Zonal Jets:
The Extra Invariant for the Rossby Wave Dynamics

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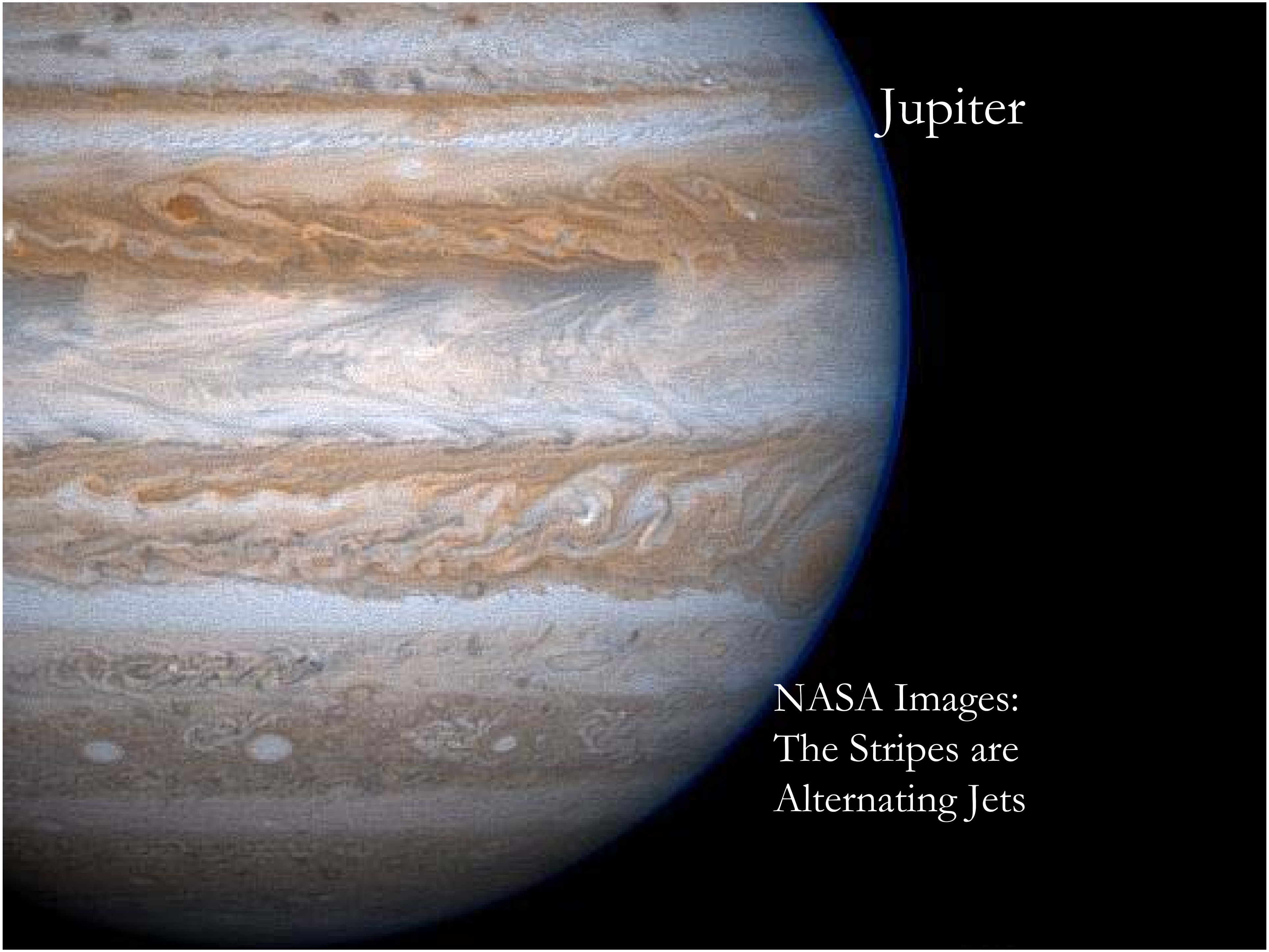

Jupiter
NASA Images:
The Stripes are
Alternating Jets

Zonal jets are well observed:

- Atmospheres and oceans on several planets (including Earth)
- Magnetized Plasmas

  Zonal jets reduce turbulent transport

  (towards the walls of a tokamak)

Zonal Jets ← Nonlinear Dynamics of Rossby waves

See in particular, Newell, JFM, 1969; Rhines, JFM, 1975; Diamond, Itoh, Itoh, Hahm, PPCF, 2005;

Rossby waves:

$$\Omega_{\mathbf{k}} = -\frac{\beta\, p}{\alpha^2 + p^2 + q^2} \qquad [\mathbf{k} = (p,q)]$$

Quasi-Geostrophic Equation (Charney-Hasegawa-Mima Equation)

$$\frac{\partial\left(\Delta\psi + \alpha^2\psi\right)}{\partial t} + \beta\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} = 0$$

Anisotropic Turbulence

The linear term in the quasi-geostrophic equation is anisotropic. So, when the wave amplitudes are not too large, we can expect anisotropic turbulence.

However, the anisotropy of linear term is the anisotropy of the first angular harmonic

$$\Omega_{\mathbf{k}} = -\frac{\beta k \cos\theta}{\alpha^2 + k^2} \qquad [\mathbf{k} = (k, \theta)]$$

At the same time zonal jets are extremely anisotropic.

Inverse cascade

An important feature of the Quasi-Geostrophic Equation is the **Inverse Cascade**.

$$\text{energy(t)} \equiv \int [(\nabla\psi)^2 + \alpha^2\psi^2]\, d\mathbf{x} = \text{const}$$

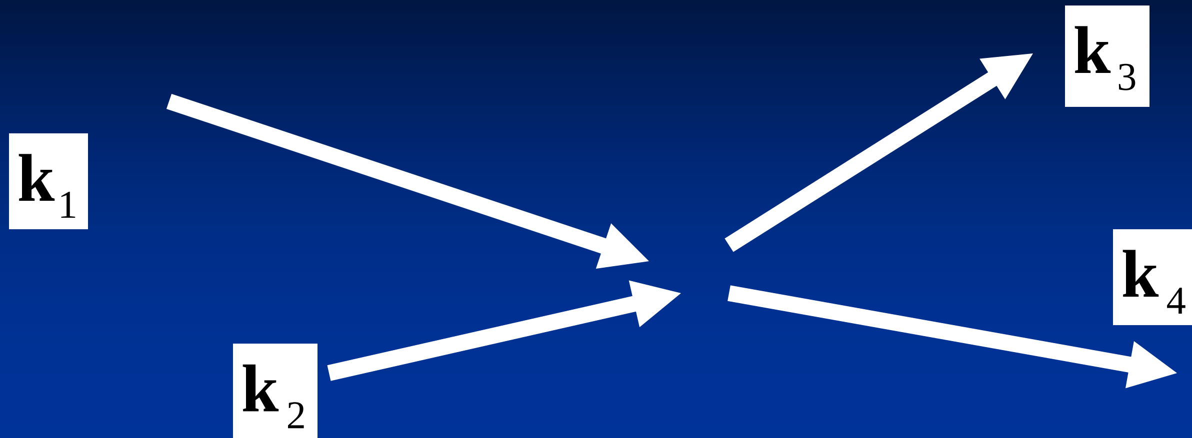
$$\text{enstrophy(t)} \equiv \int [(\Delta\psi)^2 + \alpha^2(\nabla\psi)^2]\, d\mathbf{x} = \text{const}$$

Energy flows toward the origin.

But the Large scale structure is not a Big vortex.

Zonal Jets instead

Rarefied Gas



$$\mathbf{P} = \int \mathbf{k}\, n(\mathbf{k},t)\, d\mathbf{k}$$

$$E = \int \Omega(\mathbf{k})\, n(\mathbf{k},t)\, d\mathbf{k}$$

$$N = \int n(\mathbf{k},t)\, d\mathbf{k}$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\Omega(\mathbf{k}_1) + \Omega(\mathbf{k}_2) = \Omega(\mathbf{k}_3) + \Omega(\mathbf{k}_4)$$

$$\Omega(\mathbf{k}) = \frac{\mathbf{k}^2}{2}$$

NOT

?

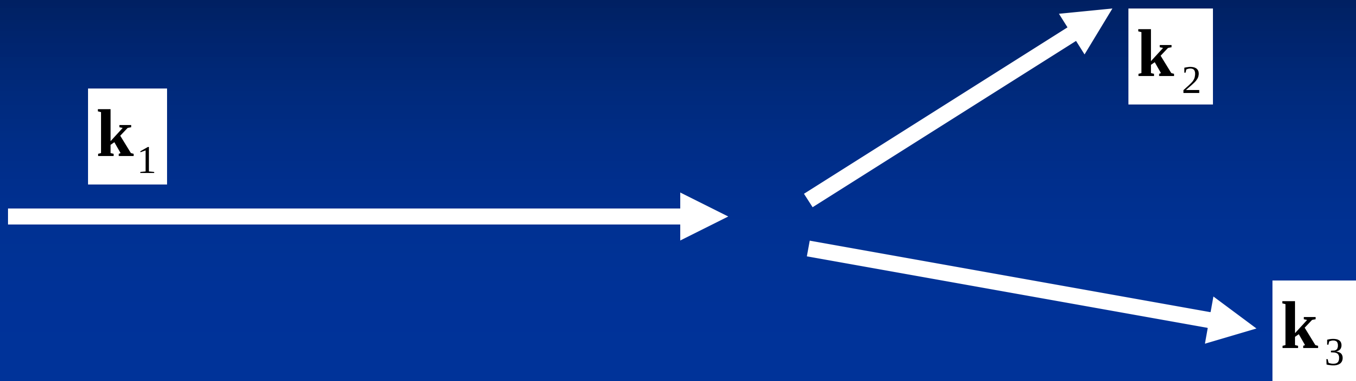
$$\Phi = \int \phi(\mathbf{k})\, n(\mathbf{k},t)\, d\mathbf{k}$$

$$\varphi(\mathbf{k}_1) + \varphi(\mathbf{k}_2) = \varphi(\mathbf{k}_3) + \varphi(\mathbf{k}_4)$$

Boltzmann, see Sercignani:

Are there more than 5 linearly-independent collision invariants for the Boltzmann equation. J. Stat. Phys. 1990

Wave Resonances



$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$$
$$\Omega(\mathbf{k}_1) = \Omega(\mathbf{k}_2) + \Omega(\mathbf{k}_3)$$

?

$$\phi(\mathbf{k}_1) = \phi(\mathbf{k}_2) + \phi(\mathbf{k}_3)$$

Zakharov, Schulman, 1980-88

$$\mathbf{P} = \int \mathbf{k}\, n(\mathbf{k}, t)\, d\mathbf{k}$$
$$E = \int \Omega(\mathbf{k})\, n(\mathbf{k}, t)\, d\mathbf{k}$$

NOT

$$\Phi = \int \phi(\mathbf{k})\, n(\mathbf{k}, t)\, d\mathbf{k}$$

The existence of extra invariant is extremely rare.

But for Rossby waves it does exist:

$$\Omega_{\mathbf{k}} = -\frac{\beta p}{\alpha^2 + k^2} \qquad [\mathbf{k} = (p, q)]$$

$$I = \int \eta(\mathbf{k})\, n(\mathbf{k}, t) d\mathbf{k}$$

$$n(\mathbf{k}, t) = \frac{E_{\mathbf{k}}(t)}{\Omega_{\mathbf{k}}}$$

$$\eta(\mathbf{k}) = \arctan\left(\frac{q - p\sqrt{3}}{k^2}\right) - \arctan\left(\frac{q + p\sqrt{3}}{k^2}\right)$$

B, Nazarenko, Zakharov, 1991; B, 1991

The extra invariant density η and the energy density Ω have the same asymptotics as $|\mathrm{k}|\to\infty$ and as $p\to 0$

Linear combination of *I* and *E*

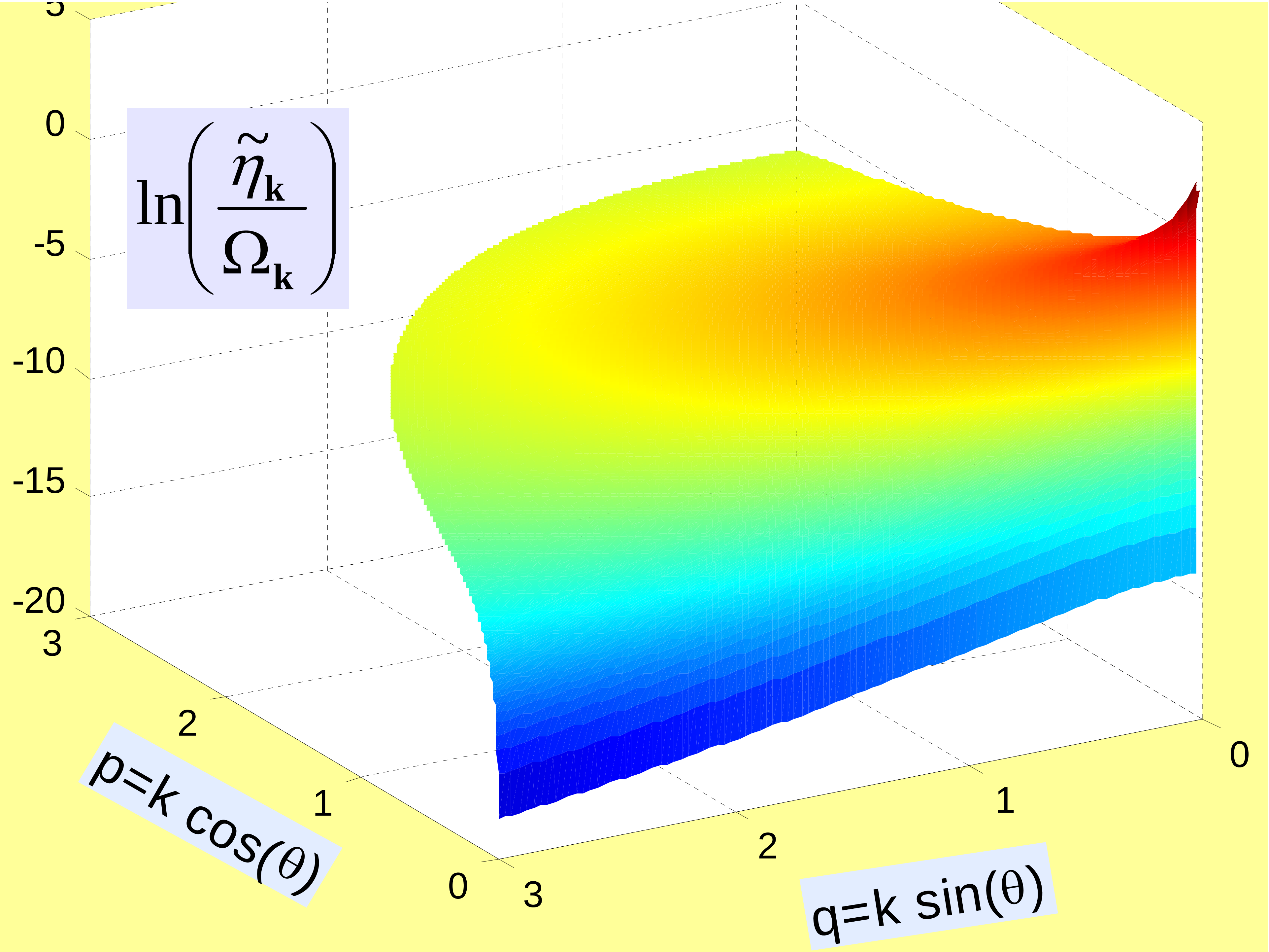
=> Cancellation of those asymptotics:

$$\tilde{I} = I - 2\sqrt{3}\,E$$

$$\tilde{\eta}_{\mathbf{k}} = \eta_{\mathbf{k}} + 2\sqrt{3}\,\frac{\Omega_{\mathbf{k}}}{\beta} \sim \frac{1}{k^5}$$

while $\eta_{\mathbf{k}} \sim \frac{1}{k}$ and $\Omega_{\mathbf{k}} \sim \frac{1}{k}$ $\left(k = |\,\mathbf{k}\,| \to \infty\right)$

B, 2005



5
0
-5
-10
-15
-20
$\ln\left(\frac{\tilde{\eta}_{\mathbf{k}}}{\Omega_{\mathbf{k}}}\right)$
3
2
1
0
p=k cos(θ)
3
2
1
0
q=k sin(θ)

Exact conservation in 3-wave resonance interactions

↓

Adiabatic conservation in all wave interactions.

I is approximately conserved over long time [B, van Heerden, 2006]

The smaller wave amplitudes, the more precisely *I* is conserved

In the course of the inverse cascade, the nonlinearity is decreasing:

$Rh = U/\beta L^2 \rightarrow 0 \qquad [\ U^2 \sim E, \quad L \rightarrow \infty\ ]$

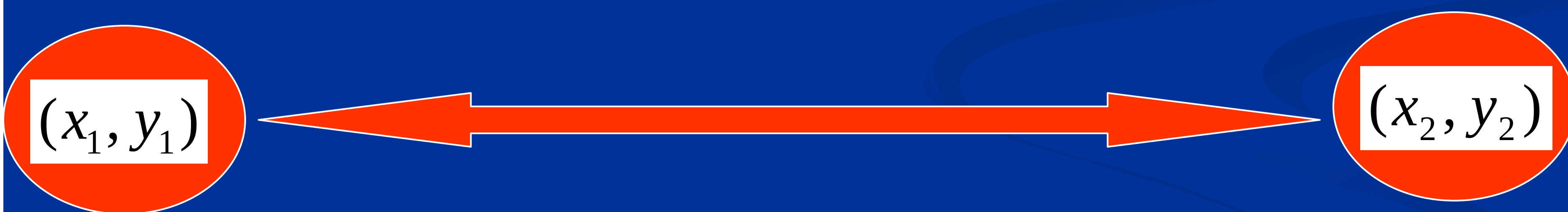
System starts to preserve the extra invariant

Is the Extra Invariant Physical?

If the invariant were meaningful, then…

$$I = \int \Xi\left(x_2 - x_1,\ y_2 - y_1\right)\, s\left(x_1,\ y_1,\ t\right)\, s\left(x_2,\ y_2,\ t\right)\, dx_1 dy_1\ dx_2 dy_2$$

$$[s = \Delta\psi - \alpha^2\psi]$$



… the kernel should vanish, when the separation distance goes to infinity

Energy and momentum are conserved in 3-wave interactions,…

$$E = \int \Omega_{\mathbf{k}}\, n(\mathbf{k})\, d\mathbf{k} \qquad \mathbf{P} = \int \mathbf{k}\, n(\mathbf{k})\, d\mathbf{k}$$

$$E = \int \frac{\Omega_{\mathbf{k}}}{p}\, s_{\mathbf{k}}(t)\, s_{-\mathbf{k}}(t)\, d\mathbf{k}$$

$$P_x = \int \frac{p}{p}\, s_{\mathbf{k}}(t)\, s_{-\mathbf{k}}(t)\, d\mathbf{k}$$

$$P_y = \int \frac{q}{p}\, s_{\mathbf{k}}(t)\, s_{-\mathbf{k}}(t)\, d\mathbf{k}$$

$$s(x,y,t) = \Delta\psi - \alpha^2\psi$$

$$s_{\mathbf{k}}(t) = (\alpha^2 + k^2)\psi_{\mathbf{k}}(t)$$

…but the y-momentum is non-local, and it is not meaningful.

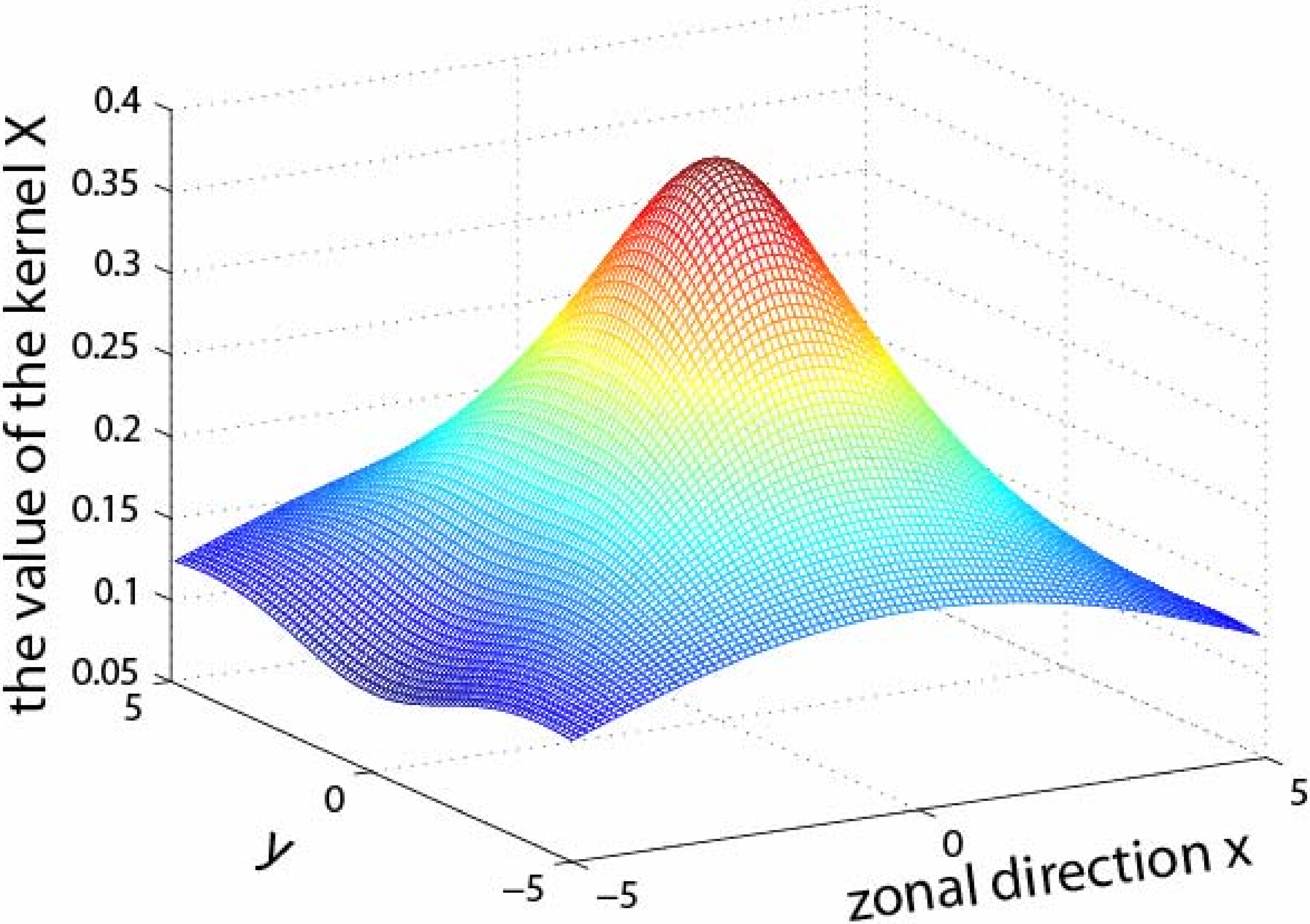
The extra invariant is local

$$I = \int X(x_2 - x_1,\ y_2 - y_1)\ s(x_1,\ y_1,\ t)\ s(x_2,\ y_2,\ t)\ dx_1 dy_1\, dx_2 dy_2$$

$$\frac{\partial X}{\partial x} = \left(\frac{2}{\sqrt{3}} \sinh \frac{\sqrt{3}\,x}{2}\ \cosh \frac{y}{2} - x \right) \frac{1}{r} K_0{}'(r)$$

$$\left[K_0{}'(r)\ \ \text{Green's function of } \Delta - 1 \right]$$

B, Yoshikawa, 2008



the value of the kernel X
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
5
0
y
−5
−5
0
5
zonal direction x

The extra invariant is not succeptible to dissipation [even to greater degree than the energy; unlike the enstrophy].

Indeed, $$\frac{\tilde{\eta}_{\mathbf{k}}}{\Omega_{\mathbf{k}}} \sim \frac{1}{k^4} \quad \text{as } k \to \infty$$

Compare with the enstrophy: $$\frac{\beta p - \Omega_{\mathbf{k}}}{\Omega_{\mathbf{k}}} \sim k^2 \quad \text{as } k \to \infty$$

$$[\mathbf{k} = (p, q); \;\; \Omega_{\mathbf{k}} = \frac{\beta p}{\alpha^2 + k^2}]$$

Shallow Water Dynamics

$$u_t + u\,u_x + v\,u_y - f(y)v = -gH_x$$

$$v_t + u\,v_x + v\,v_y + f(y)u = -gH_x$$

$$H_t + (u\,H)_x + (v\,H)_y = 0$$

Several kinds of Resonances:

- Rossby=Rossby+Rossby…………already in QG eq.
- Poincare=Poincare+Poincare……………impossible
- Rossby=Rossby+Poincare………………..impossible
- Poincare=Poincare+Rossby………………………

………………………possible, but not activated

[nonlinearity of the eq. for Rossby waves.]

$$s(x,y,t) = \frac{v_x - u_y + f(y)}{H} - \frac{f(y)}{H_0}$$

$$I = \int X_{1,2}\ s_1\ s_2\ d_{1,2}$$

$$+ \int F_{1,2}\ s_1\ u_2\ d_{1,2} + \int G_{1,2}\ s_1\ v_2\ d_{1,2}$$

$$+ \int R_{1,2,3}\ s_1\ s_2\ u_3\ d_{1,2,3} + \int R_{1,2,3}\ s_1\ s_2\ v_3\ d_{1,2,3}$$

$$+ \int Y_{1,2,3}\ s_1\ s_2\ s_3\ d_{1,2,3}$$

$X = C(y_1, y_2)\ \dfrac{\tilde{\eta}}{p}$ Need to find $C(y_1, y_2)$

There is another function conserved in the 3-wave interactions

$$p_1 + p_2 + p_3 = 0$$

$$q_1 + q_2 + q_3 = 0$$

$$\Omega(p_{1,} \, q_1) + \Omega(p_2, q_2) + \Omega(p_3, q_3) = 0$$

$$\phi(p_1, q_1) + \phi(p_2, q_2) + \phi(p_3, q_3) = 0 \quad \Leftrightarrow \quad Z(p_1, q_1)\ Z(p_2, q_2)\ Z(p_3, q_3) = 1$$

$$\phi(p, q) = \ln \frac{\alpha(q + p\sqrt{3}) + ik^2}{\alpha(q - p\sqrt{3}) + ik^2} = \ln Z(p, q)$$