Self-consistent triaxial (tidal) models for Globular Clusters

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KITP January 27, 2009

Outline

1. Flattening of globular clusters: why should we worry?
2. Observational evidence for the deviations from sphericity
3. The tidal triaxial models
   - the construction method (Bertin & Varri 2008 ApJ 689)
   - intrinsic and projected properties (Varri & Bertin in prep.)

Relevance of (tidal) nonspherical models for GCs

- It is commonly believed that the presence of a truncation radius (r_{tr}) in GCs is due to the effect of the tidal field of the hosting galaxy.
- Spherical symmetry of the GCs = 0th order approximation.
- In reality, the tidal field is nonspherical and, in fact, GCs are neither isolated nor strictly spherical.
- (partially) Self-consistent models in which the external tidal field is explicitly taken into account can be constructed, at least for a simple orbit of the cluster.
- The general case (with non-stationary tides) is not suited for a description in terms of simple analytical equilibrium models.

GCs as quasi-relaxed stellar systems

- Globular clusters: $t_{rel} < t_{age}$

  Two-stars relaxation processes should have had enough time to bring them close to a thermodynamically relaxed state, with their distribution function close to a Maxwellian.

- Elliptical galaxies: $t_{age} \ll t_{rel}$

  Only partially relaxed stellar systems; should be thought of as truly collisionless stellar systems, generally characterized by an anisotropic pressure tensor.

Simple self-consistent stationary dynamical models, i.e. equilibrium solutions of the collisionless Boltzmann equation + Poisson equation, can offer a simplified but useful representation of these stellar systems.

Origin of the flattening of GCs?

- Internal rotation
  - solid-body?

    Should be expected, by recalling the assumption of quasi-relaxation.
  - differential?

    To be used for interpreting objects with non-monotonic rotation curve and ellipticity profile.

  Few equilibrium models - for GCs: Kormendy& Anand Ap&SS 1971, Longaretti&Lagoute A&A 1996; for elliptical galaxies: Prendergast&Tomer AJ 1970, Wilson AJ 1970

- Anisotropy in velocity space

  Should be less relevant for quasi-relaxed systems as GCs, while is considered the primary cause of the flattening of bright ellipticals, as a signature of their formation process (violent relaxation). Equilibrium models based on statistical mechanics arguments: Bertin & Stiavelli Rep. Prog. Phys. 1993, Bertin & Trenti ApJ 2003

- External tidal field

  Maybe the natural expectation?

  Equilibrium models proposed by Weinberg ASPC 1993, Heggie & Ramamani MNRAS 1995

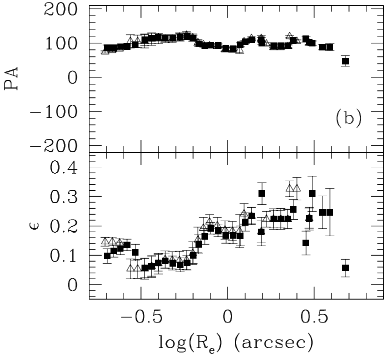
- It remains to be established which physical ingredient is the primary cause of the flattening of GCs (see van den Bergh 2008).
- Usually, the observed deviations from sphericity in the inner part of the objects are interpreted as an effect of the internal rotation (see King 1961), while the outer parts are thought to be flattened by the external tidal field.
- The recent progress in the acquisition of detailed quantitative information about the structure and the kinematics of GCs calls for renewed efforts on the side of modeling.
- To relax the assumption of spherical symmetry can be useful for deeper investigation of the structure and dynamics of these stellar systems (e.g., for elliptical galaxies, recognizing the distinction between rotation/anisotropy-supported systems is an important clue for relevant formation scenarios).

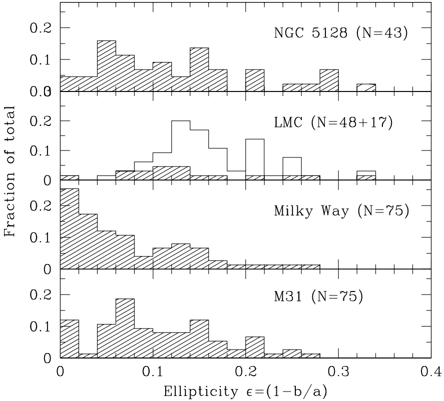
Observational evidence for the flattening of GCs

No uniformly accepted definitions of the shape parameters.
$\longrightarrow$ difficult to compare

- Galactic GCs
  - $< e >= 0.07 \pm 0.01$, PAs not correlated with direction of the galactic center

    (White&Schawl ApJ 1987, the values do not refer to a standard isophote)
  - just few ellipticity profiles are measured (Geyer et al A&A 1983)
- Extragalactic GCs
  - in LMC, SMC more flattened than GGC (Kontizas et al AJ 1989, Frenck & Fall MNRAS 1982; Kontizas et al AJ 1990)
  - M31: similar distribution of ellipticities (Barmby et al AJ 2002, 2007)
  - NGC 5128 more flattened than GGC (Harris et al AJ 2002, 2006)





from Barmby et al AJ 2002: ellipticity profile and position angle for the object 240-302 in M31

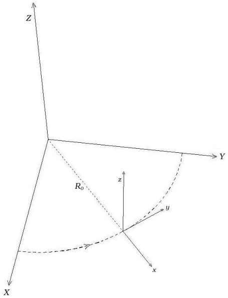
from Harris et al AJ 2002

$e = 0.16 \pm 0.01$

The triaxial (tidal) models

Simple physical model

1. $\Phi_{gal} = \Phi_{gal}(R)$ with $R^2 = X^2 + Y^2 + Z^2$
2. GC on a circular orbit in (X, Y) with $R = R_0$ and orbital frequency $\underline{\Omega} = \Omega\ \hat{e}_Z$
3. $R_0 \gg r_{tr}$
4. $M_{gal} \gg M$



Corotating frame of reference

Within the "tidal approximation", the Jacobi integral is available:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_T + \Phi_c$$

Tidal potential: $\Phi_T = \frac{1}{2}\Omega^2 \left(z^2 - \nu x^2\right)$

$\nu = 4 - \frac{\kappa^2}{\Omega^2}$ with $\nu \in [2, 3]$, typically.

Distribution function

- Focus on the generalization of the spherical King models:

  In principle, every spherical isotropic truncated model can be used.

$$f_K(H) = \begin{cases} A\left[\exp\left(-aH\right) - \exp\left(-aH_0\right)\right] & \text{if } H \leq H_0 \\ 0 & \text{if } H > H_0 \end{cases}$$

  H_0 determined by the depth of the central potential well.

- Associated density profile:

$$\rho(\psi) = \int f_K(H) d^3v = \tilde{A}\exp(\psi)\gamma\left(\frac{5}{2}, \psi\right) = \tilde{A}\hat{\rho}(\psi)$$

  with dimensionless escape energy

$$\psi(\underline{r}) = a(H_0 - [\Phi_c(\underline{r}) + \Phi_T(\underline{r})])$$

- The collisionless Boltzmann equation is satisfied; we than proceed to solve the Poisson equation.

Two-parameters family of models

The models are characterized by:

- two physical scales (i.e., free constants A and a)
- two dimensionless parameters

$$\text{Concentration} \quad \leftrightarrow \quad \Psi \equiv \psi(\underline{0}) = W_0$$

$$\text{Tidal strength} \quad \leftrightarrow \quad \epsilon \equiv \frac{\Omega^2}{4\pi G \rho_0}$$

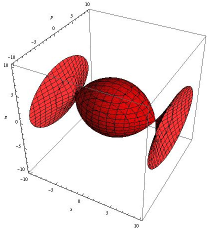
For a given value of the depth of the central potential well Ψ, there exists a (maximum) critical value for the tidal strength parameter ϵ_{cr}.

We consider only closed configurations (i.e. subcritical models, with $\epsilon \leq \epsilon_{cr}$).

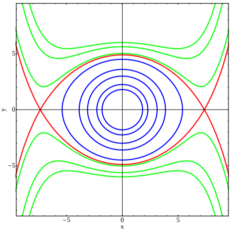
General properties of the models

- The deformation is shaped by the tidal potential and depends on ν:
  - "compression" along the $\hat{z}$-axis
  - "elongation" along the $\hat{x}$-axis
- coordinates planes = symmetry planes
- the boundary = equipotential surface for $a\Phi_{tot} = a(\Phi_c + \Phi_T)$

  Critical models: with singular Hill surface
- 2 tidal subcritical regimes:
  - weak deformation: $r_{tr} << r_J$
  - strong deformation: $r_{tr} \approx (2/3) r_J$
- For all the models, the deformation is monotonically increasing with the distance from the center.



critical Hill surface



Sections of possible Hill surfaces in the plane $(\hat{x}, \hat{y})$

The mathematical problem

- Dimensionless formulation:

$\hat{r} = r/r_0$ with $r_0 = \sqrt{9/(4\pi G \rho_0 a)}$

$$\hat{\nabla}^2\psi = -9\left[\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)} + \epsilon(1-\nu)\right] \quad \text{for} \quad \psi > 0 \quad \text{(Poisson)}$$

$$\hat{\nabla}^2\psi = -9\epsilon(1-\nu) \quad \text{for} \quad \psi < 0 \quad \text{(Laplace)}$$

- The two domains are thus separated by the boundary surface of the configuration defined by $\psi(\hat{r}) = 0$ which is unknown *a priori*.
  $\rightarrow$ we have to solve a 2nd order PDE in a free boundary problem

- BCs Poisson eq. (internal region): $\psi(\underline{0}) = \Psi$ and $\hat{\nabla}\psi(\underline{0}) = \underline{0}$
  BCs Laplace eq. (external region): $\Phi_c \rightarrow 0$ for $\hat{r} \rightarrow \infty$

- Idea: formally solve the equations and patch the solutions (and their gradient), in order to determine the free constants (note that 1 BC is missing).

Perturbative approach

- Tidal effect = (small) perturbation acting on the configuration described by the spherical King models:
  $\epsilon \ll 1 \quad \Leftrightarrow \quad t_D \ll T = 2\pi/\Omega$

$$\psi(\underline{\hat{r}}; \epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!} \psi_k(\underline{\hat{r}}) \, \epsilon^k$$

- Expansion of the general term of the series $\psi_k(\underline{r})$ in spherical harmonics $\rightarrow$ one-dimensional (radial) Cauchy problems.
- This perturbation problem is singular!
  The convergence radius of the asymptotic series vanishes $\hat{r} \rightarrow \hat{r}_{tr}$, i.e. the validity of the expansion breaks down when $\psi_0 = \mathcal{O}(\epsilon)$.
  - Introduction of an intermediate region (boundary layer)
  - Asymptotic matching for $(\psi^{(\mathrm{int})}, \psi^{(\mathrm{lay})})$ and $(\psi^{(\mathrm{lay})}, \psi^{(\mathrm{ext})})$ using the Van Dyke prescription (1975).

(Same problem arises in the study of the rigidly rotating polytropes, Smith B.L. Ap&SS 1975)

- We calculated the full explicit solution to 2 orders in ϵ.
  The relevant PDEs for $\psi_k(\hat{\underline{r}})$ are thus reduced to sets of simple (radial) ODEs for which a numerical solution is required only in the internal domain, since for the equations in the boundary layer and in the external region a formal solution is available.
  (the details on request!)

- $\psi^{(1)} = \psi_0 + \psi_1\,\epsilon \quad \leftrightarrow$ harmonics with $l = 0, 2 \quad m \geq 0$ even
  $\psi^{(2)} = \psi^{(1)} + \psi_2\,\epsilon^2/2 \leftrightarrow$ harmonics with $l = 0, 2, 4 \quad m \geq 0$ even

- From the structure of the relevant equations, we prove, *by induction*, that the k-th order solution $\psi^{(k)}$ contains only the $l = 0, 2, .., 2k$ harmonics.

The Parameter Space

- 2 dimensionless parameters (Ψ, ϵ)
- 3 relevant radii:
  - r_0 scale radius
  - r_{tr} truncation radius $\psi_0(\hat{r}_{tr}) = 0$
  - r_J tidal (Jacobi) radius

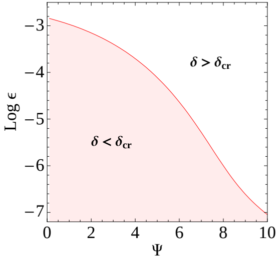
  For a fixed model, $\frac{\partial \psi}{\partial \hat{x}}(\hat{r}_J, 0, 0; \epsilon) = 0$

  0th order approximation: $r_J = \left(\frac{GM}{\Omega^2 \nu}\right)^{1/3}$
- Alternatively:
  - $C = log(r_{tr}/r_0)$ Concentration parameter
  - $\delta = r_{tr}/r_J$ "Extension" parameter
- Critical models: (Ψ, ϵ_{cr})

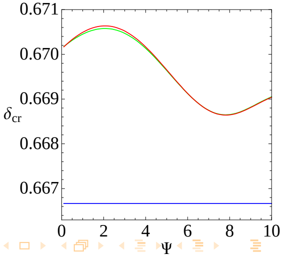
$$\begin{cases} \frac{\partial \psi}{\partial \hat{x}}(\hat{r}_J, 0, 0; \epsilon_{crit}) = 0 \\ \psi(\hat{r}_J, 0, 0; \epsilon_{crit}) = 0 \end{cases}$$

0th order approximation: $\delta_{cr} = 2/3$ (Spitzer 1987)

The critical values of $\hat{r}_J$ and ϵ slightly depend on Ψ



red line = critical 2nd order models with $\nu = 3$



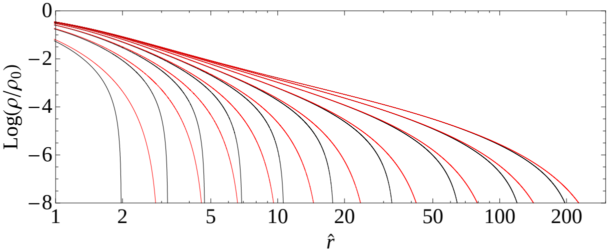
Second order models

$\Psi = 1, ..., 10$

$\epsilon = \epsilon_{cr}(\Psi)$

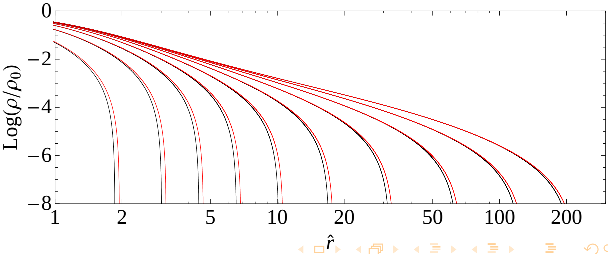
$\nu = 3$

(Keplerian galaxy)



Top panel:
red= triaxial models along $\hat{x}$-axis

black=King models



Bottom panel:
red= triaxial models along the $\hat{y}$-axis

black=along the $\hat{z}$-axis

Projected density and velocity dispersion profiles

l.o.s = $\hat{y}$-axis

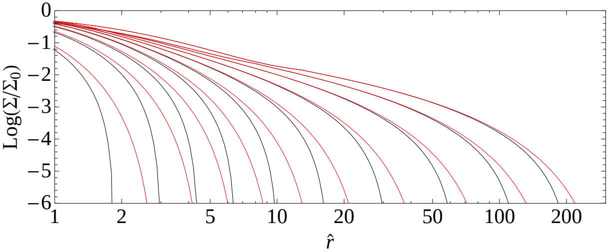
Second order models

$\Psi = 1, ..., 10$

$\epsilon = \epsilon_{cr}(\Psi)$

$\nu = 3$

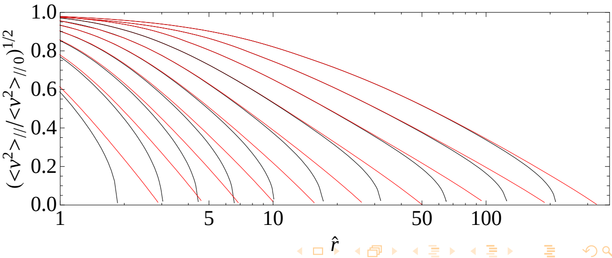
(Keplerian galaxy)



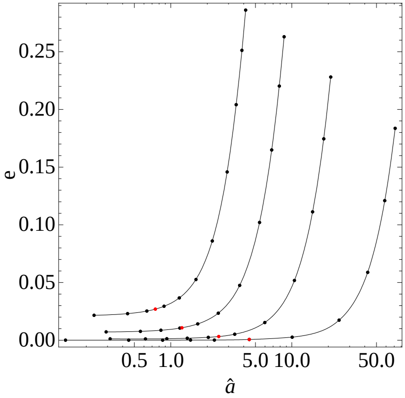
Both panels:

red= triaxial models along the $\hat{x}$-axis

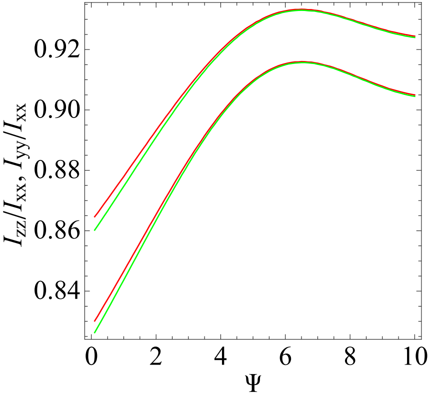
black=along the $\hat{z}$-axis



Deviations from sphericity



Ellipticity profiles of the projected image (with major axis $\hat{a}$) of the critical 2nd order models with $\Psi = 2, 4, 6, 8$ l.o.s.=$\hat{y}$-axis. Black dots: Σ/Σ_0 in the range $[0.9, 10^{-6}]$; red dot = half-light isophote.



Axis ratios of the inertia ellipsoid of the critical 2nd order models; low-Ψ critical models are the most flattened.

Conclusions

- Models of quasi-relaxed triaxial stellar systems in which the shape is due to the presence of external tides have been constructed; in the absence of the tidal field, they reduce to spherical King models.
- The same procedure has been used to extend other isotropic truncated models to the triaxial case (e.g. polytropes of index $1 < n < 5$).
- Extensions of spherical King models to the case of internal (solid body) rotation can be performed with the same mathematical approach (work in progress!)

Future developments

- Our models may be useful for the construction of ICs for N-body models of tidally perturbed clusters (also in the "underfilled" regime).
- Comparison with the observations (bearing in mind the simple assuption of circular orbit of the cluster): the moderate flattening of the models seems to be consistent with the observed values.
- Our models may be relevant for the interpretation of the "extratidal light", i.e. structure in the surface brightness profile extending beyond what is prescribed by spherical King models (van der Marel & McLaughlin ApJ 2005, Harris et al AJ 2006), and, in general, for the discussion about r_{tr} vs r_J.

Details of the mathematical method

Internal region:

$k = 0$ Cauchy problem for spherical King models

$$\begin{cases} \psi_0^{(int)\prime\prime} + (2/\hat{r})\psi_0^{(int)\prime} = -9\hat{\rho}\left(\psi_0{}^{(int)}\right)/\hat{\rho}\left(\Psi\right) \\ \psi_0{}^{(int)}(0) = \Psi \\ \psi_0{}^{(int)\prime}(0) = 0 \end{cases}$$

$$R_j(\Psi, \hat{r}) \equiv \frac{9}{\hat{\rho}(\Psi)} \frac{d^j \hat{\rho}}{d\psi^j}\bigg|_{\psi_0}$$

$k = 1$

$$\begin{cases} \left[\hat{\nabla}^2 + R_1(\Psi, \hat{r})\right] \psi_1{}^{(int)} = -9(1-\nu) \\ \psi_1{}^{(int)}|_{\hat{r}=0} = 0 \\ \hat{\nabla}\psi_1{}^{(int)}|_{\hat{r}=0} = \underline{0} \end{cases}$$

$k = 2$

$$\begin{cases} \left[\hat{\nabla}^2 + R_1(\Psi, \hat{r})\right] \psi_2{}^{(int)} = -R_2(\Psi, \hat{r})\, \psi_1{}^{(int)\,2} \\ \psi_2{}^{(int)}|_{\hat{r}=0} = 0 \\ \hat{\nabla}\psi_2{}^{(int)}|_{\hat{r}=0} = \underline{0} \end{cases}$$

General kth-order equation

$$\begin{cases} \left[\hat{\nabla}^2 + R_1(\Psi,\hat{r})\right]\psi_k = -\displaystyle\sum_{j=2}^{k-1} R_j(\Psi,\hat{r})\, X_{k,j} \\ \psi_k|_{\hat{r}=0} = 0 \\ \hat{\nabla}\psi_k|_{\hat{r}=0} = \underline{0} \end{cases}$$

- Differential operator ("shifted Laplacian") and BCs are the same at every order $k > 0$.
- Coefficients $X_{k,j}(\underline{r})$ are determined by the expansion of the r.h.s. of Poisson eq. (they are related to j-part partitions of the integer k).

Example : $k = 3$

2-part partition	$2+1$	$\rightarrow X_{3,2} = \frac{3!}{2!1!}\psi_2\psi_1 = 3\psi_2\psi_1$
3-part partition	$1+1+1$	$\rightarrow X_{3,3} = \frac{3!}{1!1!1!3!}{\psi_1}^3 = {\psi_1}^3$

Consistent with *the Faà di Bruno's formula* (1855), which generalize to higer order the chain rule for the derivative of the composite of two functions.

- Reduction to one dimensional (radial) problem: expansion of $\psi_k(\underline{r})$ in (real) spherical harmonics $Y_{l,m}(\theta, \phi)$.
- Relevant differential operator:

$$\mathcal{D}_l \equiv \frac{1}{\hat{r}^2}\frac{d}{d\hat{r}}\left(\hat{r}^2\frac{d}{d\hat{r}}\right) - \frac{l(l+1)}{\hat{r}^2} + R_1(\Psi, \hat{r})$$

For $l > 0$ also the homogeneous solutions ($\sim \hat{r}^l$ for $\hat{r} \to 0$) are non trivial and must be taken into account!

$k = 1$

$l = 0$ $\qquad f_{00} = \psi_{1\,00}/\sqrt{4\pi}$

$$\begin{cases} \mathcal{D}_0\, f_{00} = -9(1-\nu) \\ f_{00}(0) = 0 \\ f'_{00}(0) = 0 \end{cases}$$

$l > 0$ $\qquad \psi_{1lm}(\hat{r}) = A_{lm}\gamma_l(\hat{r})$ $\qquad$ Note: both equation and BCs are homogeneous!

$$\begin{cases} \mathcal{D}_l\, \psi_{1\,lm} = 0 \\ \psi_{1\,lm}(0) = 0 \\ \psi'_{1\,lm}(0) = 0 \end{cases}$$

Complete solution:

$$\psi_1(\underline{\hat{r}}) = f_{00}(\hat{r}) + \sum_{l=1}^{\infty} \sum_{m=0}^{l} A_{lm}\gamma_l(\hat{r}) Y_{lm}(\theta, \phi).$$

$k = 2$

$l \geq 0$

$$\begin{cases} \mathcal{D}_l\, \psi_{2\,lm} = -R_2(\Psi, \hat{r})\ [{\psi_1}^2]_{lm} \\ \psi_{2\,lm}(0) = 0 \\ \psi'_{2\,lm}(0) = 0 \end{cases}$$

The expansion of the r.h.s. in spherical harmonics can be performed easily by means of the 3-j Wigner symbols.

Complete solution:

particular solutions $\downarrow$ $\downarrow$ homogeneous solutions

$$\psi_2(\underline{\hat{r}}) = g_{00}(\hat{r}) + \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left[g_{lm}(\hat{r}) + B_{lm}\gamma_l(\hat{r})\right] Y_{lm}(\theta, \phi)$$

Free constants A_{lm}, B_{lm} are determined by the matching.

Boundary layer:

- New radial variable η such that: $\eta = (\hat{r}_{tr} - \hat{r})/\epsilon$.
- The solution must be scaled with respect to the tidal parameter: $\tau = \psi^{(lay)}/\epsilon$.
- For positive values of τ the Poisson eq. thus becomes:

$$\frac{\partial^2 \tau}{\partial \eta^2} - \frac{2\epsilon}{\hat{r}_{tr} - \epsilon\eta}\frac{\partial \tau}{\partial \eta} + \frac{\epsilon^2}{(\hat{r}_{tr} - \epsilon\eta)^2}\Lambda^2\tau = -\frac{9}{\hat{\rho}(\Psi)}\,\epsilon\hat{\rho}(\epsilon\tau) - 9\epsilon^2(1-\nu)\ ,$$

Λ^2 is the angular part of the Laplacian in spherical coordinates

- Asymptotic expansion of τ and calculation of the ODEs for each term (as in the internal region).
- Note that:

$$\hat{\rho}(\epsilon\tau) \sim \frac{2}{5}\tau^{5/2}\epsilon^{5/2} + \frac{4}{35}\tau^{7/2}\epsilon^{7/2} + \ldots$$

so that, within the boundary layer, the contribution of $\hat{\rho}(\epsilon\tau)$ becomes significant only beyond the tidal term.

$k = 0$

$$\frac{\partial^2 \tau_0}{\partial \eta^2} = 0$$

$$\tau_0 = F_0(\theta, \phi)\eta + G_0(\theta, \phi)$$

$k = 1$

$$\frac{\partial^2 \tau_1}{\partial \eta^2} = \frac{2}{\hat{r}_{tr}} \frac{\partial \tau_0}{\partial \eta}$$

$$\tau_1 = \frac{F_0(\theta, \phi)}{\hat{r}_{tr}}\eta^2 + F_1(\theta, \phi)\eta + G_1(\theta, \phi)$$

The free angular functions that appear in the formal solutions will be determined by the matching procedure.

External region:

$$\psi^{(ext)} = \alpha - \frac{\lambda}{\hat{r}} - \sum_{l=1}^{\infty} \sum_{m=1}^{l} \frac{\beta_{lm}}{\hat{r}^{l+1}} Y_{lm}(\theta, \phi) - \frac{9}{2} \hat{r}^2 (\cos^2\theta - \nu \sin^2\theta \cos^2\phi)\epsilon$$

$$\alpha = \alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2/2 + \dots$$
$$\lambda = \lambda_0 + \lambda_1\epsilon + \lambda_2\epsilon^2/2 + \dots$$
$$\beta_{lm} = a_{lm}\epsilon + \beta_{lm}\epsilon^2/2 + \dots$$

Free constants $\alpha_0, \lambda_0, \alpha_1, \lambda_1, a_{lm}, \alpha_2, \lambda_2, b_{lm}$ are determined by the matching.

The asymptotic matching

- We must perform separately the matching of the pairs $(\psi^{(int)}, \psi^{(lay)})$ and $(\psi^{(lay)}, \psi^{(ext)})$.
- Using the Van Dyke prescription (1975), we compare the second order expansion in ϵ of the internal/external solutions (expressed with the scaled variables) with the third order expansion in ϵ of the boundary layer solution (expressed with the unscaled variables).
- Note that the conditions for the constants with $l > 0$ (for a chosen pair (l, m)) can be expressed as a linear system: $M_{ij} u_j = v_i$

$$M = \begin{pmatrix} \gamma_l(\hat{r}_{tr}) & \hat{r}_{tr}^{-(l+1)} \\ -\gamma_l'(\hat{r}_{tr})\hat{r}_{tr} & (l+1)\hat{r}_{tr}^{-(l+1)} \end{pmatrix}$$

$$k = 1 \ (u_1, u_2) = (A_{lm}, a_{lm}) \quad (v_1, v_2) = -T_{lm}(\hat{r}_{tr})(1, -2)$$
$$k = 2 \ (u_1, u_2) = (B_{lm}, b_{lm}) \quad (v_1, v_2) = (-g_{lm}(\hat{r}_{tr}), \hat{r}_{tr} g_{lm}'(\hat{r}_{tr}))$$

$\rightarrow$ Since M is non-singular for every l, only the harmonics with non-trivial component of the tidal field / particular solutions have non-vanishing constants.

- Therefore we have:
  $k = 1$ $l = 0, 2$ with corresponding $m \geq 0$ and even
  $k = 2$ $l = 0, 2, 4$ with corresponding $m \geq 0$ and even
- Recalling that the harmonic expansion of the product of two spherical harmonics (l_1, m_1) and (l_2, m_2) can be expressed by means of 3-j Wigner symbols, we note that the composed harmonic (l, m) must satisfy the following *selection rules*:

  (i) $|l_1 - l_2| \leq l \leq l_1 + l_2$ ("triangular inequality")
  (ii) $m_1 + m_2 = m$
  (iii) $l_1 + l_2 + l$ must be even

  $\rightarrow$ the kth-order term is characterized by harmonics with $l = 0, 2, ..., 2k$ and corresponding positive and even values of m