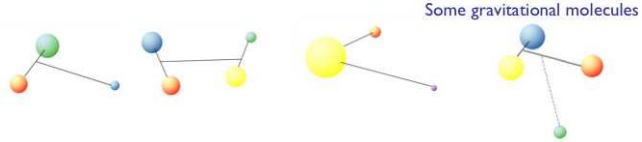
Stability and Resonance in the General Three-Body Problem

Rosemary Mardling

School of Mathematical Sciences
Monash University

KITP – February 19, 2009



Astrophysical Applications

Processes not modelled accurately by the *restricted* three-body problem (circular coplanar with $m_2 \ll m_1$ and $m_3 = 0$)

Pure three-body

- Decaying triples in young (and old) star clusters
- Planet scattering
- Forming the moon
- MBH + stellar binary
- Binary MBHs + star (or cluster)
- Triple MBHs

Astrophysical Applications

Three-body + $\Phi(\mathbf{r}, t)$ + dissipation

- MBH + stellar binary
- Binary MBHs + star (or cluster)
- Triple MBHs
- Planetesimals and planets in the presence of a disk
- Galaxy triples
- Tidal stability of globular clusters on eccentric galactic orbits
- Four-body hierarchies...

Stability

An unstable system ≡ one body will escape to infinity ≡ "Lagrange unstable"

Desirable features of a stability criterion

- Clear formulation from first principals
- Works for all initial conditions
- Expressible in terms of orbital parameters
- Simple to use
- Easy to include extra potentials etc

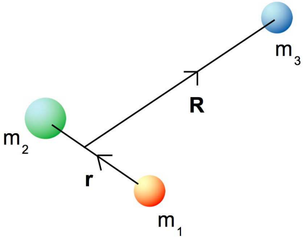
All stability criteria for the general problem have been (semi-)empirical

Equations of motion

Hierarchical (Jacobi) coordinates

$$\mu_i \ddot{\mathbf{r}} + \frac{G m_1 m_2}{|\mathbf{r}|^2}\hat{\mathbf{r}} = \frac{\partial \Phi}{\partial \mathbf{r}}$$

$$\mu_o \ddot{\mathbf{R}} + \frac{G m_{12} m_3}{|\mathbf{R}|^2}\hat{\mathbf{R}} = \frac{\partial \Phi}{\partial \mathbf{R}}$$



The interaction potential (disturbing function):

$$\Phi = -\frac{G m_{12} m_3}{|\mathbf{R}|} + \frac{G m_2 m_3}{|\mathbf{R} - \alpha_1 \mathbf{r}|} + \frac{G m_1 m_3}{|\mathbf{R} + \alpha_2 \mathbf{r}|}$$

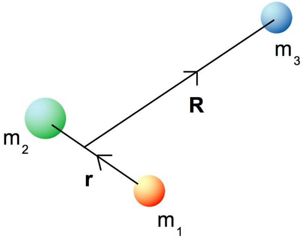
$\alpha_i = m_i/m_{12}$, $m_{12} = m_1 + m_2$, $m_{123} = m_1 + m_2 + m_3$,
$\mu_i = m_1 m_2/m_{12}$, $\mu_o = m_{12} m_3/m_{123}$

Equations of motion

Hierarchical (Jacobi) coordinates

$$\mu_i \ddot{\mathbf{r}} + \frac{G m_1 m_2}{|\mathbf{r}|^2}\hat{\mathbf{r}} = \frac{\partial \Phi}{\partial \mathbf{r}}$$

$$\mu_o \ddot{\mathbf{R}} + \frac{G m_{12} m_3}{|\mathbf{R}|^2}\hat{\mathbf{R}} = \frac{\partial \Phi}{\partial \mathbf{R}}$$



The interaction potential (disturbing function):

$$\Phi = -\frac{G m_{12} m_3}{|\mathbf{R}|} + \frac{G m_2 m_3}{|\mathbf{R} - \alpha_1 \mathbf{r}|} + \frac{G m_1 m_3}{|\mathbf{R} + \alpha_2 \mathbf{r}|}$$

$\alpha_i = m_i/m_{12},\ m_{12} = m_1 + m_2,\ m_{123} = m_1 + m_2 + m_3,$

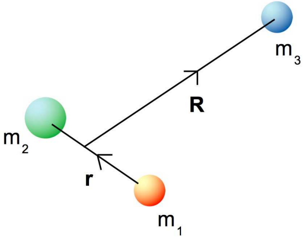
$\mu_i = m_1 m_2/m_{12},\ \mu_o = m_{12} m_3/m_{123}$

Equations of motion

Hierarchical (Jacobi) coordinates

$$\mu_i\ddot{\mathbf{r}} + \frac{Gm_1m_2}{|\mathbf{r}|^2}\hat{\mathbf{r}} = \frac{\partial\Phi}{\partial\mathbf{r}}$$

$$\mu_o\ddot{\mathbf{R}} + \frac{Gm_{12}m_3}{|\mathbf{R}|^2}\hat{\mathbf{R}} = \frac{\partial\Phi}{\partial\mathbf{R}}$$



Numerical solution

This is a 12th-order system - need to specify 12+2 parameters:

$\mathbf{r}$, $\dot{\mathbf{r}}$, $\mathbf{R}$, $\dot{\mathbf{R}}$

OR

2x[e, a, λ, (ϖ, Ω, I)]

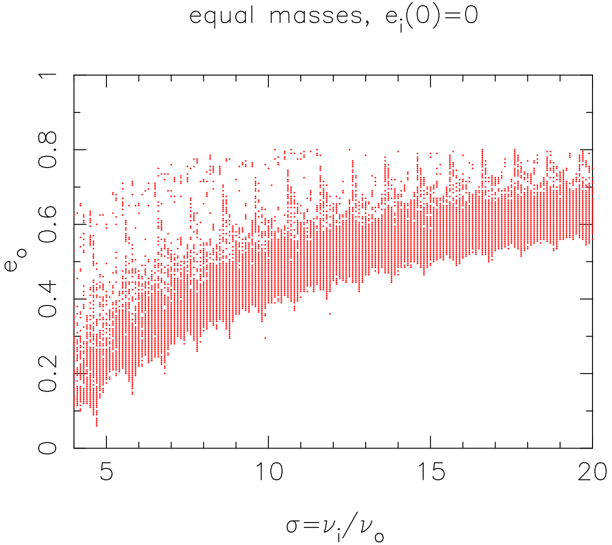
+2 mass ratios

A Stable Triple

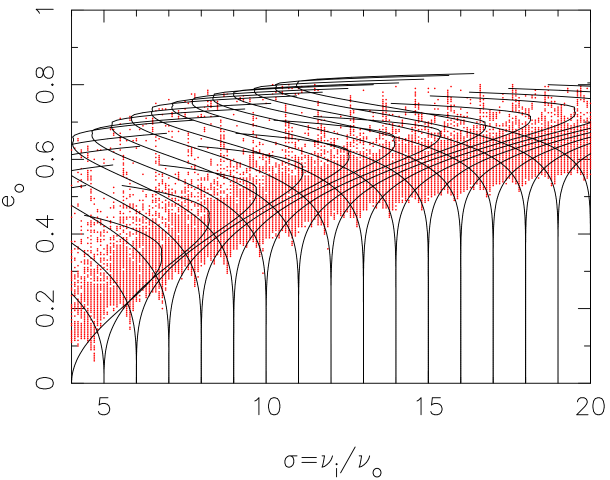
While energy is exchanged during an outer orbit, after one whole orbit the nett exchange is exponentially small.

An Unstable Triple

A finite amount of energy is exchanged each outer orbit, as the outer body random-walks its way out of the system.


equal masses, e_i(0)=0
e_o
0
0.2
0.4
0.6
0.8
1
5
10
15
20
σ=ν_i/ν_o

equal masses, $e_i(0)=0$



The interaction potential

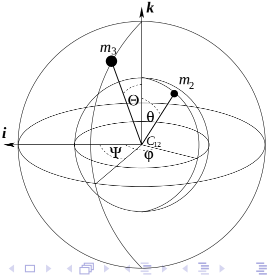
Spherical harmonic expansion

$$\Phi = -\frac{Gm_{12}m_3}{|\mathbf{R}|} + \frac{Gm_2 m_3}{|\mathbf{R} - \alpha_1 \mathbf{r}|} + \frac{Gm_1 m_3}{|\mathbf{R} + \alpha_2 \mathbf{r}|}$$

$$= G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \mathcal{M}_l \left(\frac{r^l}{R^{l+1}} \right) Y_{lm}(\theta, \varphi) Y_{lm}^*(\Theta, \psi)$$

$$\mathcal{M}_l = \frac{m_1^{l-1} + (-1)^l m_2^{l-1}}{m_{12}^l}$$

$\mathcal{M}_2 = 1$
$\mathcal{M}_3 = 0$ if $m_1 = m_2$.



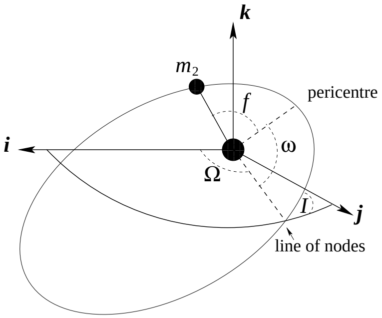
The interaction potential

Coplanar systems: $\theta = \Theta = \pi/2$

$$\Phi = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \mathcal{M}_l \left(\frac{r^l}{R^{l+1}} \right) Y_{lm}(\pi/2, \varphi) Y^*_{lm}(\pi/2, \psi)$$

$$\varphi = f_i + \varpi_i$$
$$\psi = f_o + \varpi_o$$

$$\varpi_i = \omega_i + \Omega_i$$
$$\varpi_o = \omega_o + \Omega_o$$



The interaction potential

Coplanar systems: $\theta = \Theta = \pi/2$

$$\Phi = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \mathcal{M}_l \left(\frac{r^l}{R^{l+1}} \right) Y_{lm}(\pi/2, \varphi) Y^*_{lm}(\pi/2, \psi)$$

$$\varphi = f_i + \varpi_i, \qquad \psi = f_o + \varpi_o$$

$$\Phi = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} c^2_{lm} \mathcal{M}_l \, e^{im(\varpi_i - \varpi_o)} \left(r^l \, e^{imf_i} \right) \left(\frac{e^{-imf_o}}{R^{l+1}} \right)$$

∼periodic: freq ν_i ∼periodic: freq ν_o

$c^2_{lm} = \frac{4\pi}{2l+1}[Y_{lm}(\pi/2, 0)]^2$. Eg. $c^2_{22} = 3/8$, $c^2_{21} = 0$

The interaction potential

Coplanar systems

$$\Phi = G\mu_i m_3 \sum_{l=2}^{\infty} \sum_{m=-l,2}^{l} c_{lm}^2 \mathcal{M}_l \, e^{im(\varpi_i - \varpi_o)} \left(r^l \, e^{imf_i} \right) \left(\frac{e^{-imf_o}}{R^{l+1}} \right)$$

$$= 2G\mu_i m_3 \sum_{lmnn'} c_{lm}^2 \mathcal{M}_l \left(\frac{a_i^l}{a_o^{l+1}} \right) s_{n'}^{(lm)}(e_i) F_n^{(lm)}(e_o) \cos \phi_{mnn'}$$

where

$$\phi_{mnn'} = n'\lambda_i - n\lambda_o + (m - n')\varpi_i - (m - n)\varpi_o$$

is a *resonance angle.*

The interaction potential

Coplanar "moderate-mass ratio" systems:
$n:1$ "quadrupole" resonances ($l=m=2$ and $n'=1$)

$$n=[\nu_i/\nu_o]$$

$$\Phi_n=\frac{3}{4}\frac{G\mu_i m_3}{a_i}\left(\frac{a_i}{a_o}\right)^3 s_1^{(22)}(e_i)F_n^{(22)}(e_o)\cos\phi_{2n1}$$

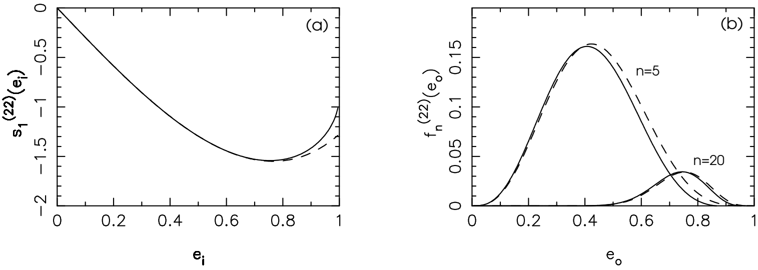
$$\phi_{2n1}\equiv\phi_n=\lambda_i-n\lambda_o+\varpi_i-(2-n)\varpi_o$$

Moderate-mass ratio systems:
both $m_2/m_1\gtrsim 0.01$ and $m_3/m_1\gtrsim 0.01$
OR at least one of $m_2/m_1\gtrsim 0.05$ *or* $m_3/m_1\gtrsim 0.05$

Eccentricity functions

$$s_1^{(22)}(e_i) = -3e_i + \frac{13}{8}e_i^3 + \frac{5}{192}e_i^5 - \frac{227}{3072}e_i^7 + \mathcal{O}(e_i^9),$$

$$F_n^{(22)}(e_o) \simeq \frac{4}{3\sqrt{2\pi}} \frac{(1-e_o^2)^{3/4}}{e_o^2}\, n^{3/2} e^{-n\xi(e_o)}$$



The interaction potential

Energy exchange

The interaction potential governs energy transfer between orbits:

$$\frac{\dot{E}_i}{E_i} = \frac{2}{3}\frac{\dot{\nu}_i}{\nu_i} = -\frac{\dot{a}_i}{a_i} = -\frac{2}{\mu_i \nu_i a_i^2}\frac{\partial \Phi}{\partial \lambda_i}$$

Can show that energy exchanged during one outer orbit is

$$\boxed{\Delta E_o \simeq -\Delta E_i \sim e_i\, \sigma^{5/2} e^{-\sigma \xi(e_o)}}$$

Asymptotic expression for "overlap integral"

$\sigma = \nu_i/\nu_o$, $\xi = \mathrm{Cosh}^{-1}(1/e_o) - \sqrt{1-e_o^2}$

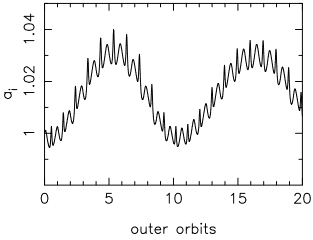
On average "no" energy is exchanged in a non-resonant triple

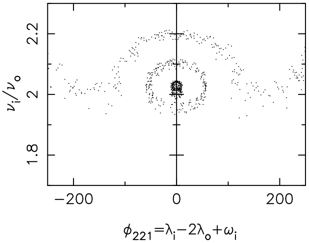
The interaction potential

Energy exchange

Significant energy is exchanged in a resonant triple

eg. GJ 876: a 2:1 resonant planetary system



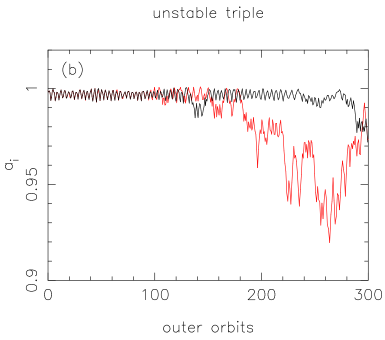


The interaction potential

Energy exchange

Significantly more energy is exchanged in a *unstable* triple

eg. $R_p/a_i = 3.6,\ e_i = 0,\ e_o = 0.5$



Resonance

Nonlinear resonance in weakly interacting systems

$$H = H_1\left[\nu_1, \nu_2, \ldots\right] + H_2\left[\omega_1, \omega_2, \ldots\right] + \Phi[\nu_1, \nu_2, \ldots, \omega_1, \omega_2, \ldots]$$

$$\Phi = \sum_{n_i' n_i} \Phi_{n_i' n_j}(c_1, c_2, \ldots) \cos(n_1'\theta_1 + n_2'\theta_2 + \ldots - n_1\varphi_1 - n_2\varphi_2 - \ldots),$$

$$\dot{\theta}_i = \nu_i,\ \dot{\varphi}_j = \omega_j$$

resonance angle

$$\phi_{n_1' n_2' \ldots n_1 n_2 \ldots} = n_1'\theta_1 + n_2'\theta_2 + \ldots - n_1\varphi_1 - n_2\varphi_2 - \ldots$$

$$\boxed{\ddot{\phi} \simeq -\Omega^2 \sin\phi}$$

Resonance

Hierarchical triples are weakly interacting systems

$$H = H_i\,[\nu_i, \varpi_i, \Omega_i] + H_o\,[\nu_o, \varpi_o, \Omega_o] + \sum_n \Phi_n(m_j, e_j, I_j, \sigma)\cos\phi_n$$

Coplanar resonance angle

$$\phi_n = \lambda_i - n\lambda_o + \varpi_i - (2-n)\varpi_o$$

$$\dot\phi_n = \nu_i - n\nu_o + \dot\varpi_i - (2-n)\dot\varpi_o$$
$$\simeq \nu_i - n\nu_o$$ for moderate-mass systems.

$$\ddot\phi_n \simeq -\Omega_n^2 \sin\phi_n$$

Resonance overlap

1892 Poincaré: *Les méthodes nouvelles de la méchanique céleste*

1954 Kolmogorov, 1963 Arnol'd, 1966 Moser

1955 Fermi, Pasta, Ulam

- 1969 Walker and Ford:
  Amplitude instability and ergodic behaviour for conservative nonlinear oscillator systems
- 1979 Chirikov:
  A universal instability of many-dimensional oscillator systems
- 1980 Wisdom:
  The resonance overlap criterion and the onset of stochastic behavior in the restricted three-body problem

Resonance overlap

Resonance angles

$$\phi_n = \lambda_i - n\lambda_o + \varpi_i - (2-n)\varpi_o$$

$$\dot\phi_n = \nu_i - n\,\nu_o$$

$$\ddot\phi_n = \dot\nu_i - n\,\dot\nu_o$$

$$\frac{\dot\nu_i}{\nu_i} = -\frac{3}{2}\frac{\dot a_i}{a_i} = -\frac{3}{\mu_i\nu_i a_i^2}\frac{\partial\Phi}{\partial\lambda_i}, \qquad \frac{\dot\nu_o}{\nu_o} = -\frac{3}{\mu_o\nu_o a_o^2}\frac{\partial\Phi}{\partial\lambda_o}$$

$$\Phi \simeq \frac{3}{4}\frac{G\mu_i m_3}{a_i}\left(\frac{a_i}{a_o}\right)^3 s_1^{(22)}(e_i)F_n^{(22)}(e_o)\cos\phi_n, \qquad n \simeq \nu_i/\nu_o$$

$$\ddot\phi_n \simeq \Omega_n^2 \sin\phi_n$$

Resonance overlap

Resonance angles

$$\phi_n = \lambda_i - n\lambda_o + \varpi_i - (2-n)\varpi_o$$

$$\dot\phi_n = \nu_i - n\,\nu_o$$

$$\ddot\phi_n = \dot\nu_i - n\,\dot\nu_o$$

$$\frac{\dot\nu_i}{\nu_i} = -\frac{3}{2}\frac{\dot a_i}{a_i} = -\frac{3}{\mu_i\nu_i a_i^2}\frac{\partial\Phi}{\partial\lambda_i}, \qquad \frac{\dot\nu_o}{\nu_o} = -\frac{3}{\mu_o\nu_o a_o^2}\frac{\partial\Phi}{\partial\lambda_o}$$

$$\Phi \simeq \frac{3}{4}\frac{G\mu_i m_3}{a_i}\left(\frac{a_i}{a_o}\right)^3 s_1^{(22)}(e_i)F_n^{(22)}(e_o)\cos\phi_n, \qquad n \simeq \nu_i/\nu_o$$

$$\ddot\phi_n \simeq -\Omega_n^2 \sin\phi_n$$

Resonance overlap

Resonance angles

$$\phi_n = \lambda_i - n\lambda_o + \varpi_i - (2-n)\varpi_o$$

$$\dot\phi_n = \nu_i - n\,\nu_o$$

$$\ddot\phi_n = \dot\nu_i - n\,\dot\nu_o$$

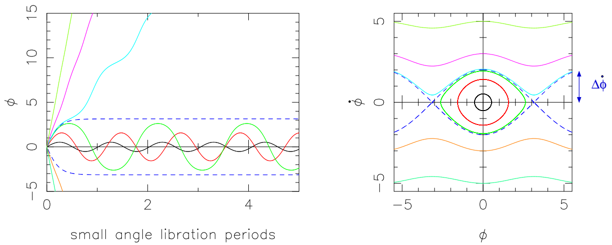
$$\frac{\dot\nu_i}{\nu_i} = -\frac{3}{2}\frac{\dot a_i}{a_i} = -\frac{3}{\mu_i\nu_i a_i^2}\frac{\partial\Phi}{\partial\lambda_i}, \qquad \frac{\dot\nu_o}{\nu_o} = -\frac{3}{\mu_o\nu_o a_o^2}\frac{\partial\Phi}{\partial\lambda_o}$$

$$\Phi \simeq \frac{3}{4}\frac{G\mu_i m_3}{a_i}\left(\frac{a_i}{a_o}\right)^3 s_1^{(22)}(e_i)F_n^{(22)}(e_o)\cos\phi_n, \qquad n \simeq \nu_i/\nu_o$$

$$\ddot\phi_n \simeq -\Omega_n^2 \sin\phi_n, \qquad \Omega_n^2 = -\tfrac{9}{4}\nu_o^2 s_1^{(22)}(e_i)F_n^{(22)}(e_o)\left[M_i^{(2)} + n^{2/3}M_o^{(2)}\right]$$

Resonance overlap

Pendulums and resonance



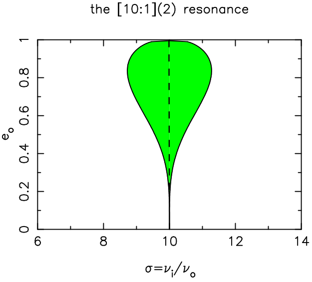
Separatrix: $\dot\phi = \pm 2\Omega\cos(\phi/2)$

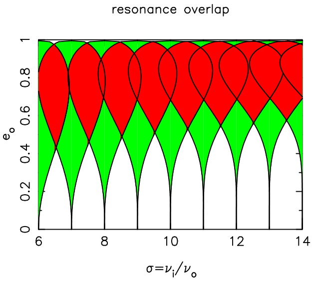
A system is defined to be in $n:1$ resonance if ϕ_n librates.

Resonance width: $\Delta\dot\phi_n = 2\Omega_n$

Resonance overlap

$$\Delta\dot{\phi} = \nu_o(\sigma - n)$$

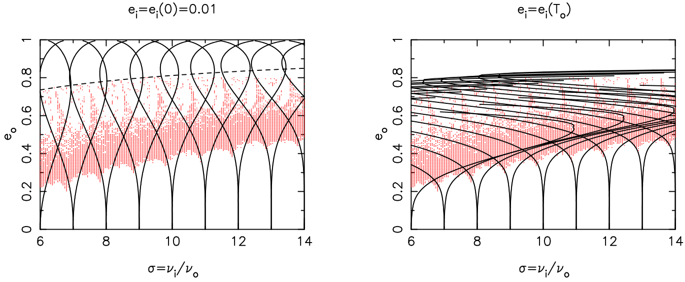




An equal-mass system with $e_i(0) = 0.01$.

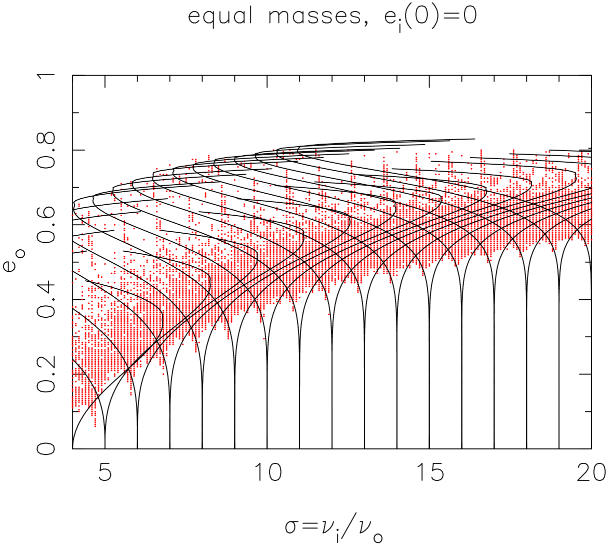
Resonance overlap

Induced eccentricity

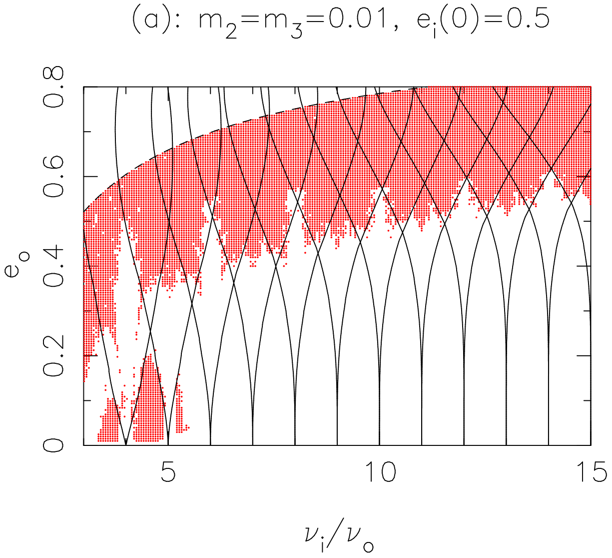


Red dots from direct three-body integrations: dot=unstable system

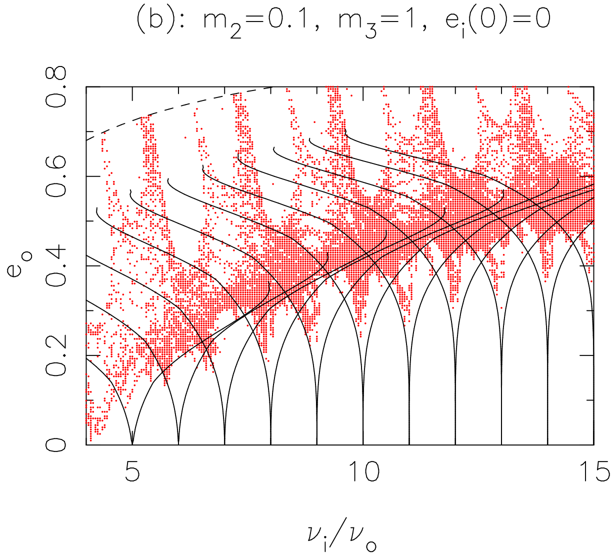
Results



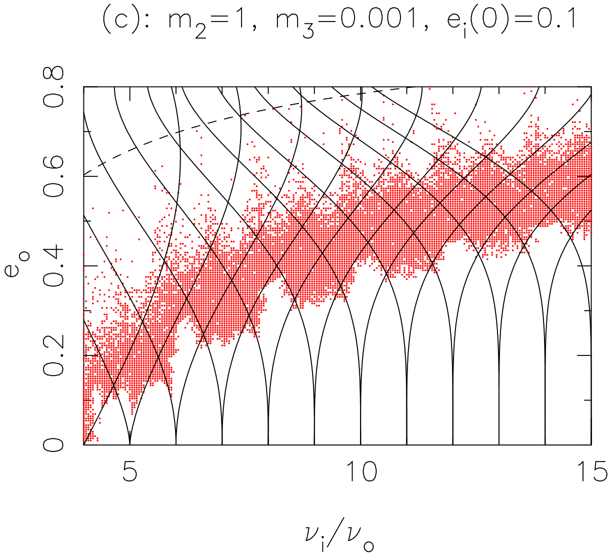
Results



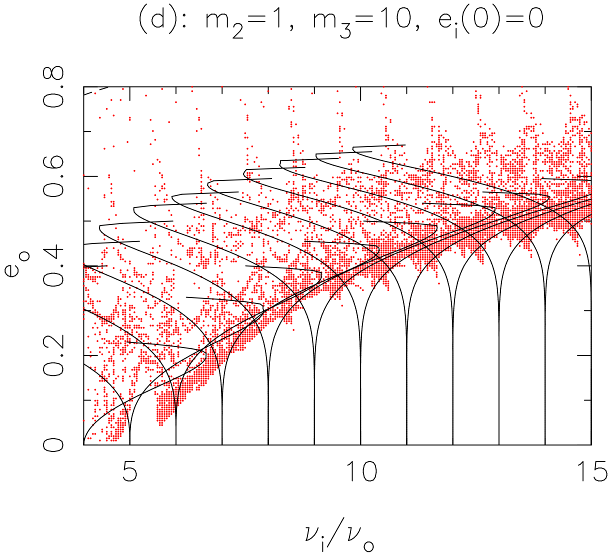
Results



Results

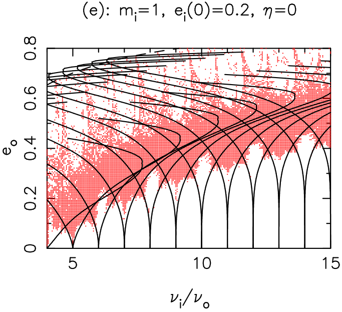


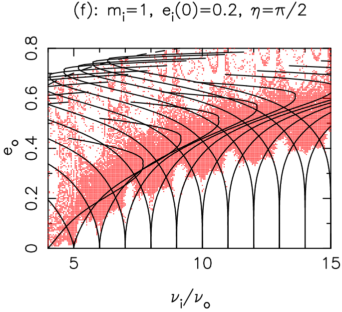
Results



Results

Changing the relative orientation





$$\eta = \varpi_i - \varpi_o$$

Stability Algorithm

1. Determine which n:1 resonance the system is near:
   $n = \lfloor \nu_i/\nu_o \rfloor$
2. Calculate distance from exact resonance:
   $(\delta\sigma)_n = \nu_i/\nu_o - n$
3. Calculate "pendulum energies" $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$ with $\phi_n = 0$:
   $\mathcal{E}_n = \frac{1}{2}(\delta\sigma)_n^2 - (\Omega/\nu_o)^2(1+\cos\phi_n)$

If both $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$ are negative, system is *unstable*

Stability Algorithm