A 2D Perspective on the SYK model

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KITP -- Chaos and Order program

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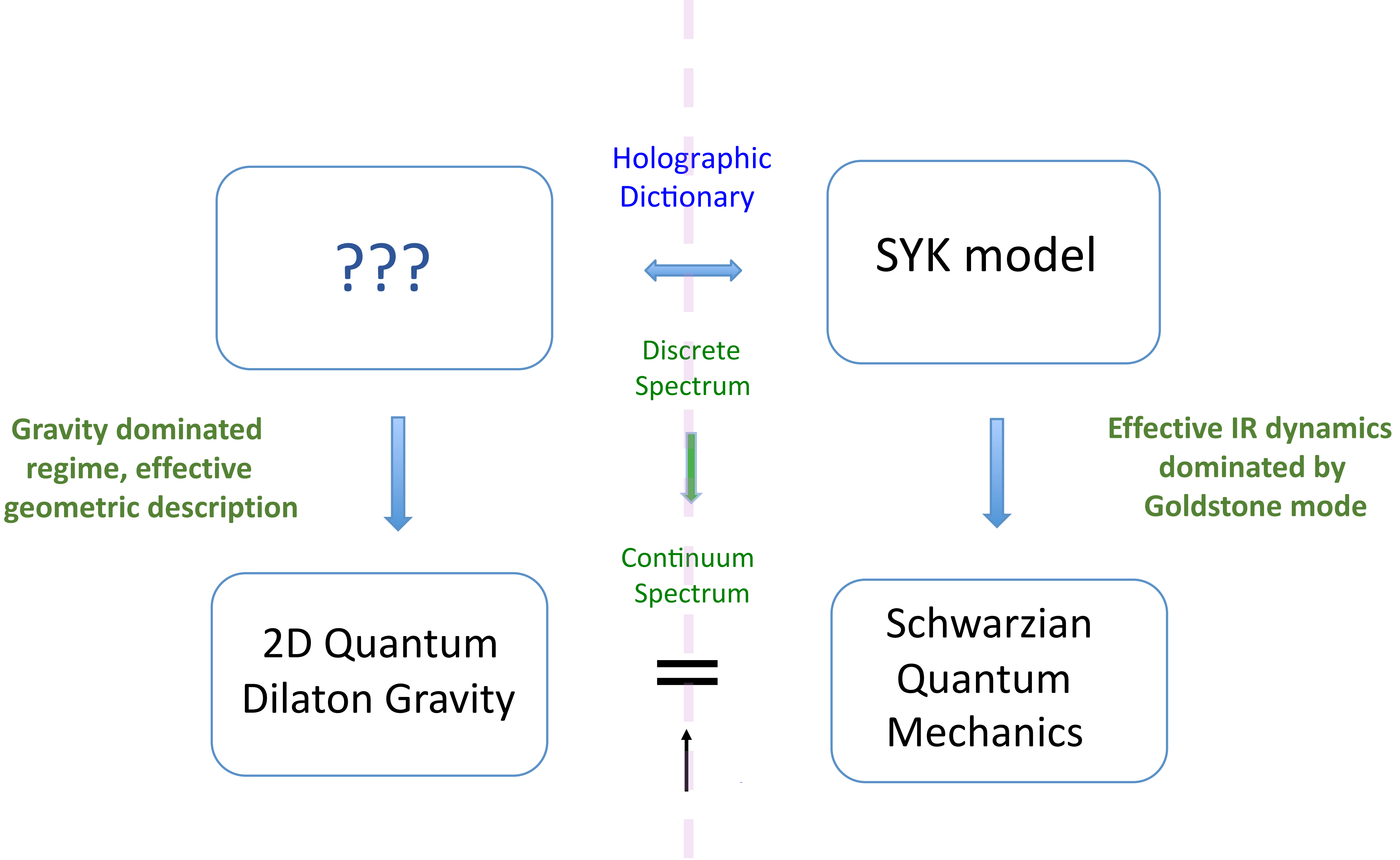
Based on: arXiv:1705.08408, T. Mertens, J. Turiaci, HV + work in progress

Anticipated in: arXix:1412.5205, with S. Jackson, L. McGough, HV NPB 901 (2015) 382


Holographic
Dictionary
???
SYK model
Discrete
Spectrum
Gravity dominated
regime, effective
geometric description
Effective IR dynamics
dominated by
Goldstone mode
Continuum
Spectrum
2D Quantum
Dilaton Gravity
=
Schwarzian
Quantum
Mechanics

Low dimensional holography

SYK model ←→ 2D dilaton gravity

$$S_{2D} = \int d^2x \sqrt{-g}\, \Phi(R+\Lambda) \; + \; S_{\text{matter}}$$

Almheiri, Polchinski; Jensen; Maldacena, Stanford, Yang; Engelsoy, Mertens, HV; Kitaev

equivalent to:

charged particle
on hyperbolic plane
w/ constant B-field

SYK model = 1D many body QM with maximal chaos

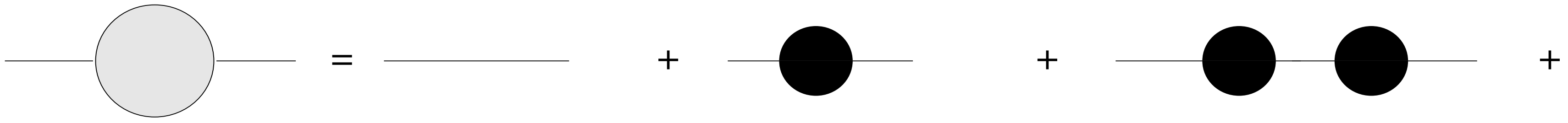
$$H = \sum_{ijk\ell} J_{ijk\ell}\psi^i\psi^j\psi^k\psi^\ell$$

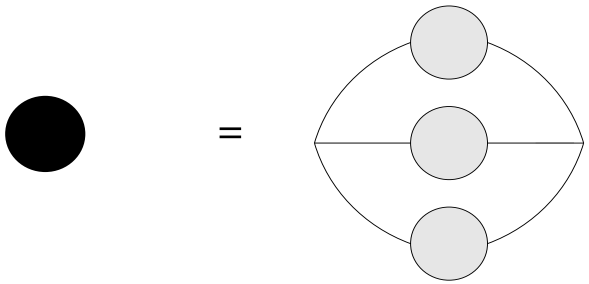
random couplings

$$\{\psi^i, \psi^j\} = \delta^{ij}$$

N majorana variables

$$G(\tau_1, \tau_2) \equiv \frac{1}{N}\sum_i \langle \psi_i(\tau_1)\psi_i(\tau_2)\rangle,$$





Large N limit of SD equations = soluble

Dominated by `pumpkinic' diagrams

Dynamical Mean Field Theory

$$-S_E/N = \frac{1}{2}\mathrm{Tr}\log\left(\partial_\tau - \Sigma\right) - \frac{1}{2}\int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{\mathcal{J}^2}{q^2}G(\tau_1,\tau_2)^q\right]$$

at large q reduces to

$$G(\tau_1,\tau_2) \equiv \frac{1}{N}\sum_i \langle\psi_i(\tau_1)\psi_i(\tau_2)\rangle = \frac{\mathrm{sgn}(\tau_{12})}{2}\left(1 + \frac{1}{q}g(\tau_1,\tau_2)\right)$$

Liouville CFT on kinematic space!

$$S_{\mathrm{eff}} = \frac{N}{8q^2}\int d\tau_1 d\tau_2 \left[\partial_{\tau_1}g\partial_{\tau_2}g - 4\mathcal{J}^2 \exp g(\tau_1,\tau_2)\right].$$

$$c = \frac{12\pi N}{q^2}.$$

IR limit of SD equations

$$\int d\tau' G(\tau,\tau')\Sigma(\tau',\tau'') = -\delta(\tau-\tau'') \ , \qquad \Sigma(\tau,\tau') = J^2 \left[G(\tau,\tau')\right]^{q-1}$$

are invariant under 1D diffeomorphisms

$$G(\tau,\tau') \to \left[f'(\tau)f'(\tau')\right]^{\Delta} G(f(\tau),f(\tau')) \ , \qquad \Sigma(\tau,\tau') \to \left[f'(\tau)f'(\tau')\right]^{\Delta(q-1)} \Sigma(f(\tau),f(\tau'))$$

→ IR effective theory is dominated by a dynamical

Goldstone mode = 1D reparametrizations $f(\tau)$

$$S[f] = -C\int_0^\beta d\tau\left(\{f,\tau\} + \frac{2\pi^2}{\beta^2}f'^2\right) \qquad \{f,\tau\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

$$= -C\int_0^\beta d\tau\,\{F,\tau\}, \qquad F \equiv \tan\left(\frac{\pi f(\tau)}{\beta}\right) \qquad F \to \frac{aF+b}{cF+d}$$

Schwarzian QM = exactly solvable

➔

should be able to compute anything we want!

Canonical formulation:

Engelsoy, Mertens, HV

$$L = \pi_\phi \dot{\phi} + \pi_f \dot{f} - (\pi_\phi^2 + \pi_f e^\phi)$$

$$[f, \pi_f] = i$$

$$[\phi, \pi_\phi] = i$$

SL(2,R) symmetry: $f \to \frac{af+b}{cf+d}$ → generators $[\ell_a, \ell_b] = i\epsilon_{abc}\ell_c$

Hamiltonian = Casimir:

$$H = \pi_\phi^2 + \pi_f e^\phi = \ell_0^2 - \tfrac{1}{2}\{\ell_{-1}, \ell_1\}$$

$$j = -\tfrac{1}{2} + ik \qquad E(k) = -j(j+1) = \frac{1}{4} + k^2$$

$$Z(\beta) \;=\; \int_{\mathcal{M}} \mathcal{D}f\, e^{-S[f]}$$

Partition function

$$\mathcal{M} \;=\; \mathrm{Diff}(S^1)/SL(2,\mathbb{R})$$

integral over energy E = ¼ + k^2

with continuous spectral density

$$\rho(E) = \sinh\!\Big(2\pi\sqrt{E-1/4}\Big)$$

Stanford, Witten

$$Z(\beta) \;=\; \int_0^\infty d\mu(k)\, e^{-\beta E(k)}, \qquad d\mu(k) = dk^2 \sinh(2\pi k).$$

Partition function = integral over a symplectic manifold ← can be quantized!

$$Z(\beta) = \int_{\mathcal{M}} \mathcal{D}f \, e^{-S[f]}$$

Identity representation

$$= \lim_{\substack{c\to\infty \\ q\to 1}} \mathrm{Tr}\left(q^{L_0}\right),$$

$$q^{\frac{c}{24}} = e^{-\frac{\pi^2}{\beta}} = \text{fixed}.$$

$$= e^{S_0+\beta E_0} \left(\frac{\pi}{\beta}\right)^{3/2} \exp\left(\frac{\pi^2}{\beta}\right).$$

$$\mathcal{M} = \mathrm{Diff}(S^1)/SL(2,\mathbb{R})$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m}$$

$$L_n = \frac{\beta c}{48\pi^2} \int_0^\beta d\tau \, e^{2\pi i n\tau/\beta} \{F,\tau\}.$$

Identity character

$$\mathrm{Tr}\left(q^{L_0}\right) \equiv \chi_0(q) = \frac{q^{\frac{1-c}{24}}(1-q)}{\eta(\tau)}$$

This is an exact result

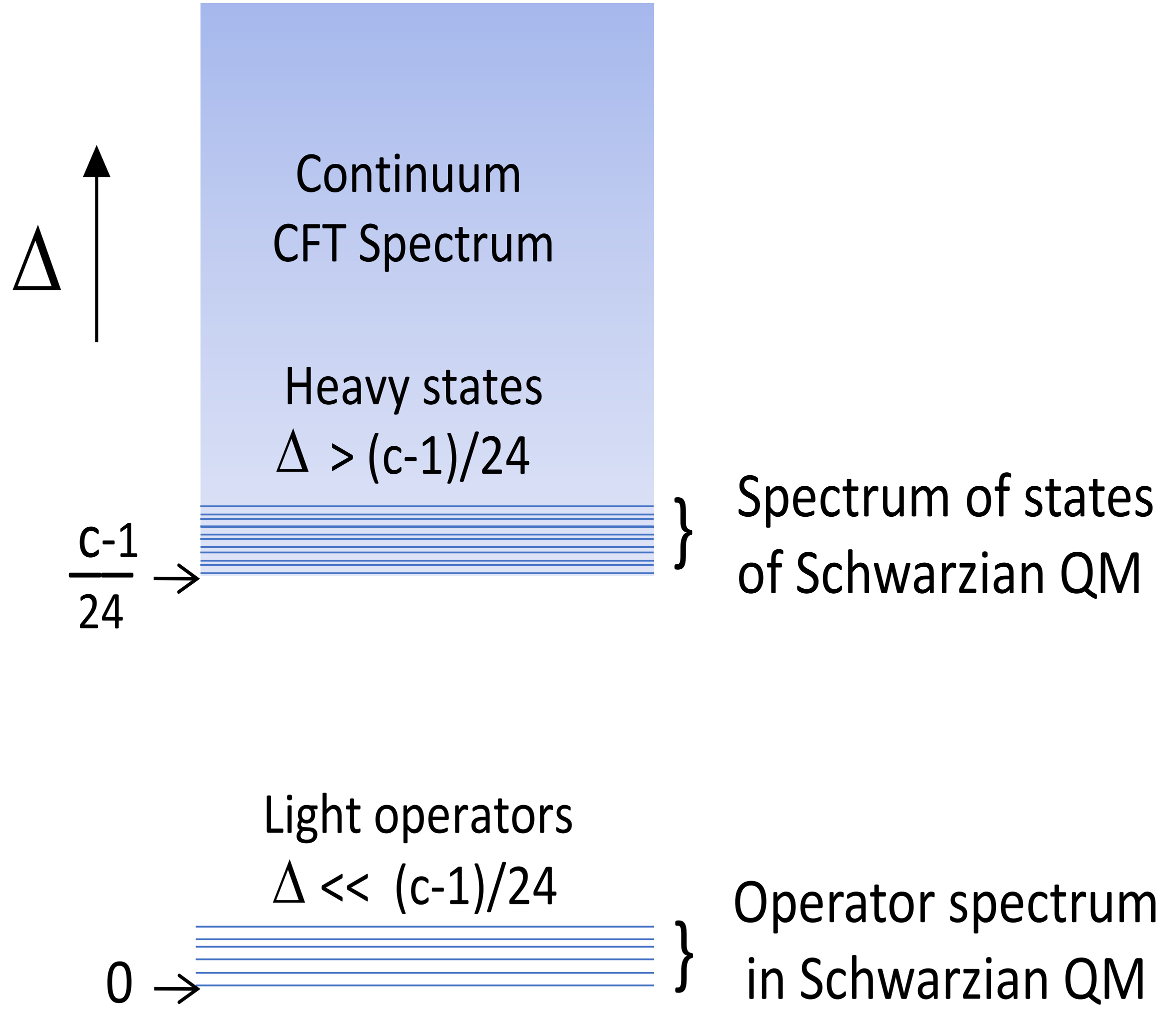
c.f. Stanford, Witten
Bagrets, Altland, Kamenev

$$\chi_0(q) = \int_0^\infty dP\ S_0^P\ \chi_P(\tilde{q})$$

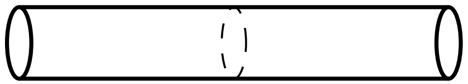
$$S_0^P = 4\sqrt{2}\sinh(2\pi bP)\sinh\left(\frac{2\pi P}{b}\right).$$

$$c = 1 + 6Q^2 = 1 + 6\left(b + b^{-1}\right)^2$$

$$\Delta(P) = \frac{Q^2}{4} + P^2$$

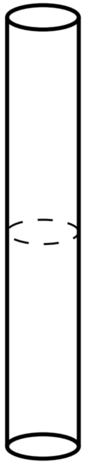


$$\mathrm{Tr}\left(q^{L_0}\right) \; =$$



$$\longrightarrow$$

$\langle \mathrm{ZZ}|$



$|\mathrm{ZZ}\rangle$

$$= \; \langle ZZ|\, \tilde{q}^{L_0}\, |ZZ\rangle$$

$$q \;\to\; 1 \qquad\qquad \tilde{q} \to 0$$

- Boundary State:

$$|ZZ\rangle = \int_0^\infty dP \;\Psi_{\mathrm{ZZ}}(P)\; ||P\rangle\rangle \qquad\qquad |\Psi_{\mathrm{ZZ}}(P)|^2 = S_0^P$$

- Schwarzian Limit:

$$||P\rangle\rangle \to |P\rangle$$

- $b \to 0$
- $P = kb$

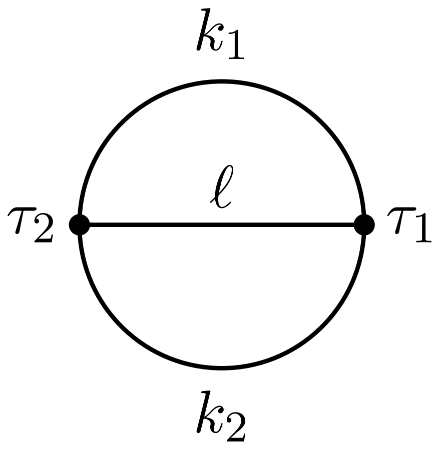
$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{D}f \, e^{-S[f]} \, \mathcal{O}_1 \dots \mathcal{O}_n$$

Correlation functions

$$\mathcal{O}_\ell(\tau_1, \tau_2) \equiv \left(\frac{\sqrt{f'(\tau_1) f'(\tau_2)}}{\frac{\beta}{\pi} \sin \frac{\pi}{\beta} [f(\tau_1) - f(\tau_2)]} \right)^{2\ell}$$

Two-point function

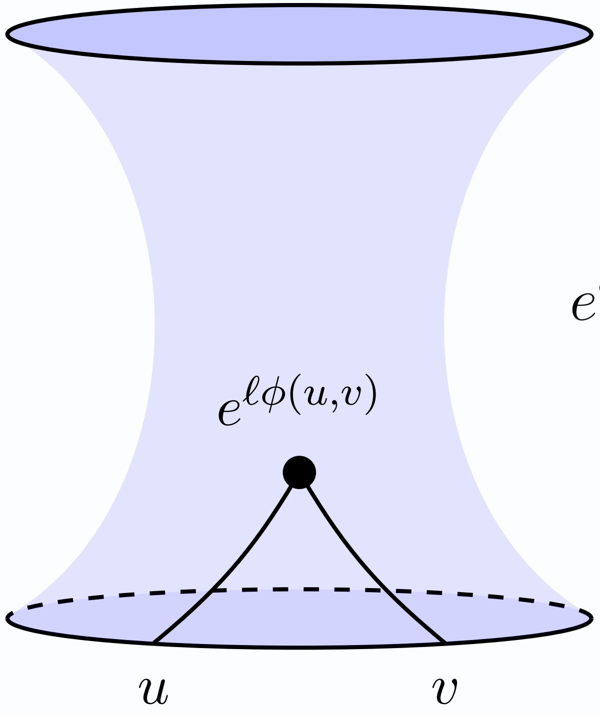
$$\langle \mathcal{O}_\ell(\tau_1, \tau_2) \rangle = \int \prod_{i=1}^{2} d\mu(k_i) \; \mathcal{A}_2(k_i, \ell, \tau_i).$$



Liouville theory on hyperbolic cylinder → reduces to dilaton gravity for c → ∞

$$S = \frac{c}{192\pi}\int d\tau \int_0^{\pi} d\sigma \,\left[(\partial\phi)^2 + 4\mu e^{2\phi}\right]$$

$$\partial_u\partial_v\phi(u,v) = e^{2\phi(u,v)}.$$



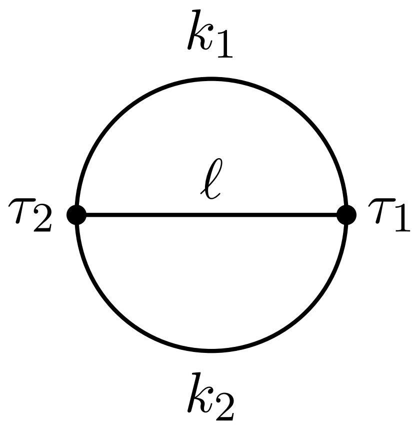
$$e^{\phi_{cl}(u,v)} = \frac{\sqrt{f'(u)f'(v)}}{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}[f(u)-f(v)]}.$$

Insertion of $\mathcal{O}_\ell(\tau_1,\tau_2)$ in Schwarzian $\leftrightarrow$ Insertion of $V_\ell = e^{2\ell\phi(\tau_1,\tau_2)}$ in Liouville CFT

Two point function

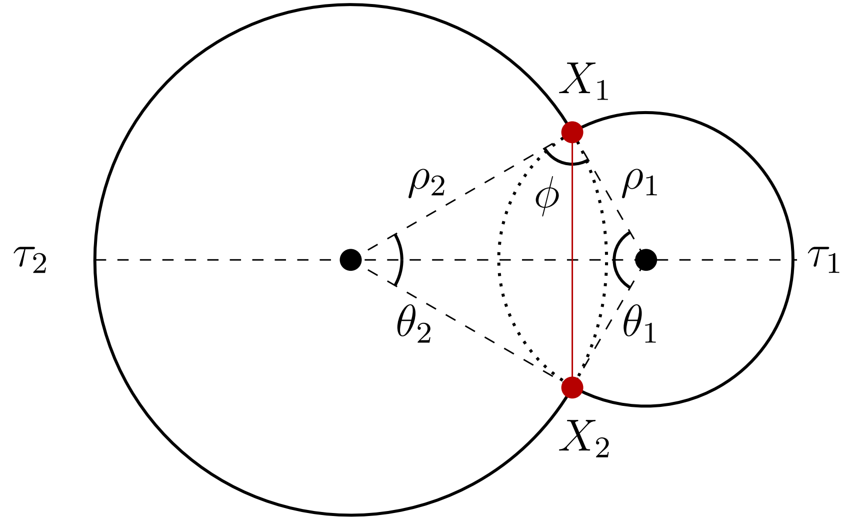
$$\langle \mathcal{O}_\ell(\tau_1, \tau_2) \rangle = \int \prod_{i=1}^{2} d\mu(k_i)\ \mathcal{A}_2(k_i, \ell, \tau_i).$$

$$\mathcal{A}_2(k_i, \ell, \tau_i) =$$



$$\mathcal{A}_2(k_i, \ell, \tau_i) = e^{-(\tau_2-\tau_1)k_1^2-(\beta-\tau_2+\tau_1)k_2^2}\ \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}$$

Mertens, Turiaci, HV
c.f. Bagrets et al



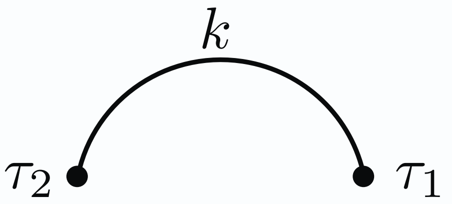
Semi-classical interpretation of two-point function

$$\begin{aligned}\langle \mathcal{O}(\tau)\mathcal{O}(0)\rangle_\beta &= \int \prod_{i=1,2} dk_i \rho(k_i)\ e^{-\frac{k_1^2}{2C}\tau-\frac{k_2^2}{2C}(\beta-\tau)} \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}, \\ &= \int \prod_{i=1,2} dk_i d\theta_i\ e^{-I(k_i,\theta_i,\tau,\ell)},\end{aligned}$$

where the 'action' appearing in the exponent is given by

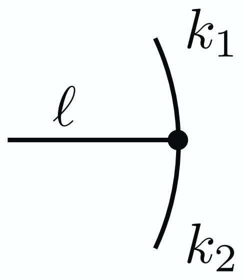
$$I(k_i,\theta_i,\tau,\ell) = \sum_{i=1,2}\left(\frac{k_i^2}{2C}\tau_i + \theta_i k_i - \log\rho(k_i)\right) + \ell\log\left(\cos\frac{\theta_1}{2}+\cos\frac{\theta_2}{2}\right)^2 + I_0(\ell)$$

The exact non-perturbative answer for the 2n-point functions can be summarized via a simple set of diagrammatic rules:



$$= \quad e^{-\,k^2\,(\tau_2-\tau_1)}$$

\`propagator’



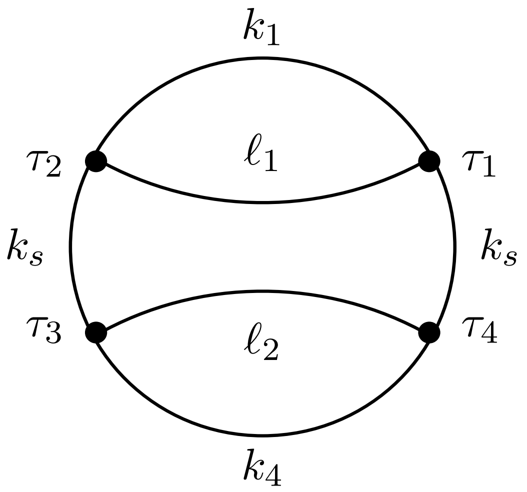
$$= \quad \gamma_\ell(k_1, k_2)$$

\`vertex’

$$\gamma_\ell(k_1, k_2) \; = \; \sqrt{\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}}.$$

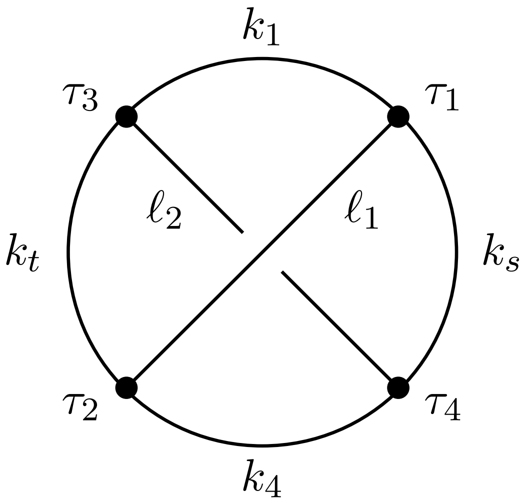
Four-point function

$$\left\langle \mathcal{O}_{\ell_1}(\tau_1,\tau_2)\,\mathcal{O}_{\ell_2}(\tau_3,\tau_4)\right\rangle \;=$$

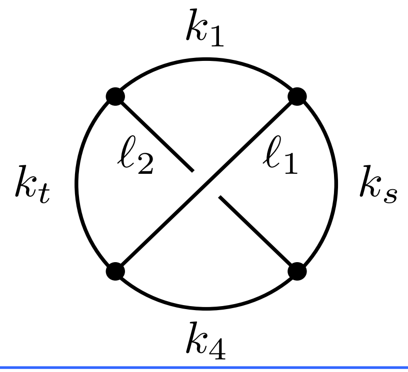


OTO four-point function

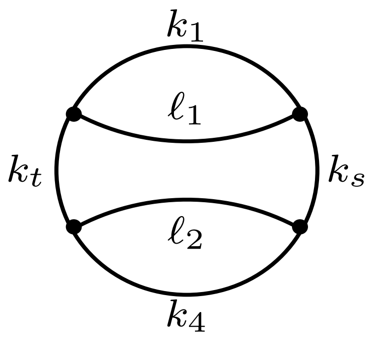
$$\left\langle \mathcal{O}_{\ell_1}(\tau_1,\tau_2)\,\mathcal{O}_{\ell_2}(\tau_3,\tau_4)\right\rangle_{\mathrm{OTO}} \;=$$



R-matrix



$$= R_{k_s k_t}\left[\begin{smallmatrix} k_4 & \ell_2 \\ k_1 & \ell_1 \end{smallmatrix}\right]$$



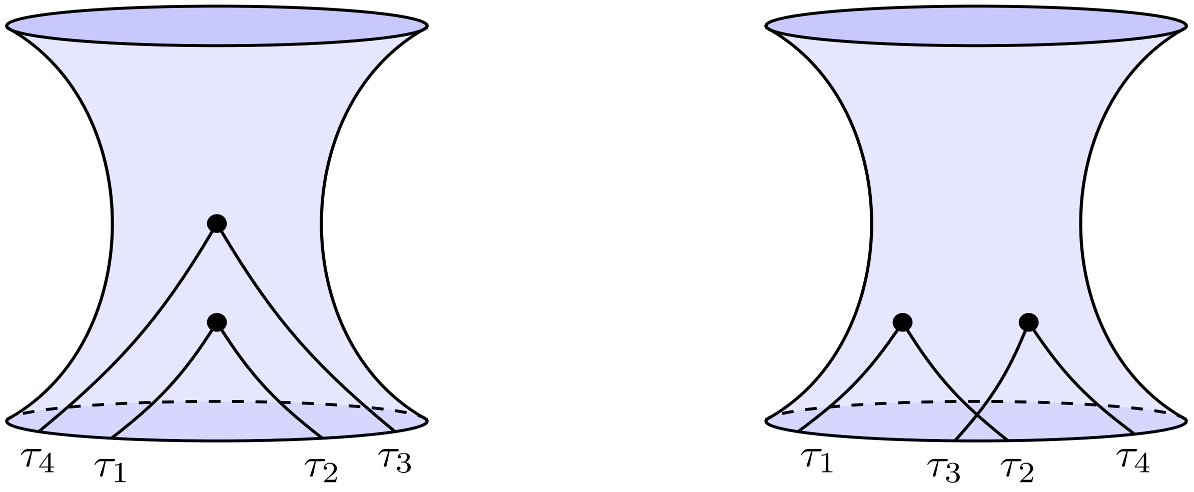
The R-matrix of the Schwarzian is found to be equal to a classical 6j-symbol of SU(1,1)

$$R_{k_s k_t}\left[\begin{smallmatrix} k_4 & \ell_2 \\ k_1 & \ell_1 \end{smallmatrix}\right] = \begin{Bmatrix} \ell_1 & k_4 & k_s \\ \ell_2 & k_1 & k_t \end{Bmatrix} = \sqrt{\Gamma(\ell_1 \pm ik_2 \pm ik_s)\Gamma(\ell_3 \pm ik_2 \pm ik_t)\Gamma(\ell_1 \pm ik_4 \pm ik_t)\Gamma(\ell_3 \pm ik_4 \pm ik_s)}$$
$$\times\, \mathbb{W}(k_s, k_t; \ell_1 + ik_4, \ell_1 - ik_4, \ell_3 - ik_2, \ell_3 + ik_2),$$

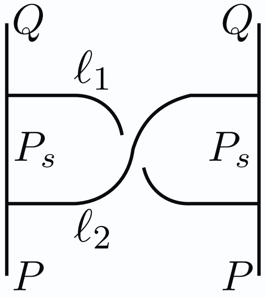
$\mathbb{W}$ = Wilson function
linear combination of ${}_4F_3$

Groenevelt

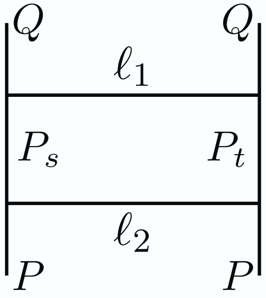
Matches with the gravitational shockwave amplitude

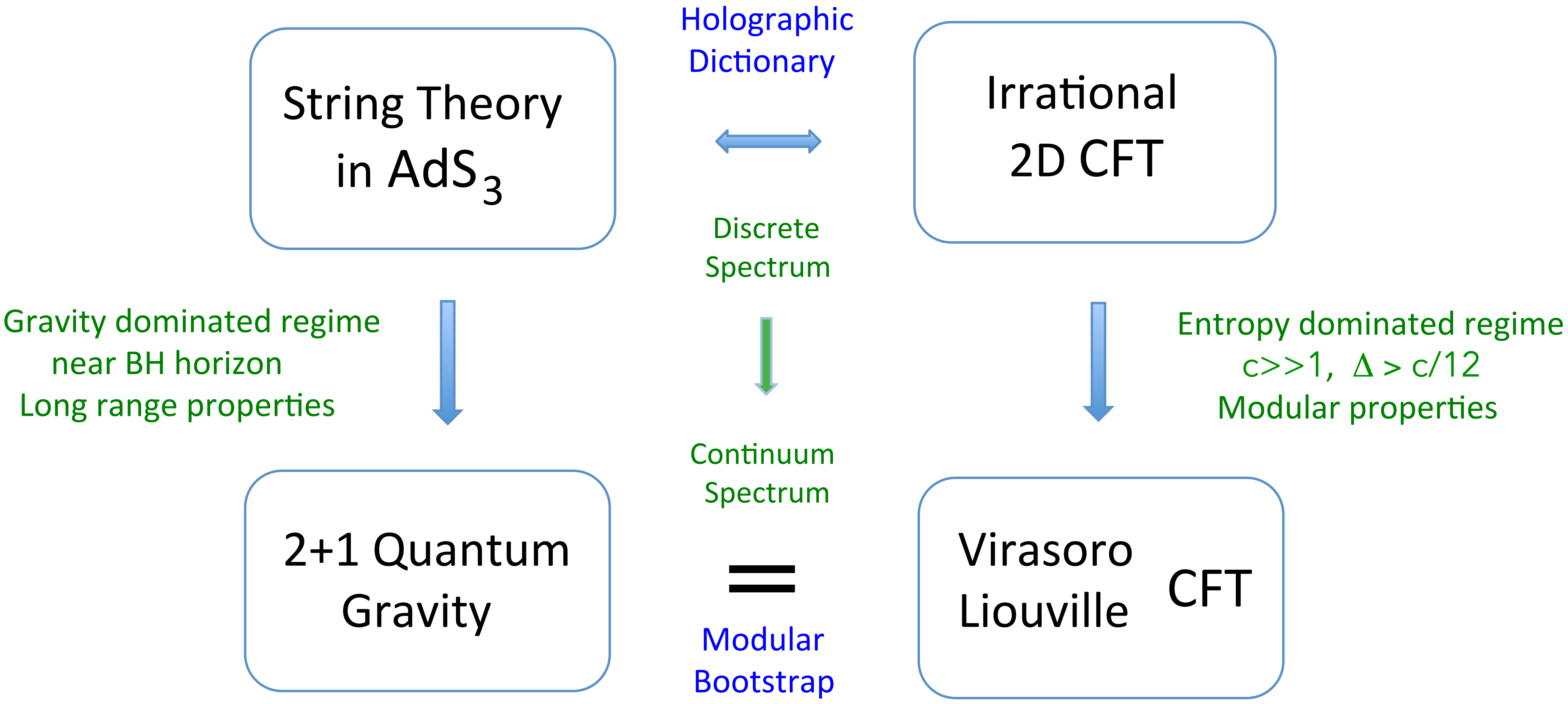


$$G^{\text{OTO}}_{\ell_1\ell_2} = \int dP dQ\ \Psi^\dagger_{\text{ZZ}}(P)\Psi_{\text{ZZ}}(Q) \times \int dP_s$$



$$= \int dP dQ\ \Psi^\dagger_{\text{ZZ}}(P)\Psi_{\text{ZZ}}(Q) \times \int dP_s dP_t\ \ R_{P_s P_t}$$





S. Jackson, L. McGough, HV NPB 901 (2015) 382; J. Turiaci, HV, JHEP 1612 (2016) 110

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We study the geometric quantization of Teichmuller space and show that the physical state conditions take the form of conformal Ward identities that define the space of Virasoro conformal blocks in 2-d CFT.
Possible applications of these results to the [conformal bootstrap] are indicated.

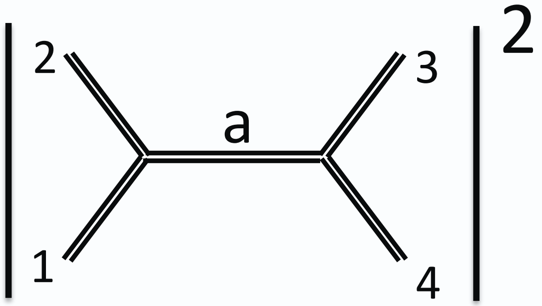
Hilbert state of the (2 + 1)-dimensional gravity theory

$$\Psi \in \mathscr{H}^{+} \otimes \mathscr{H}^{-} \tag{6.13}$$

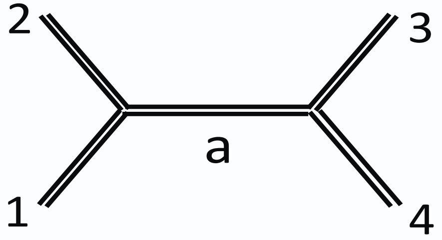
can be decomposed into a sum of left and right conformal blocks as

$$\Psi = \sum_{I,J} N^{IJ} \Psi_I^{+} \otimes \Psi_{\bar{J}}^{-} , \tag{6.14}$$

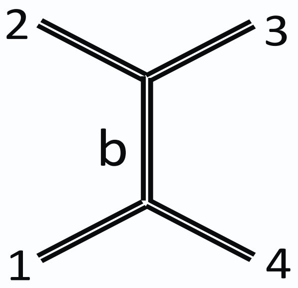
$$\left\langle \mathcal{O}_1(0)\,\mathcal{O}_2(1)\,\mathcal{O}_3(z,\bar{z})\,\mathcal{O}_4(\infty)\right\rangle = \sum_a$$



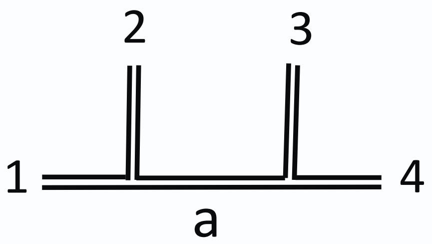
Conformal blocks



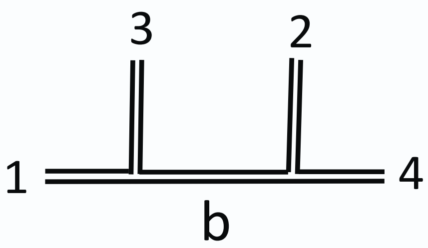
$$= \sum_b F_{ab}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$



F = Fusion matrix R= Braid matrix



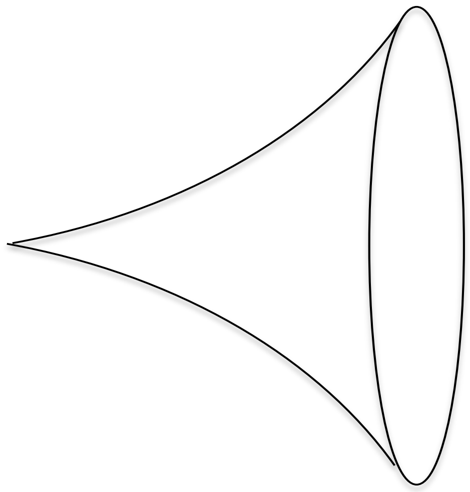
$$= \sum_b R^{\varepsilon}_{ab}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$



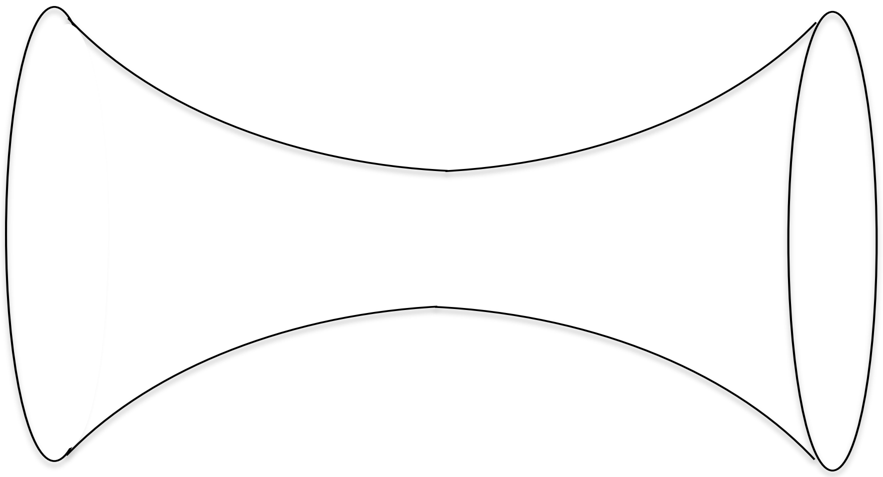
2D Virasoro CFT = 2D Quantum Hyperbolic Geometry

$$T(z) = \sum_{i=1}^{n-1}\left(\frac{\Delta_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right)$$

Stress-energy tensor

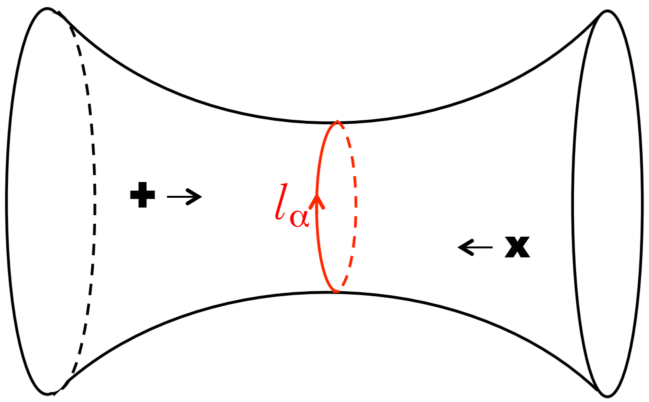


Elliptic

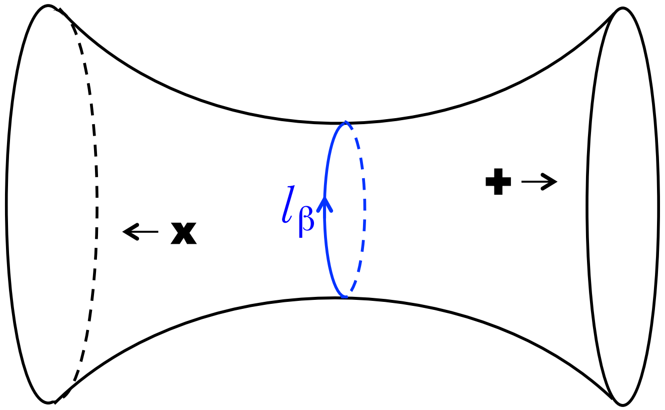


Hyperbolic

2+1-D AdS Gravity = 2D Quantum Hyperbolic Geometry



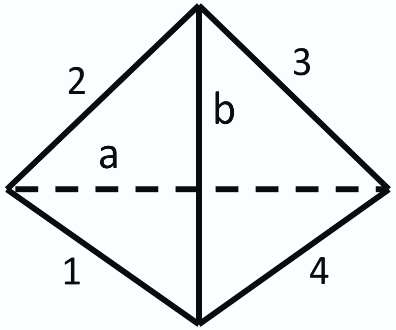
$$\hat{l}_\alpha |\alpha\rangle = l_\alpha |\alpha\rangle.$$



$$\hat{l}_\beta |\beta\rangle = l_\beta |\beta\rangle\,.$$

$$\mathcal{R}_{\alpha\beta} = \exp\left(\frac{i}{\hbar} S_{\alpha\beta}(l_\alpha, l_\beta)\right) = \langle\beta|\alpha\rangle$$

$$S_{\alpha\beta} = \mathrm{Vol}\left(T\begin{bmatrix} 1 & 2 & \alpha \\ 3 & 4 & \beta \end{bmatrix}\right)$$



Volume of a hyperbolic tetrahedron

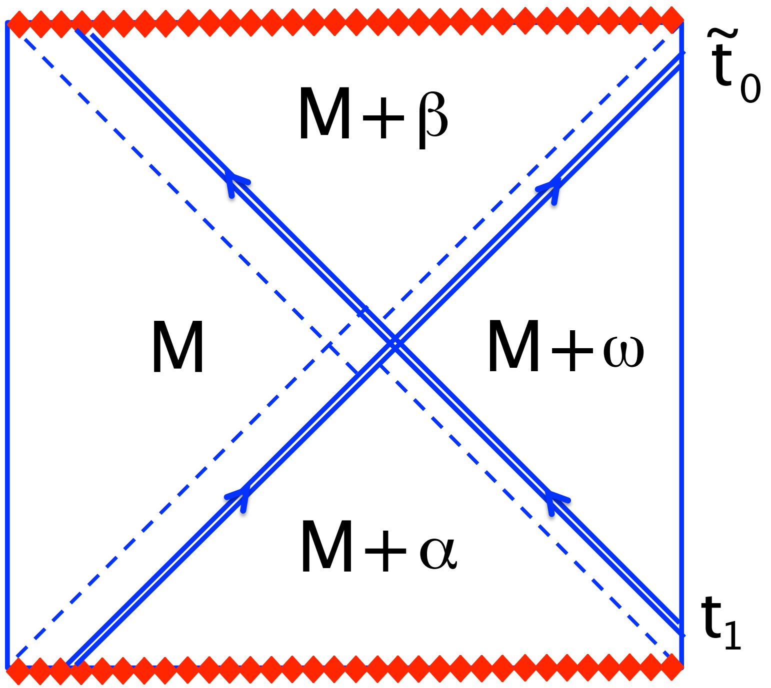
Ponsot-Teschner

6j-symbol of $SL(2)_q$

$$\phi_{\omega-\alpha}(t_1)\,\phi_\alpha(t_0) \;=\; e^{\frac{i}{\hbar}S_{\alpha\beta}}\,\phi_{\omega-\beta}(\tilde{t}_0)\,\phi_\beta(\tilde{t}_1).$$

Exchange relation for localized wave-packets

→ contains the gravitational scattering amplitude
→ spectral decomposition of OTO four-point function
→ scattering phase determined via geometric optics



c.f. Stanford
Shenker

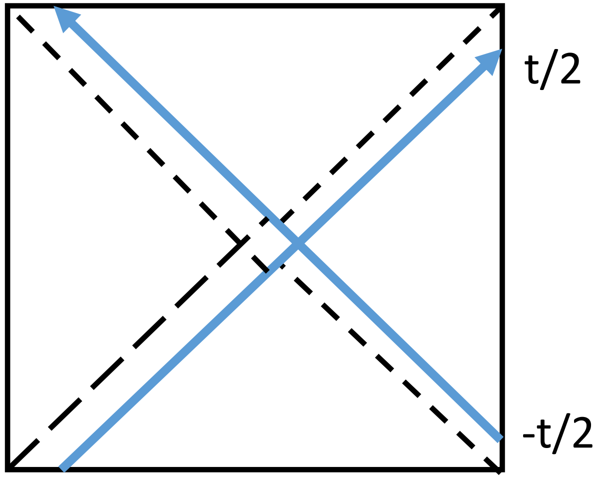
arXix:1412.5205, with S. Jackson, L. McGough, HV -- NPB 901 (2015) 382

Semiclassical limit of OTO 4pt function

$$C \sim G_N^{-1} \to \infty$$

[Shenker, Stanford]

$$\langle V_1 W_3 V_2 W_4 \rangle = \int_0^\infty dq_+ \int_0^\infty dp_- \; \Psi_1^*(q_+)\, \Phi_3^*(p_-)\, \mathcal{S}(p_-, q_+)\, \Psi_2(q_+)\, \Phi_4(p_-)$$



$$\mathcal{S} = \exp\left(\frac{i\beta}{4\pi C}\, p_- q_+\right)$$

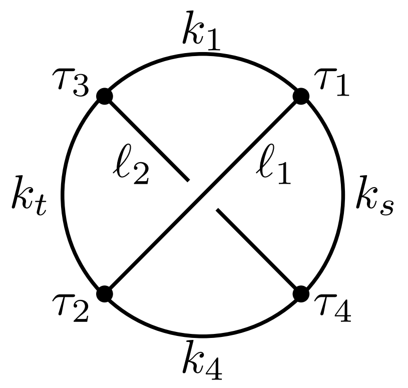
↓

Dray-'t Hooft S-matrix

Semiclassical limit of OTO 4pt function

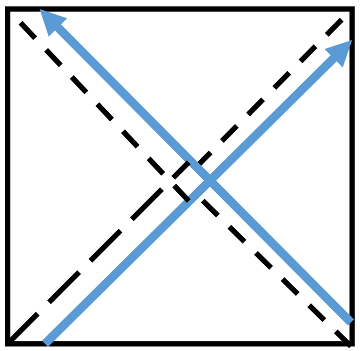
Large C
high temperature

$$\langle V_1 W_3 V_2 W_4 \rangle = \prod_{i=1}^{4} \int \frac{d\omega_i}{2\pi} \, \Psi_1^*(\omega_1)\, \Psi_3^*(\omega_3)\, \mathcal{S}(\omega_1, \omega_2, \omega_3, \omega_4)\, \Psi_2(\omega_2)\, \Psi_4(\omega_4).$$

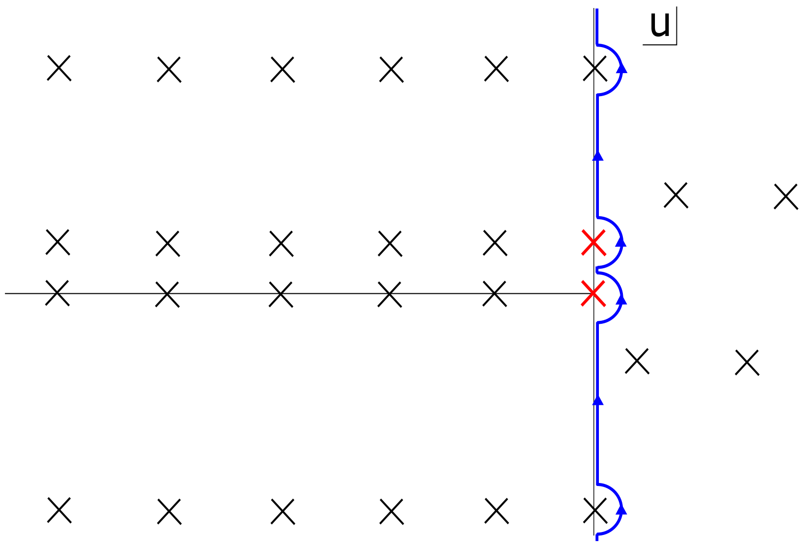


Schwarchild S-matrix

$$\mathcal{S}(\omega_1, \omega_3; \omega_2, \omega_4) = \frac{\beta}{(2\pi)^2} \delta(\omega_1 + \omega_3 - \omega_2 - \omega_4) \frac{\Gamma\left(i\nu_1 - i\nu_2\right)}{\left(\frac{4\pi i C}{\beta}\right)^{i(-\nu_1+\nu_2)}},$$



$$\mathcal{S} = \exp\left(\frac{i\beta}{4\pi C}\, p_- q_+\right)$$



Microscopic understanding of Lyapunov and fast thermalizing behavior?

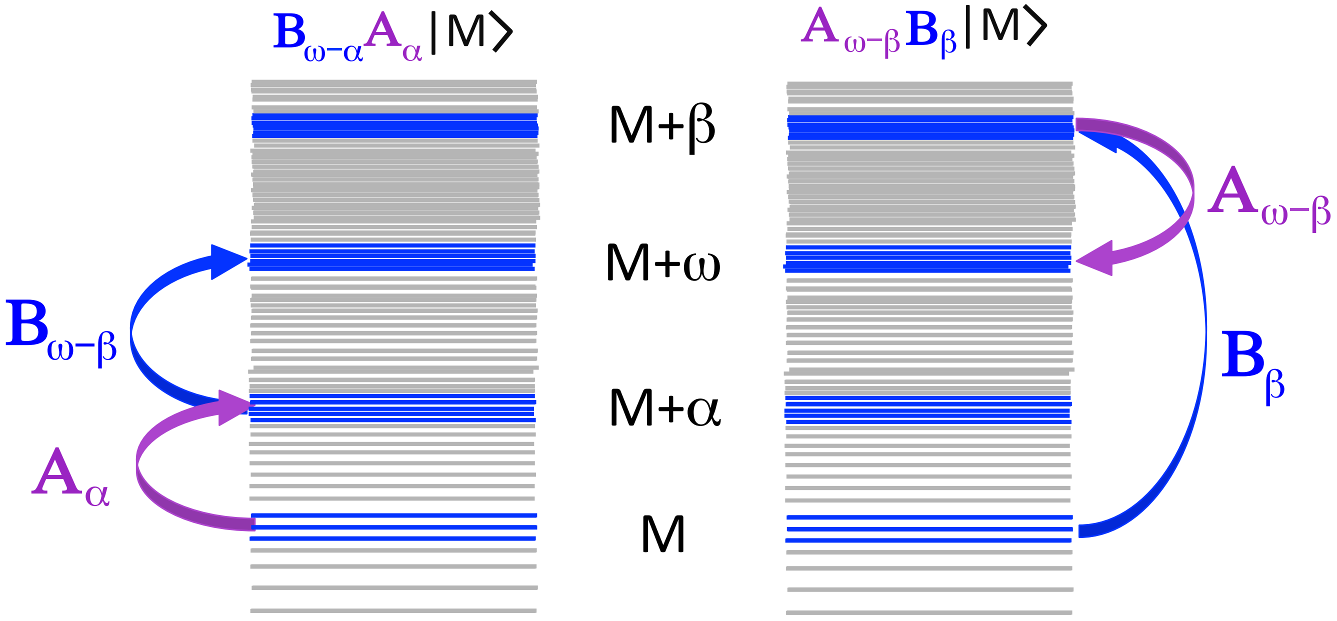


Figure 4: The scrambling of a signal (operator A) due to the a perturbation (operator B) at some earlier time $t_1 < t_0$. An observer that measures the state can detect signal A only if A acts on the state from the left. Passing A through B produces a new intermediate channel with energy β, which for $t_0 - t_1 > t_{\rm crit}$ exceeds ω. Signal A becomes scrambled: its coherent phase information get washed out by the large entropy region of the spectrum near $M + \beta$.