)^3$ invariant sector and can lead to small degeneracies.
- Z_2 parity transformation within each group like

$$\psi^{1bc} \to -\psi^{1bc}$$

- Interchanges of the groups flip the energy

$$P_{23}\psi^{abc}P_{23} = \psi^{acb} \ , \qquad P_{12}\psi^{abc}P_{12} = \psi^{bac}$$

$$P_{23}HP_{23} = -H \ , \qquad P_{12}HP_{12} = -H$$

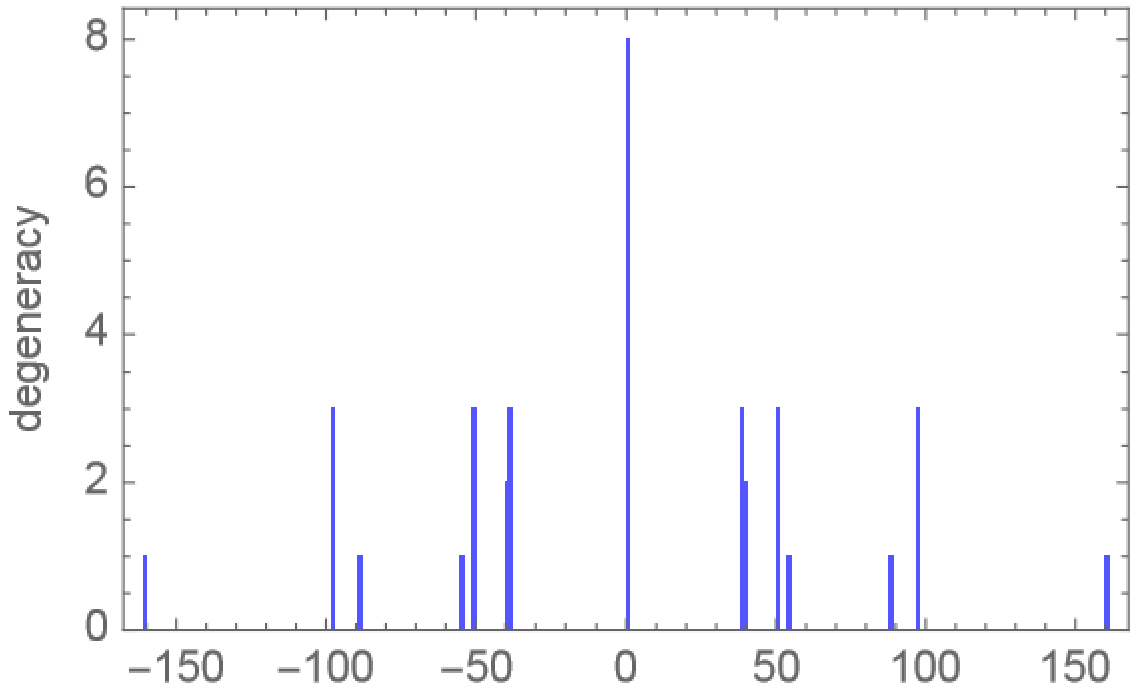
- Z_3 symmetry generated by $P = P_{12}P_{23}$, $\quad P^3 = 1$

$$P\psi^{abc}P^\dagger = \psi^{cab} \ , \qquad PHP^\dagger = H$$

- At non-zero energy the gauge singlet states transform under the group $A_4 \times Z_2$.
- The 36 states are labeled by E and the three parities

E	P_1	P_2	P_3	E	P_1	P_2	P_3
−160.140170	1	1	1	160.140170	1	1	1
−97.019491	1	1	−1	97.019491	1	1	−1
−97.019491	−1	1	1	97.019491	−1	1	1
−97.019491	1	−1	1	97.019491	1	−1	1
−88.724292	−1	−1	−1	88.724292	−1	−1	−1
−54.434603	1	1	1	54.434603	1	1	1
−50.549167	1	1	−1	50.549167	1	1	−1
−50.549167	−1	1	1	50.549167	−1	1	1
−50.549167	1	−1	1	50.549167	1	−1	1
−39.191836	1	1	1	39.191836	1	1	1
−39.191836	1	1	1	39.191836	1	1	1
−38.366652	1	−1	−1	38.366652	1	−1	−1
−38.366652	−1	1	−1	38.366652	−1	1	−1
−38.366652	−1	−1	1	38.366652	−1	−1	1
0.000000	1	1	1	0.000000	−1	−1	−1
0.000000	−1	1	1	0.000000	1	−1	−1
0.000000	1	−1	1	0.000000	−1	1	−1
0.000000	1	1	−1	0.000000	−1	−1	1

Energy Distribution for N=4



- For N=6 there will be over 595 million states packed into energy interval <1932. So, the gaps will be tiny.

Exact Eigenvalues

- The maximum degeneracy at non-zero energy is 3.
- The results were so precise that they allowed us to deduce the exact expressions in terms of square root.
- The ground state is non-degenerate and has energy in units of g/4

$$E_0 = -\sqrt{32\left(447+\sqrt{125601}\right)}$$

- It is not far from our lower bound -166.277

Complex Tensor Model

- The action

$$S = \int dt \Big(i\bar{\psi}^{abc}\partial_t\psi^{abc} + \frac{1}{4} g \psi^{a_1b_1c_1}\bar{\psi}^{a_1b_2c_2}\psi^{a_2b_1c_2}\bar{\psi}^{a_2b_2c_1} \Big)$$

has SU(N)xO(N)xSU(N)xU(1) symmetry.

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- Gauge invariant two-particle operators

$$\mathcal{O}_2^n = \bar{\psi}^{abc}(D_t^n\psi)^{abc} \qquad n = 0, 1, \ldots$$

including $\bar{\psi}^{abc}\psi^{abc}$

Spectrum of two-particle operators

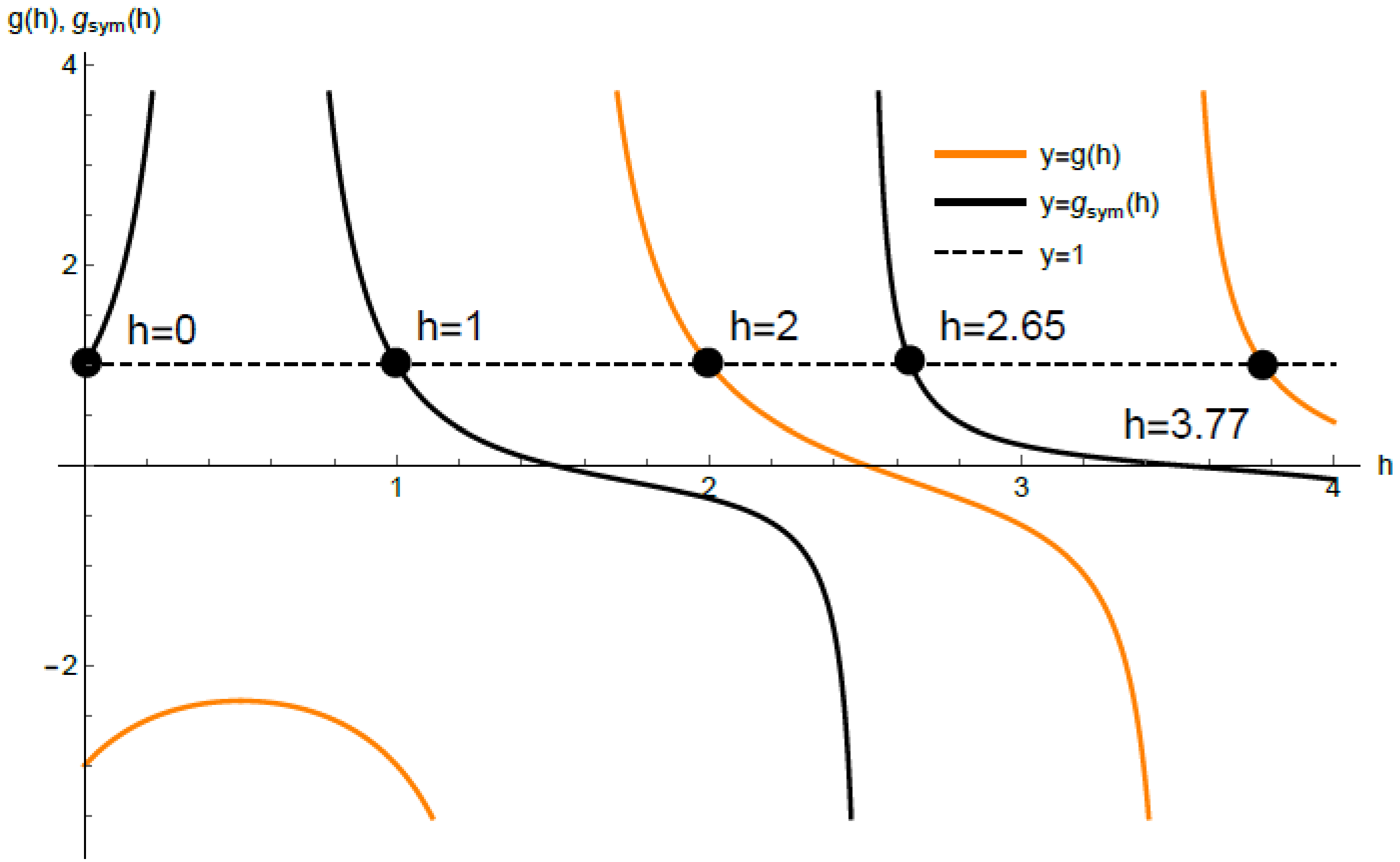
- The integral equation also admits symmetric solutions

$$v(t_1,t_2)=\frac{1}{|t_1-t_2|^{1/2-h}}$$

- Calculating the integrals we get

$$g_{\mathrm{sym}}(h)=-\frac{1}{4\pi}l^{-}_{\frac{3}{2}-h,\frac{1}{2}}l^{+}_{1-h,\frac{1}{2}}=-\frac{1}{2}\frac{\tan(\frac{\pi}{2}(h+\frac{1}{2}))}{h-1/2}$$

- The first solution is h=1 corresponding to U(1) charge $\bar{\psi}^{abc}\psi^{abc}$



- The additional scaling dimensions

$$h \approx 2.65,\ 4.58,\ 6.55,\ 8.54$$

approach $$h_n = n + \frac{1}{2} + \frac{1}{\pi n} + \mathcal{O}(n^{-3})$$

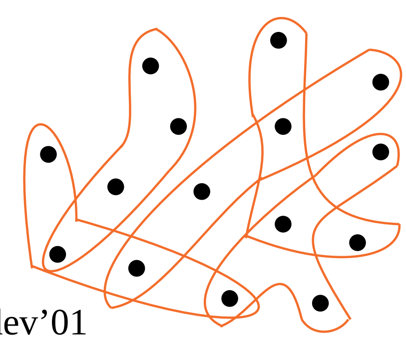
Sachdev-Ye-Kitaev Model

$$H = \frac{1}{4!} \sum_{i_1, i_2, i_3, i_4 = 1}^{N} J_{i_1 i_2 i_3 i_4} \chi_{i_1} \chi_{i_2} \chi_{i_3} \chi_{i_4}$$

- Majorana fermions $\{\chi_i, \chi_j\} = \delta_{ij}$
- $J_{i_1 i_2 i_3 i_4}$ are Gaussian random

$$\langle J^2_{i_1 i_2 i_3 i_4} \rangle = 3! \frac{J^2}{N^3} \quad \langle J_{i_1 i_2 i_3 i_4} \rangle = 0$$

- Has $O(N_{SYK})$ symmetry after averaging over disorder



Sachdev, Ye '93,
Georges, Parcollet, Sachdev'01
Kitaev '15

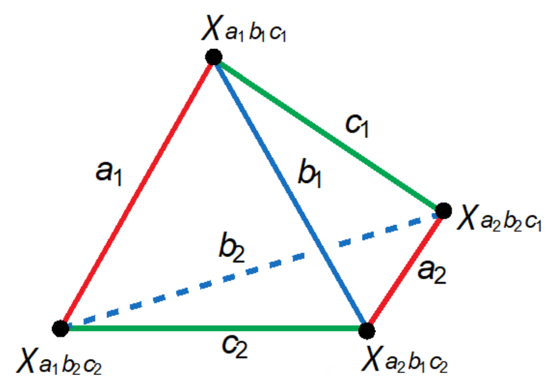
$O(N)^3$ Tensor Model

$$H = \frac{1}{4} \sum_{a_1, \ldots, c_2 = 1}^{N} \frac{J}{N^{3/2}} \chi_{a_1 b_1 c_1} \chi_{a_1 b_2 c_2} \chi_{a_2 b_1 c_2} \chi_{a_2 b_2 c_1}$$

- Majorana fermions

$$\{\chi_{abc}, \chi_{a'b'c'}\} = \delta_{aa'} \delta_{bb'} \delta_{cc'}$$

- No disorder
- Has $O(N)_a$ x $O(N)_b$ x $O(N)_c$ symmetry



IK, Tarnopolsky'16

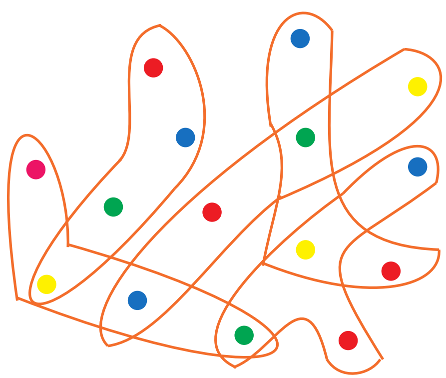
Gross-Rosenhaus Model q=4, f=4

$$H = \sum_{i_1,i_2,i_3,i_4=1}^{N} J_{i_1 i_2 i_3 i_4} \chi^0_{i_1} \chi^1_{i_2} \chi^2_{i_3} \chi^3_{i_4}$$

- Majorana fermions $\{\chi^a_i, \chi^b_j\} = \delta_{ij}\delta^{ab}$
- $J_{i_1 i_2 i_3 i_4}$ are Gaussian random

$$\langle J^2_{i_1 i_2 i_3 i_4} \rangle = 4^4 \frac{J^2}{N^3} \quad \langle J_{i_1 i_2 i_3 i_4} \rangle = 0$$

- Has O(N_{SYK}) x O(N_{SYK}) x
- O(N_{SYK}) x O(N_{SYK}) symmetry



Gross, Rosenhaus' 16

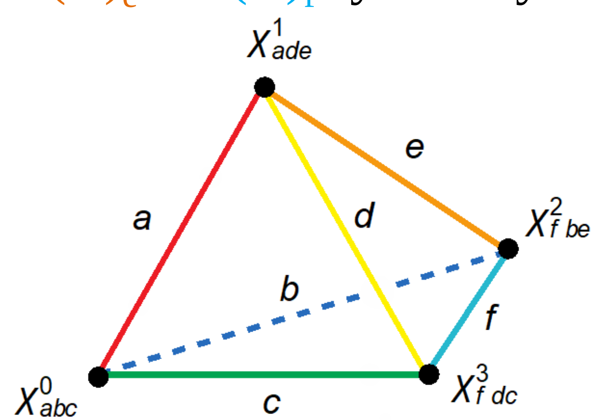
Gurau-Witten Model

$$H = \sum_{a,\dots,f=1}^{N} \frac{J}{N^{3/2}} \chi^0_{abc} \chi^1_{ade} \chi^2_{fbe} \chi^3_{fdc}$$

- Majorana fermions

$$\{\chi^A_{abc}, \chi^B_{a'b'c'}\} = \delta_{aa'}\delta_{bb'}\delta_{cc'}\delta^{AB}$$

- No disorder
- Has $O(N)_a$ x $O(N)_b$ x $O(N)_c$ x $O(N)_d$ x $O(N)_e$ x $O(N)_f$ symmetry



Gurau '10
Witten'16

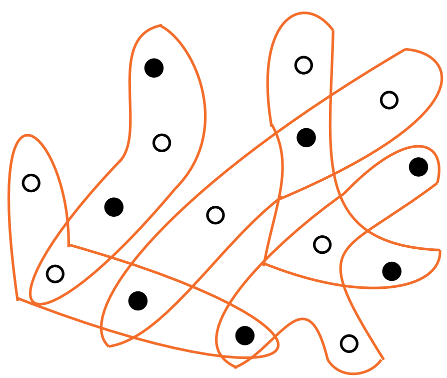
Complex SYK Model

$$H = \frac{1}{4!} \sum_{i_1,i_2,i_3,i_4=1}^{N} J_{i_1 i_2 i_3 i_4} \chi^\dagger_{i_1} \chi^\dagger_{i_2} \chi_{i_3} \chi_{i_4}$$

- Complex fermions $\{\chi_i, \chi_j^\dagger\} = \delta_{ij}$
- $J_{i_1 i_2 i_3 i_4}$ are Gaussian random

$$\langle J^2_{i_1 i_2 i_3 i_4} \rangle = 3! \frac{J^2}{N^3} \quad \langle J_{i_1 i_2 i_3 i_4} \rangle = 0$$

- Has $U(N_{SYK})$ symmetry after averaging over disorder



Sachdev '15

Davison, Fu, Gu, Georges, Jensen, Sachdev '16

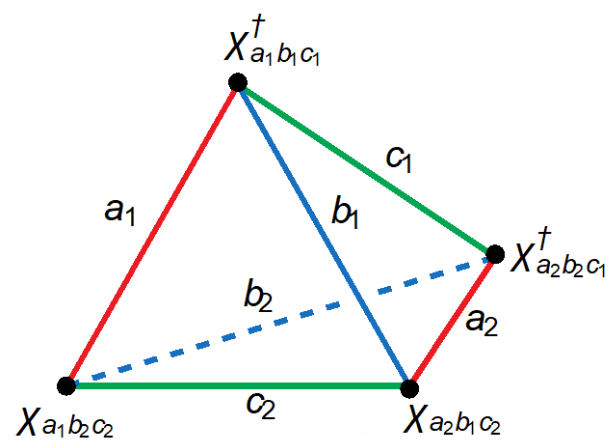
Complex Tensor Model

$$H = \frac{1}{4} \sum_{a_1,\ldots,c_2=1}^{N} \frac{J}{N^{3/2}} \chi^\dagger_{a_1 b_1 c_1} \chi^\dagger_{a_2 b_2 c_1} \chi_{a_1 b_2 c_2} \chi_{a_2 b_1 c_2}$$

- Complex fermions

$$\{\chi_{abc}, \chi^\dagger_{a'b'c'}\} = \delta_{aa'} \delta_{bb'} \delta_{cc'}$$

- Has $SU(N)_a$ x $SU(N)_b$ x $O(N)_c$ x $U(1)$ symmetry and no disorder



IK, GT'16

Conclusions

- The vector and matrix large N limits have been used extensively for many years in various theoretical physics problems.
- The tensor large N limits for rank 3 and higher are relatively new.
- The $O(N)^3$ fermionic tensor quantum mechanics seems to be the closest counterpart of the basic SYK model for Majorana fermions. Yet, there are some important differences between the two.

- Gauging the $SO(N)^3$ symmetry leaves interesting spectra of operators and eigenstates.
- Found the complete spectrum of the gauged N=4 model, where there are 36 states.
- Energy gaps should become very small already for N=6, where there are over 595 million states.

- Vector: CFTs are dual to higher spin quantum gravity in AdS; e.g. the O(N) Wilson-Fisher Model coupled to Chern-Simons is dual to the Vasiliev theory in AdS_4. One Regge trajectory.
- Matrix: $\mathcal{N}$=4 Super-Yang-Mills is dual string theory on AdS_5 x S^5. An infinite number of Regge trajectories.
- Tensor: Vastly more operators than in the matrix case. Hagedorn temperature vanishes for large N.

What quantum gravity theories are they dual to?