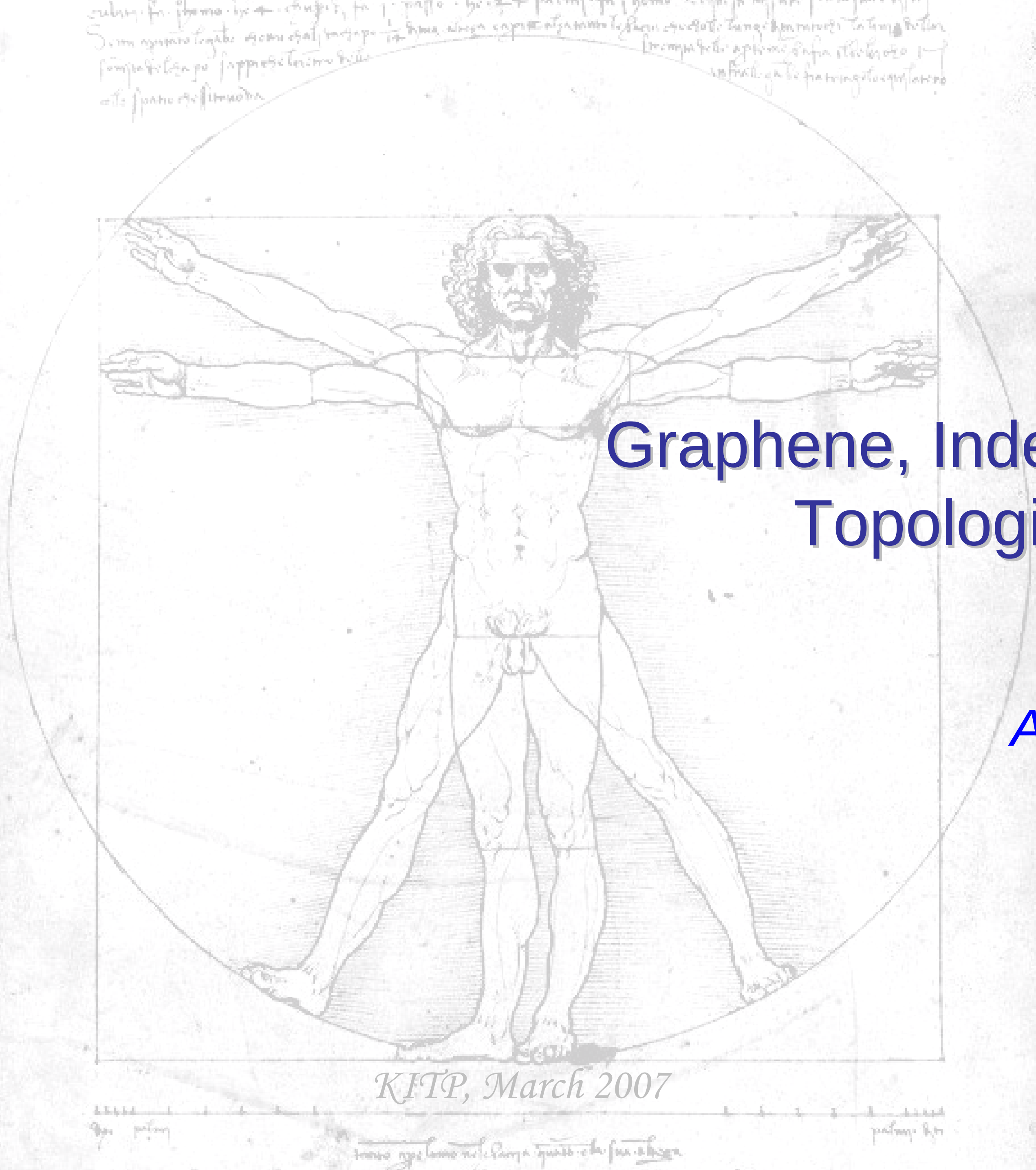


UNIVERSITY OF LEEDS

Graphene, Index Theorem and Topological Degeneracy

JKP

Agapitos Hatzinikitas

Michael Stone

cond-mat/0607394
quant-ph/0701203

THE ROYAL SOCIETY

KITP, March 2007

Experimental verification of anyonic statistics with photons

- JKP
- Christian Schmid
- Witlef Wieczorek
- Reinhold Pohlner
- Nikolai Kiesel
- Harald Weinfurter

MAX-PLANCK-INSTITUT FÜR QUANTENOPTIK GARCHING

MPQ

xqp

Kitaev's honeycomb lattice model

- Energy of vortices
- Energy of fermions
- Vortex interactions
- Breakdown of non-abelian topological phase
- $B=0$ and $B\neq 0$

Frustration of cold atoms in optical lattices

- 1 Dim optical lattice that simulates a ladder
- Exactly solvable limits
- Breakdown of superfluidity for large tunneling couplings
- Fragmentation of superfluid

Chiral interactions in triangular lattices

$$H=-\sum_{<i,j>}\vec{\sigma}_i\cdot\vec{\sigma}_j-\lambda\sum_{<i,j,k>}\vec{\sigma}_i\cdot\vec{\sigma}_j\times\vec{\sigma}_k$$

- Ground state degeneracy
- Chiral spin liquid states
- …

James Wootton, Ville Lahtinen, Kristan Temme --- Juanjo Ripoll, Dimitris Tsomokos

Quantum Information, Physics and Topology

- **Encoding and manipulating QI** in small physical systems is pledged by **decoherence** and **control errors**.

- **Error correction** can be employed to resolve this problem by using a (huge) overhead of qubits and quantum gates.

- An alternative method is to employ **intrinsically error protected systems** such as **topological** ones => properties are described by integer numbers!

  protected by macroscopic properties: hard to destroy.

- E.g. you can employ system with **degenerate ground states**:

  - Make sure **degeneracy is protected by topological properties** (V)
  - Make sure degenerate states are **locally indistinguishable** (X?)
  - Encode information in these **degenerate levels**

TOPOLOGICAL DEGENERACY

Anyons

for your great kindness in the matter of the names respecting which I applied to you; but I hoped to have met you last Saturday at Kensington and therefore delayed expressing my obligations.

I have taken your advice and the names used are <u>anode</u> <u>cathode</u> <u>anions</u> <u>cations</u> and <u>ions</u> the last I shall have but little occasion for. I had some hot objections made to them here and found myself very much in the condition of the man with his son and Ass who tried to please every body; but when

Letter from Faraday to Whewell (1834)

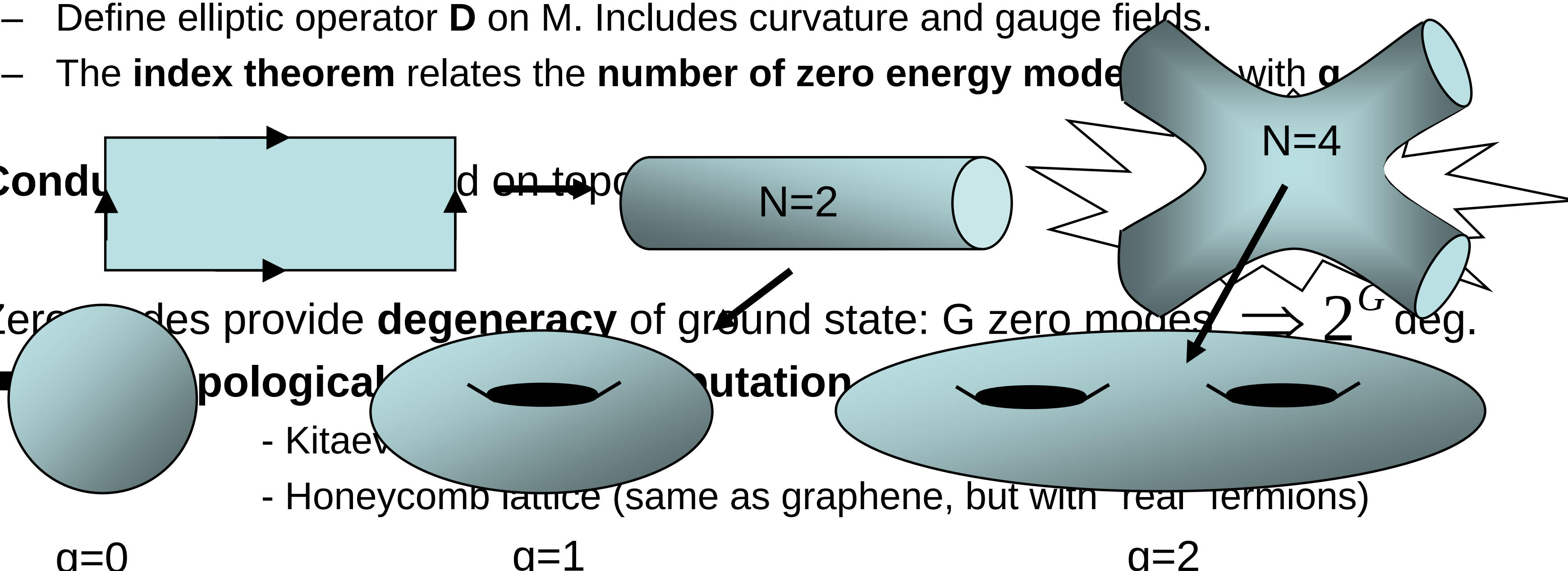
Overview

- **Graphene:** two dimensional layer of graphite –honeycomb lattice of C atoms
  - Fullerene: C60, C70
  - Nanotubes
- Conducting properties of these materials: zero energy modes.
- Can be used as miniaturized elements of circuits.

- **Index theorem** (Atiyah-Singer)
  - Smooth, orientable, compact, Riemannian manifolds, M, with genus, **g.**
  - Define elliptic operator **D** on M. Includes curvature and gauge fields.
  - The **index theorem** relates the **number of zero energy mode** ... with **g**

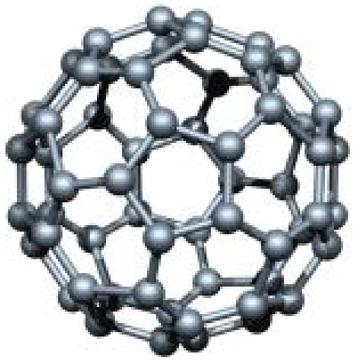
- **Condu**... d on topo...

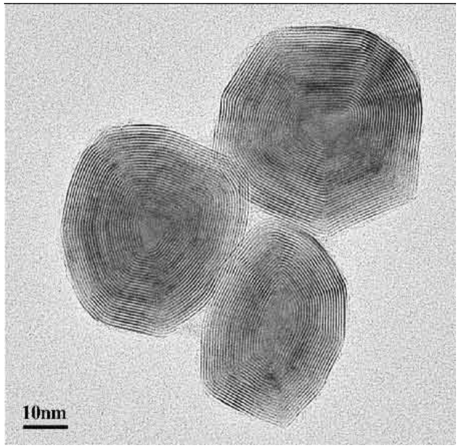
- Zer... des provide **degeneracy** of ground state: G zero modes $\Rightarrow 2^G$ deg.
  - ■ ...**pological** ... **utation**
    - Kitaev...
    - Honeycomb lattice (same as graphene, but with "real" fermions)



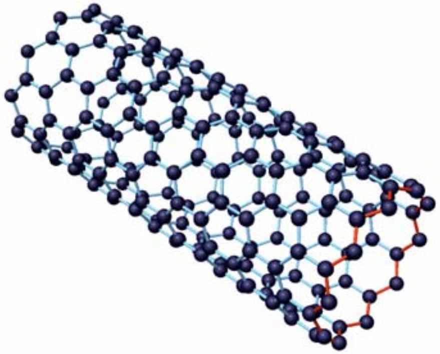
Different geometries of Graphene

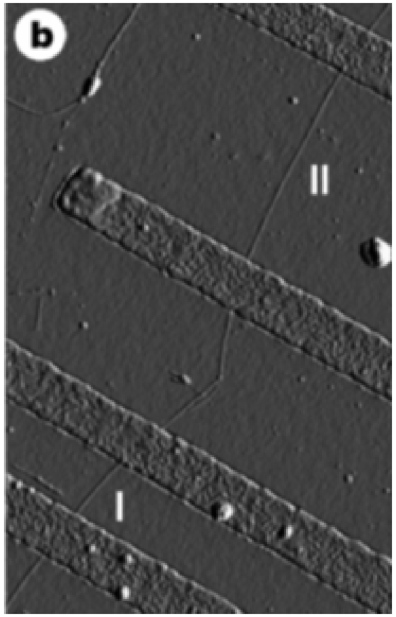
Fullerene (C60):

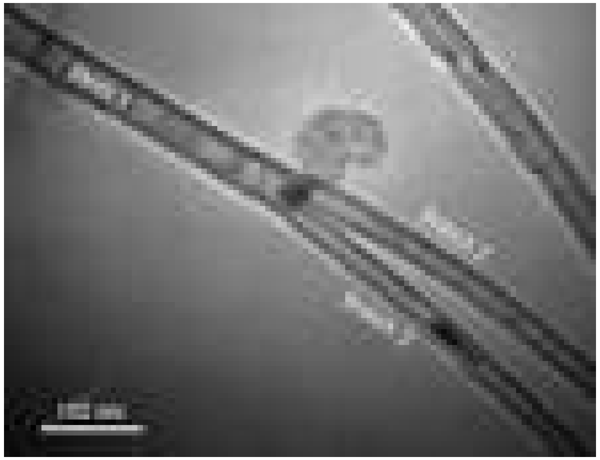




Nanotubes:





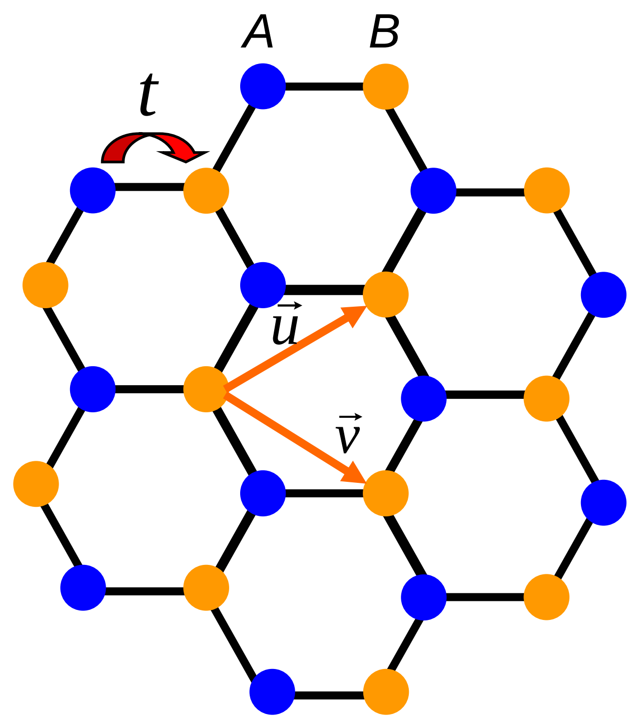


Graphene: structure

The Hamiltonian of graphene is given by

$$H = -t\sum_{<i,j>} a_i^{+} a_j = -\frac{t}{2}\sum_{<i,j>} (a_i^{+} b_j + b_j^{+} a_i)$$

a_i fermionic modes



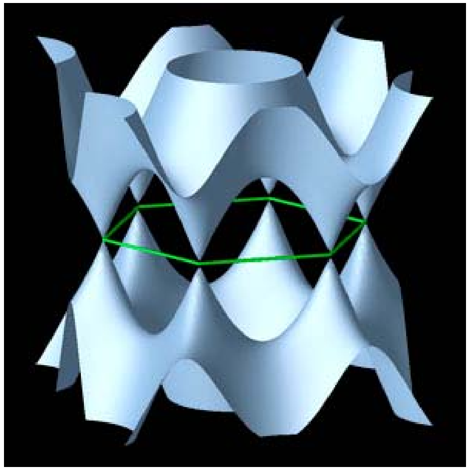
Fourier transformation:

$$H_{\vec{k}} = \begin{pmatrix} 0 & -t(1+e^{-i\vec{k}\cdot\vec{u}}+e^{-i\vec{k}\cdot\vec{v}}) \\ -t(1+e^{i\vec{k}\cdot\vec{u}}+e^{i\vec{k}\cdot\vec{v}}) & 0 \end{pmatrix}$$

$$E(\vec{k}) = \pm t\sqrt{3+2\cos\vec{k}\cdot\vec{u}+2\cos\vec{k}\cdot\vec{v}+2\cos\vec{k}\cdot(\vec{u}-\vec{v})}$$

Fermi points: *E(k)=0*

E(kx,ky):

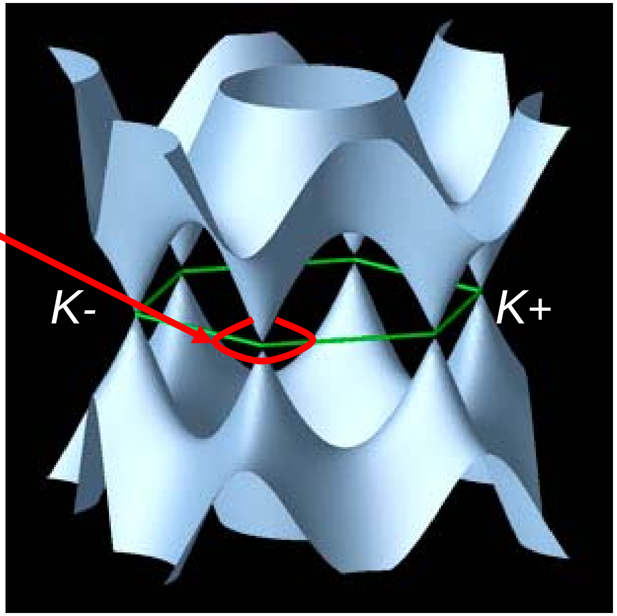


Graphene: structure

$$E(\vec{k}) = \pm t\sqrt{3 + 2\cos\vec{k}\cdot\vec{u} + 2\cos\vec{k}\cdot\vec{v} + 2\cos\vec{k}\cdot(\vec{u}-\vec{v})}$$

Linearise energy $E(\vec{k})$ around a conical point,

$$\vec{k} = \vec{K} + \vec{p}$$



$$H_{\vec{p}} \approx \pm\frac{3t}{2}\begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \pm\frac{3t}{2}\vec{\sigma}\cdot\vec{p}$$

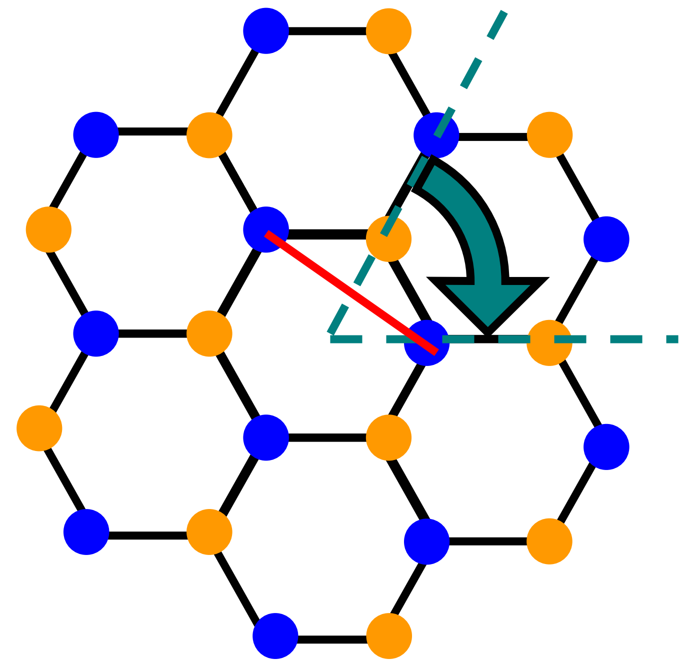
Relativistic Dirac equation at the tip of a pencil!

Two types of spinors: $\begin{pmatrix} |K_+, A\rangle \\ |K_+, B\rangle \end{pmatrix}, \begin{pmatrix} |K_-, A\rangle \\ |K_-, B\rangle \end{pmatrix}$

$K_\pm$ are the Fermi points and A and B are the two triangular sub-lattices

Note: σ^z rotation maps to states with the same energy, but opposite momenta

Graphene: curvature



To introduce curvature:

cut $\pi/3$ sector and reconnect sites.

This creates a single **pentagon** with no other deformations present.

Results in a **conical configuration**.

To preserve continuity of the spinor field when circulating the pentagon one can introduce **two additional fields**:

-Spin connection *Q*: $$\oint Q_\mu dx^\mu = -\frac{\pi}{6}\sigma^z$$ Mixes A and B components

-Non-abelian gauge field, *A*: $$\oint A_\mu dx^\mu = -\frac{\pi}{2}\tau^y$$ Mixes + and – spinors

Resulting 4x4 Dirac equation can be decoupled by simple rotation to a pair of 2x2 Dirac equations (k=1,2):

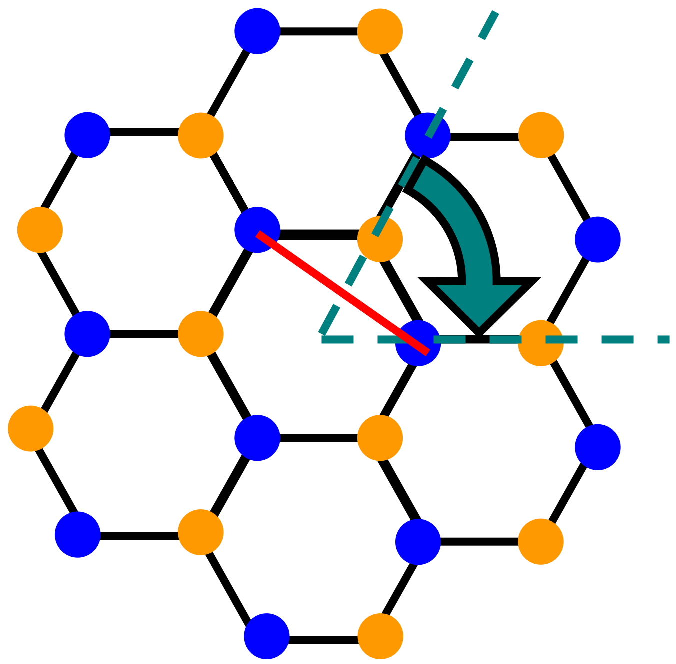
elliptic operator

$$\frac{3t}{2}\sum_\mu \gamma^\mu (p_\mu - iQ_\mu - iA_\mu^k)\psi^k = E\psi^k$$

$$\oint A_\mu^k dx^\mu = \pm\frac{\pi}{2}$$

Graphene: curvature

$$\frac{3t}{2}\sum_{a,\mu}\gamma^{\mu}(p_{\mu}-iQ_{\mu}-iA_{\mu}^{k})\psi^{k}=E\psi^{k}$$



$$F_{\mu\nu}^{k}=\partial_{\mu}A_{\nu}^{k}-\partial_{\nu}A_{\mu}^{k}$$

$$\gamma^{\mu}=e_{a}^{\mu}\gamma^{a},\; g^{\mu\nu}=e_{a}^{\mu}e_{b}^{\nu}\eta^{ab}$$

$$\Gamma_{\mu\nu}^{\sigma}=\frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho}+\partial_{\nu}g_{\mu\rho}-\partial_{\rho}g_{\mu\nu})$$

$$R^{\mu}{}_{\nu\rho\sigma}=\partial_{\sigma}\Gamma_{\nu\rho}^{\mu}-\partial_{\rho}\Gamma_{\nu\sigma}^{\mu}+\Gamma_{\nu\rho}^{\lambda}\Gamma_{\lambda\sigma}^{\mu}-\Gamma_{\nu\sigma}^{\lambda}\Gamma_{\lambda\rho}^{\mu}$$

$$R_{\mu\nu}=R^{\rho}{}_{\mu\nu\rho},\; R=g^{\mu\nu}R_{\mu\nu}$$

Continuous limit: Small energies => large wavelengths =>
insensitive to lattice spacing, conical singularity,…

Index Theorem

Consider operators, $P,\ P^+$ $\quad V_+ \xrightarrow{\ P\ } V_-\,,\ V_- \xrightarrow{\ P^+\ } V_+$

$$\text{For } \lambda \neq 0, \quad P^+Pu = \lambda u \Rightarrow (PP^+)Pu = \lambda Pu$$

non-zero modes come in pairs

Define (Dirac op.) $\quad D\!\!\!/ = \begin{pmatrix} 0 & P^+ \\ P & 0 \end{pmatrix}, \quad D\!\!\!/^2 = \begin{pmatrix} P^+P & \\ 0 & PP^+ \end{pmatrix}$ $\quad V = V_+ \oplus V_-$

Same number of zero modes as *D*

Define operator: $\quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad$ with eigenvalues +1, -1 for V_+, V_-

Consider $\nu_+,\ \nu_-$ the dimension of the **null** subspace of V_+, V_-

Then
$$Tr(\gamma_5 e^{-tD\!\!\!/^2}) = \sum_{Sp(P^+P)} e^{-t\lambda^2} - \sum_{Sp(PP^+)} e^{-t\lambda^2} = \nu_+ - \nu_- \equiv \text{index}(D\!\!\!/)$$

Non-zero eigenvalues cancel in pairs. **Expression is *t* independent.**

Index Theorem

$\not{D}$ can describe a general 2-dimensional **Dirac** operator defined over a **compact** surface coupled with a gauge field.

One can evaluate that

$$\not{D}^2 = -g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} - \frac{1}{4}R$$

metric covariant derivative gauge field curvature scalar

Heat kernel expansion (2-dims):

$$Tr(fe^{-tD}) = \frac{1}{4\pi t}\sum_{k\geq 0} t^{k/2} a_k(f,D)$$

For $f = \gamma_5,\ D = \not{D}^2$ the only non-zero coefficient is

$$a_2 = Tr\left\{\gamma_5\left(\frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} - \frac{1}{4}R\right)\right\} = 2\iint F \Rightarrow Tr(\gamma_5 e^{-t\not{D}^2}) = \frac{1}{2\pi}\iint F$$

Index Theorem

We have

$$Tr(\gamma_5 e^{-t\not{D}^2}) = \sum_{Sp(P^+P)} e^{-t\lambda^2} - \sum_{Sp(PP^+)} e^{-t\lambda^2} = \nu_+ - \nu_- \equiv \text{index}(\not{D})$$

Also

$$Tr(\gamma_5 e^{-t\not{D}^2}) = \frac{1}{2\pi}\iint F_{xy} d^2x$$

The **Index theorem** states: *for a (compact) manifold…*

$$\text{index}(D) = \nu_+ - \nu_- = \frac{1}{2\pi}\iint F$$

integer!

It is a **topological number**: small deformations do not change its value.

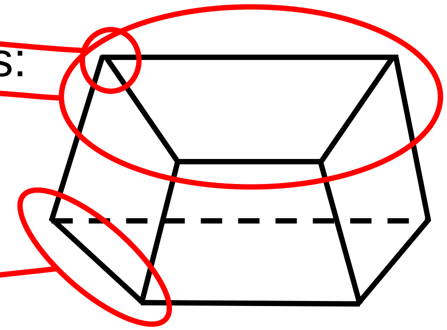
From this theorem you can obtain the **least number of zero modes.** The exact number is obtained if ν_+ or ν_- is equal to zero.

[Atiyah and Singer, Ann. of Math. 87, 485 (1968);...]

Index Theorem: Euler characteristic

Euler characteristic for lattices on “smooth” surfaces:

$$\chi = V - E + F = 2(1-g) - N$$



Consider folding of graphene in a compact manifold. The **minimal** violation is obtained by insertion of **pentagons** or **heptagons** that contribute positive or negative curvature respectively. Consider

- n_5 number of pentagons
- n_6 number of hexagons
- n_7 number of heptagons

$$\Rightarrow \quad V = (5n_5 + 6n_6 + 7n_7)/3$$

$$E = (5n_5 + 6n_6 + 7n_7)/2$$

$$F = n_5 + n_6 + n_7$$

From the Euler characteristic formula:

$$n_5 - n_7 = 6\chi = 12(1-g) - 6N$$

Fullerenes: $g = 0, N = 0 \Rightarrow n_5 = 12$

Nanotubes: $g = 0, N = 2 \Rightarrow n_5 - n_7 = 0$

Index Theorem: Graphene application

$$\iint F = \oint A$$

Stokes's theorem

$$\frac{1}{2\pi}\left(\pm\frac{\pi}{2}\right)(n_5 - n_7) = \pm\frac{3}{2}\chi$$

$$\text{index}(D) = \nu_+ - \nu_- = \frac{1}{2\pi}\iint F$$

Thus, one obtains:

$$\nu_+ - \nu_- = \begin{cases} \frac{3}{2}\chi, \text{ for } k=1 \\ -\frac{3}{2}\chi, \text{ for } k=2 \end{cases}$$

Least number of zero modes:

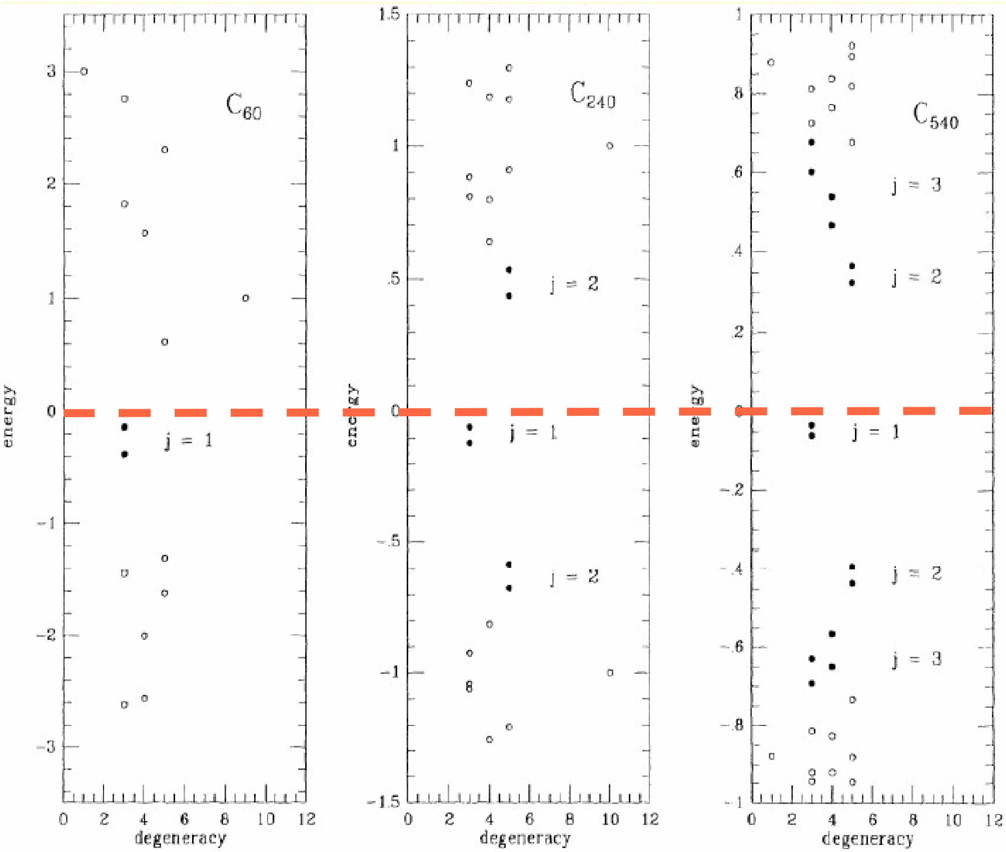
$$3\chi = 6|1-g| + 3N$$

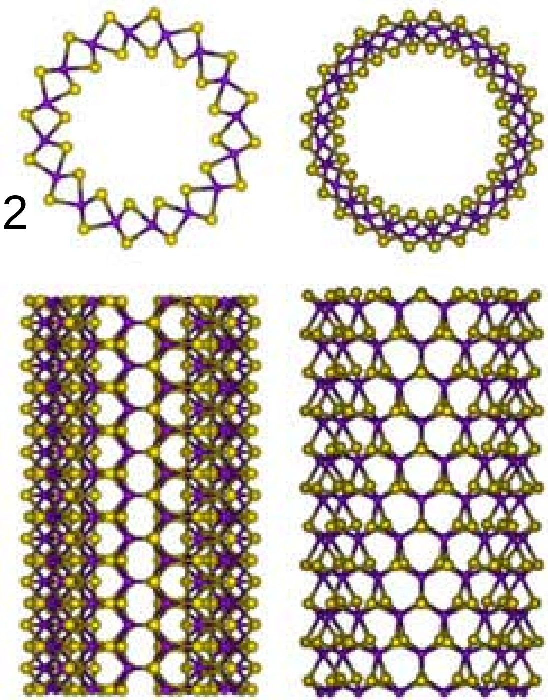
Index Theorem: Graphene application

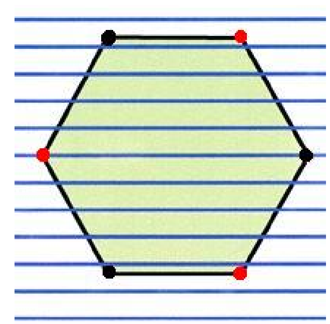
$$\mathrm{index}(D) = \nu_+ - \nu_- = 6\,|1-g\,| + 3N$$

C60: g=0, N=0

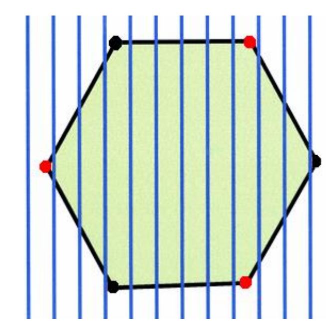
Nanotubes: g=0, N=2







Zero mode pairs



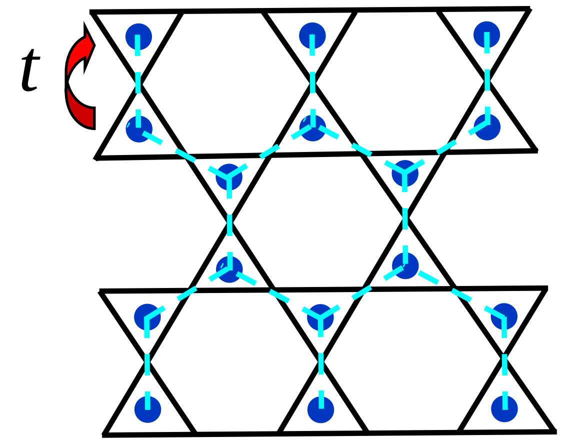
No zero modes

[J. Gonzalez *et al.* Phys. Rev. Lett. 69, 172 (1992)]

Ultra-cold Fermi atoms and optical lattices

Single species ultra cold **Fermi atoms** superposed by **optical lattices** that form a hexagonal lattice.

[Duan *et al.* Phys. Rev. Lett. 91, 090402 (2003)]



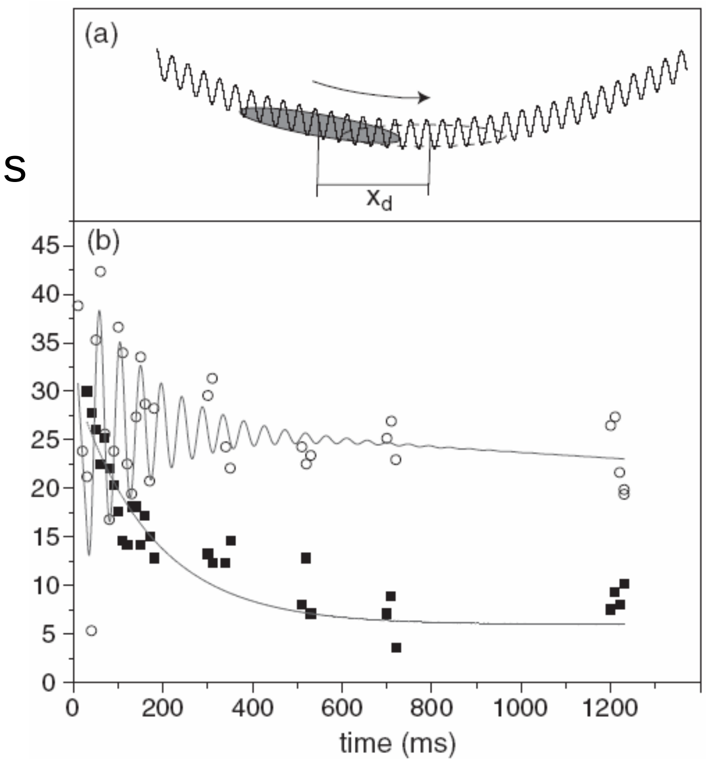
- Very low temperatures: T~0.1TF
- Arbitrary filling factors: e.g. 1/2

See dependence of conductivity on **disorder**, **impurities** and lattice **defects**: e.g. insert pentagons at the edge of the lattice or effect of empty sites.

Similar **index theorem** can be devised for open boundary conditions.

Measurement of conductivity in Fermi lattices has already been performed in the laboratory:

[Ott *et al.* Phys. Rev. Lett. 92, 160601 (2004)]



Conclusions

- **Index Theorem** for various graphene configurations.
- Agrees well with known models of **fullerenes** and **nanotubes**.
- Gives conductivity properties for **more complex models**:

  sideways connected nanotubes.
- Predicts **stability** of spectrum under small deformations.

- Relate to **topological models**:
  - obtain **topologically related degeneracy:** $2^{6|1-g|+3N}$
  - encode and manipulate **quantum information**.
  - apply **reverse engineering** to find new models with specific degeneracy properties.

- Related experiments with **ultra-cold Fermi atoms** can give insight to the properties of graphene. May be easier to implement than solid state setup.

[cond-mat/0607394, quant-ph/0701203]

Thank you for your attention!

Postdoc positions on Topological Quantum Information

University of Leeds

available now!

Email: j.k.pachos@leeds.ac.uk