Entanglement entropy of spin chains with strong randomness

Gil Refael (Caltech)

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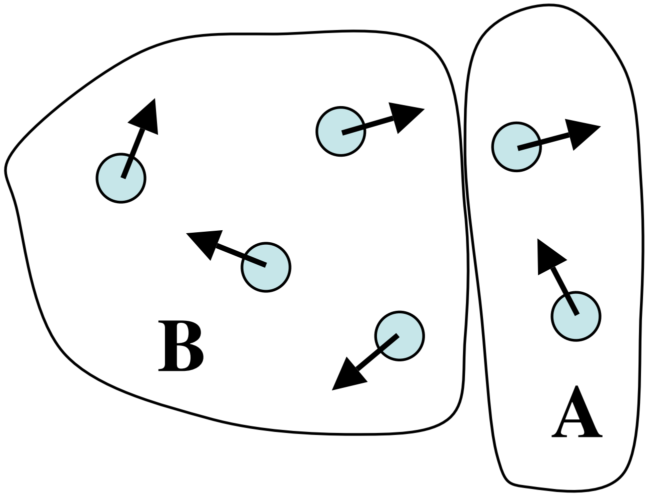
- Quantum information:
  - Definition of entanglement entropy.
  - Examples of universal entropy.
- Conformal invariance:

  Entanglement entropy of critical CFT's.
- Strong randomness in quantum magnetism:
  - Random singlet phase.
  - Entanglement of spin-1/2 Heisenberg model.
  - Entanglement of spin-1 Heisenberg model at criticality.

Entanglement entropy - Introduction

- Density matrix:

$$\rho = |GS\rangle\langle GS|$$



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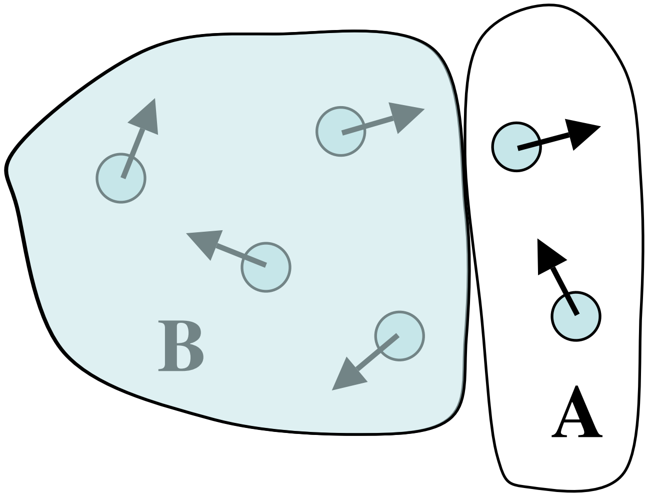

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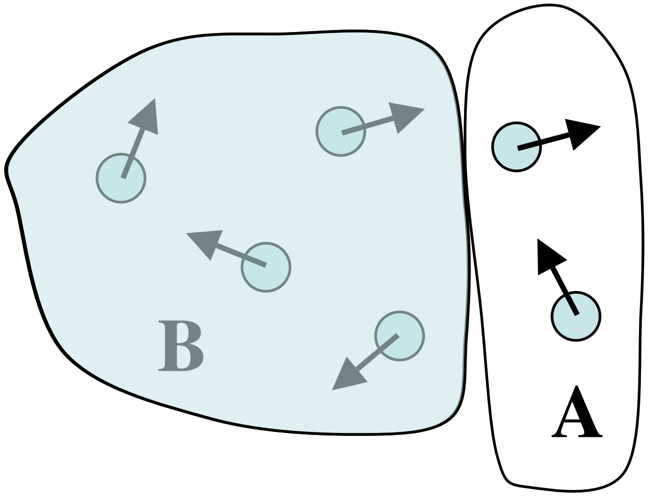

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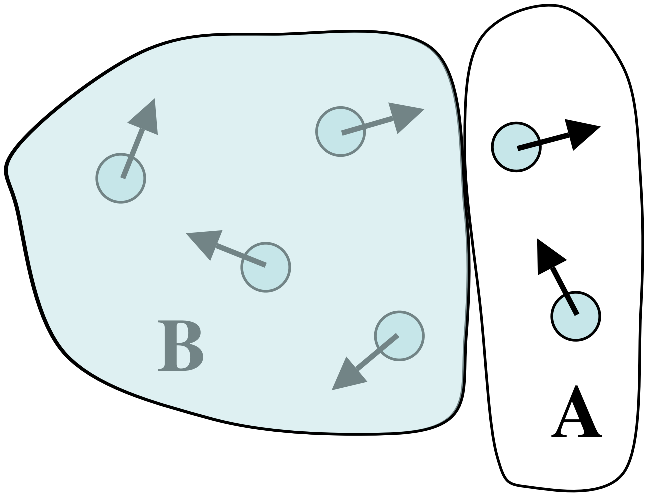

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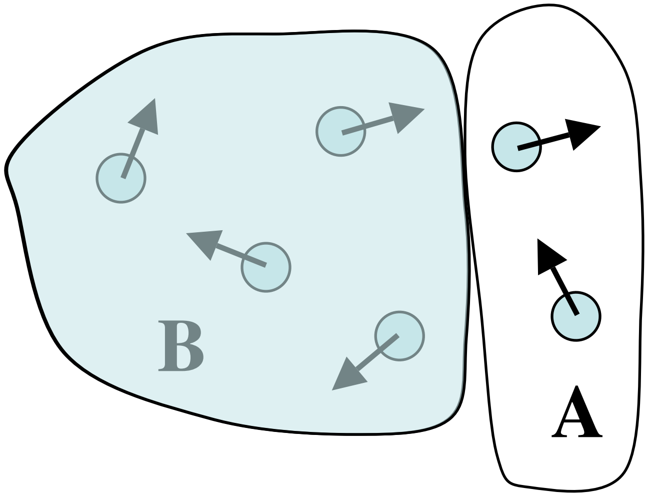

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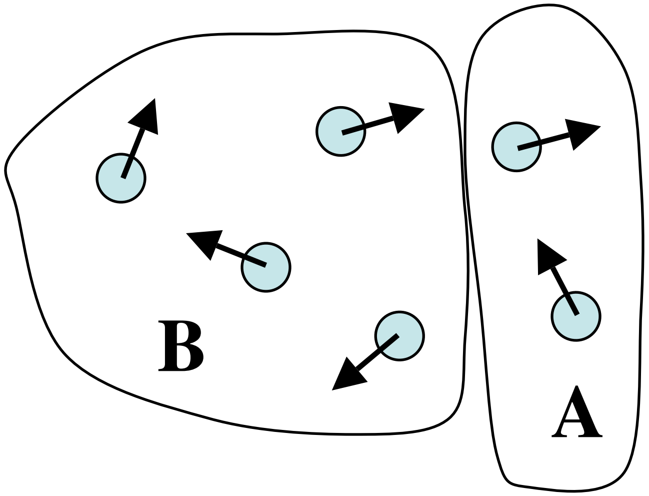

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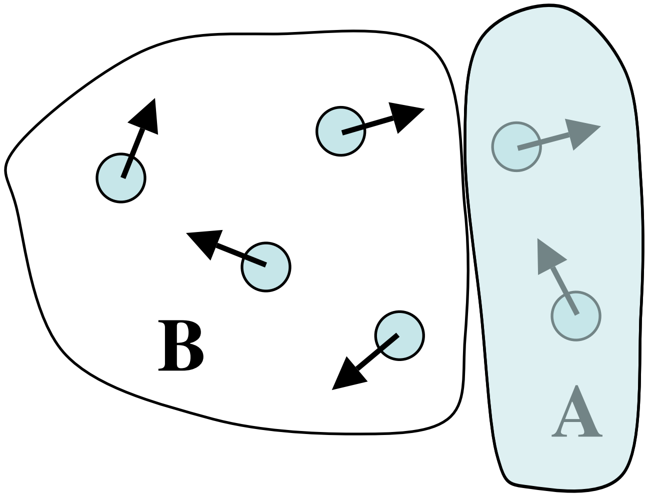

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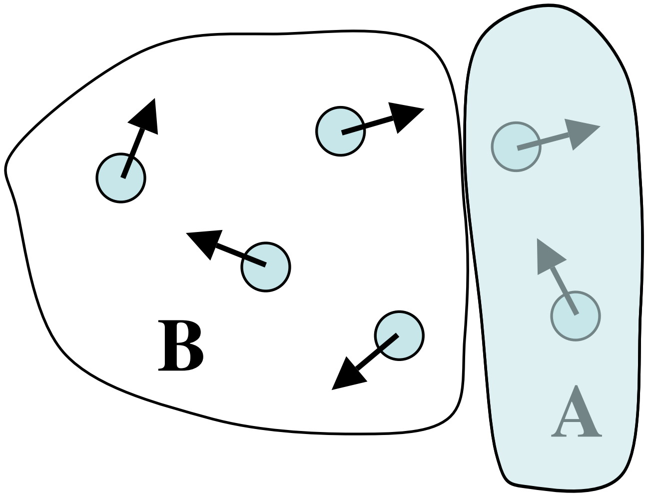

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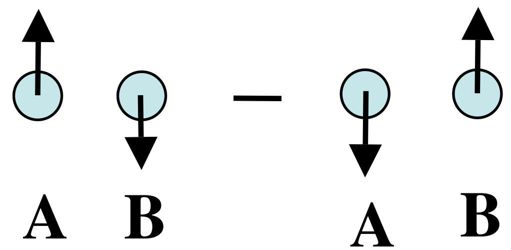
How many states in subsystem A are determined by subsystem B?

Entanglement entropy of two spins in a singlet

- Singlet (or triplet):

$$\frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle|\downarrow_B\rangle - |\downarrow_A\rangle|\uparrow_B\rangle\right)$$



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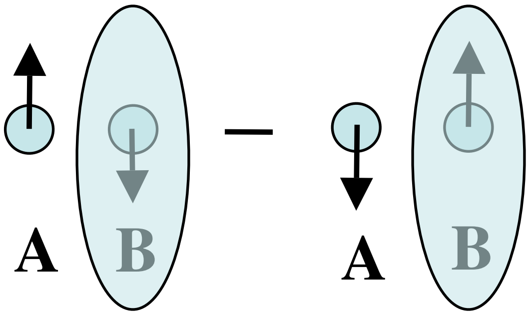

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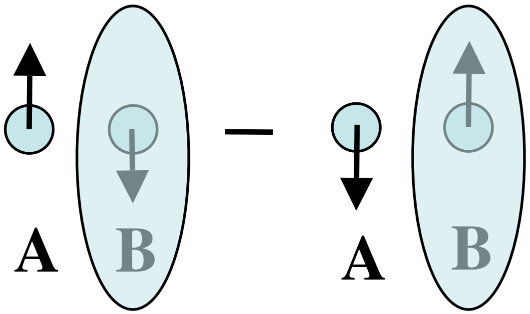

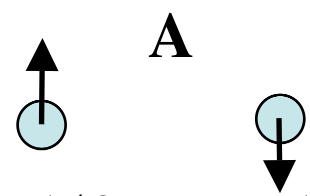

$$p_\uparrow = 1/2 \qquad p_\downarrow = 1/2$$

$$\rho_A = \frac{1}{2}|\uparrow_A\rangle\langle\uparrow_A| + \frac{1}{2}|\downarrow_A\rangle\langle\downarrow_A|$$

$$= \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$

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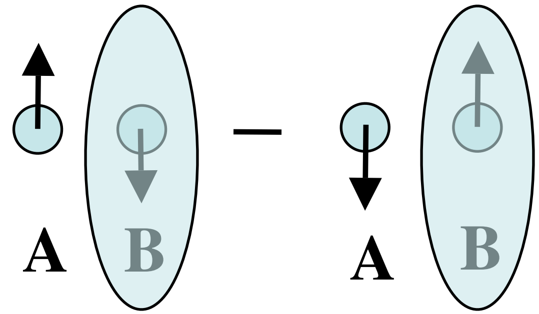

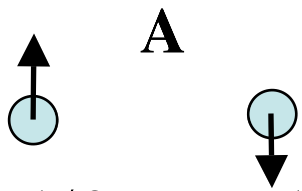

$$p_\uparrow = 1/2 \qquad p_\downarrow = 1/2$$

$$E_{AB} = -\sum_i p_i \log_2 p_i = 1$$

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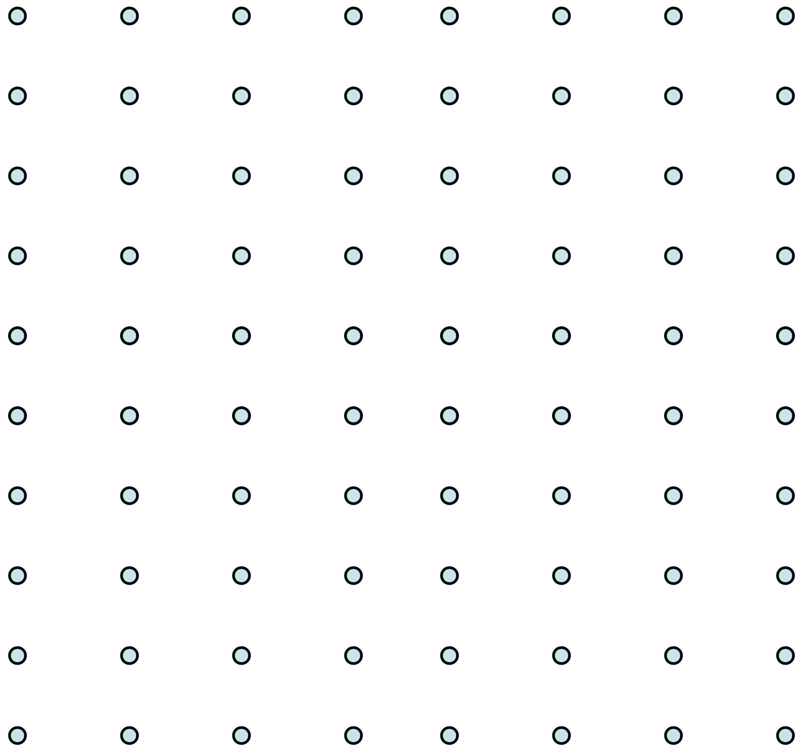
Entanglement as a universal measure



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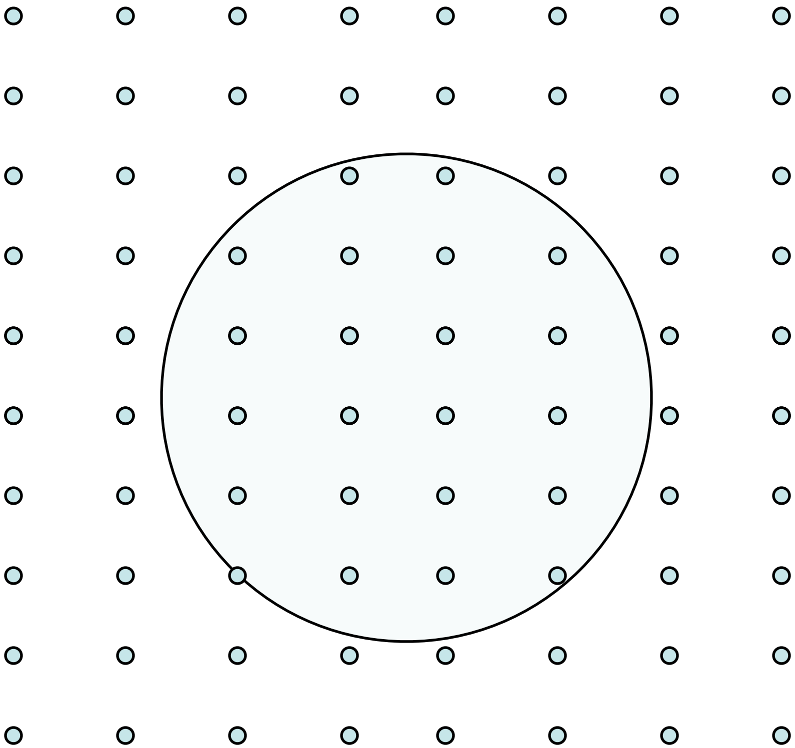

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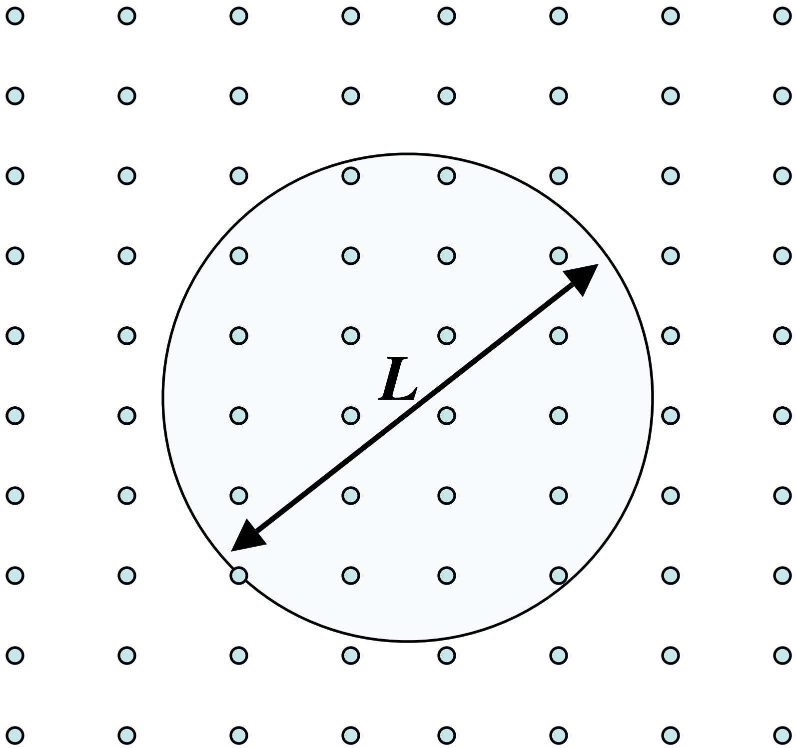
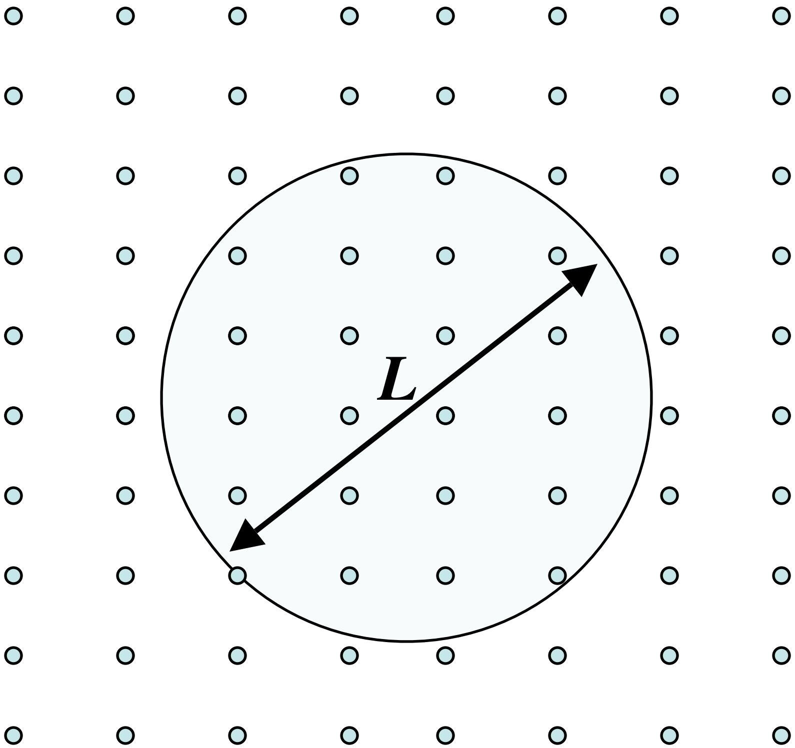


Entanglement as a universal measure

- *$d \geq 2$ – Contour area law:* $E = aL^{d-1} + \ldots$ (e.g., count singlets on perimeter)



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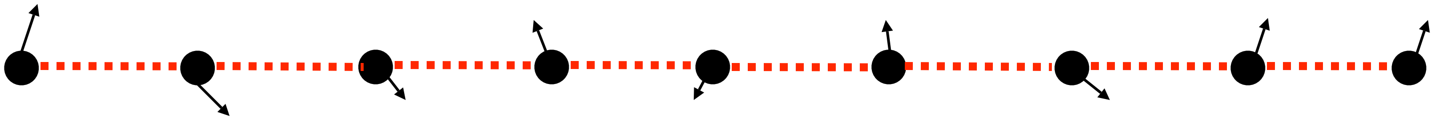
- 2D Gaussian quantum critical points:

$$E = aL + \alpha \ln L + \ldots$$

With α being a universal coefficient that depends on partition topology (how many corners, etc.).

(Moore and Fradkin, 2006)

Conformal spin chains – Heisenberg model



$$H = J \sum_i \vec{S}_i \times \vec{S}_{i+1} \qquad s = 1/2$$

- Space-time invariance $\frac{1}{E} \sim \tau \sim x$
(*z=1*,).
- Spin liquid states with algebraically decaying correlations.

- Central charge c :

$$f \sim -\frac{\pi}{6} c T^2 \qquad C_V \sim \frac{1}{3} \pi \, c \, T$$

Engtanglement and CFT's central charge

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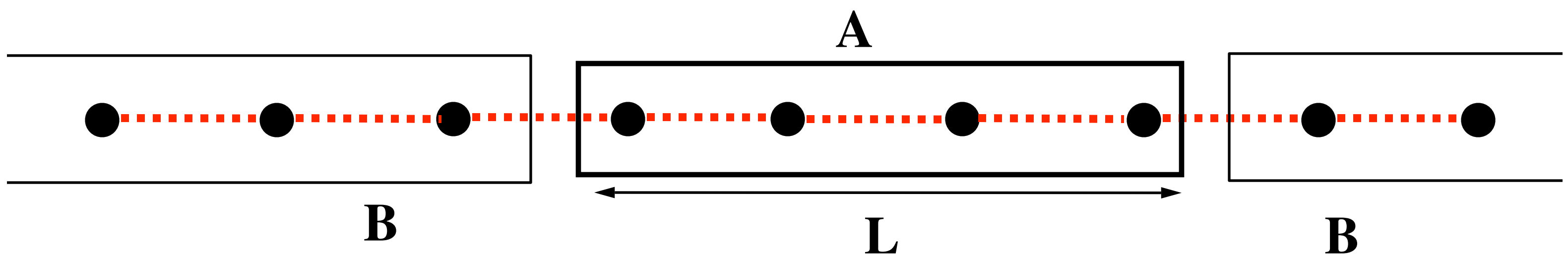

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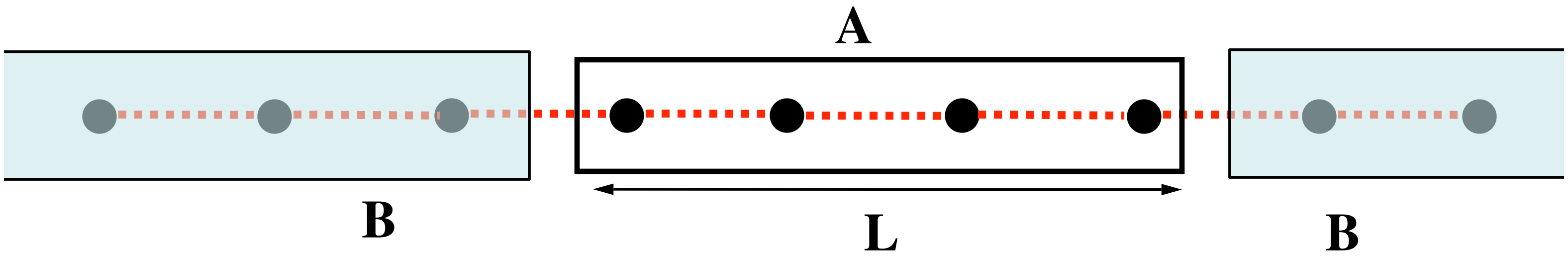

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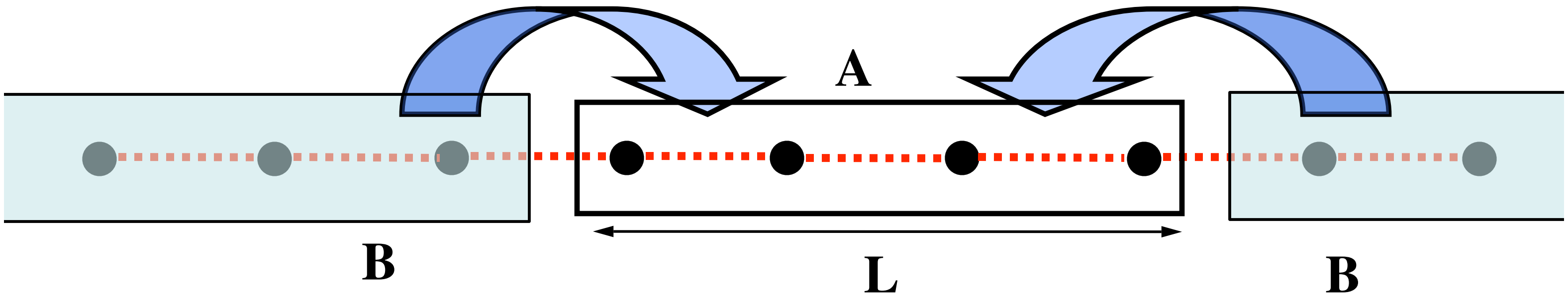

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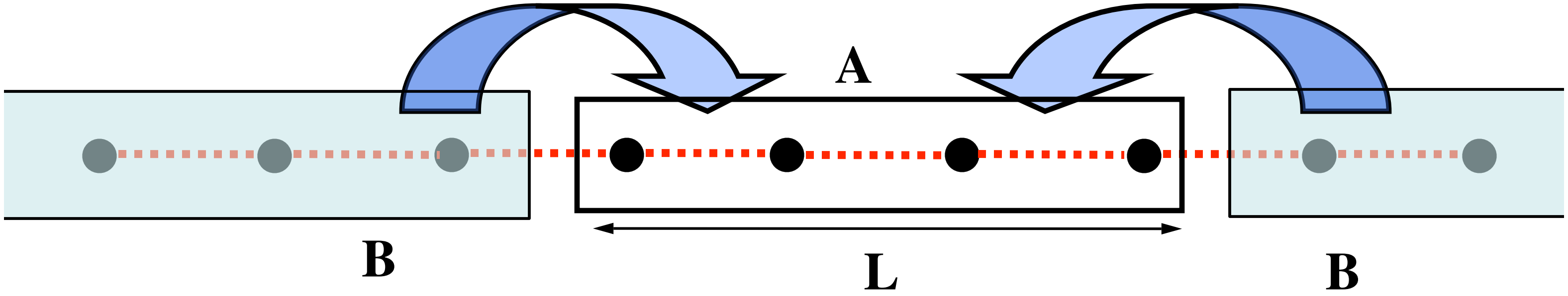

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Holzhey, Larsen, Wilczek (1994).

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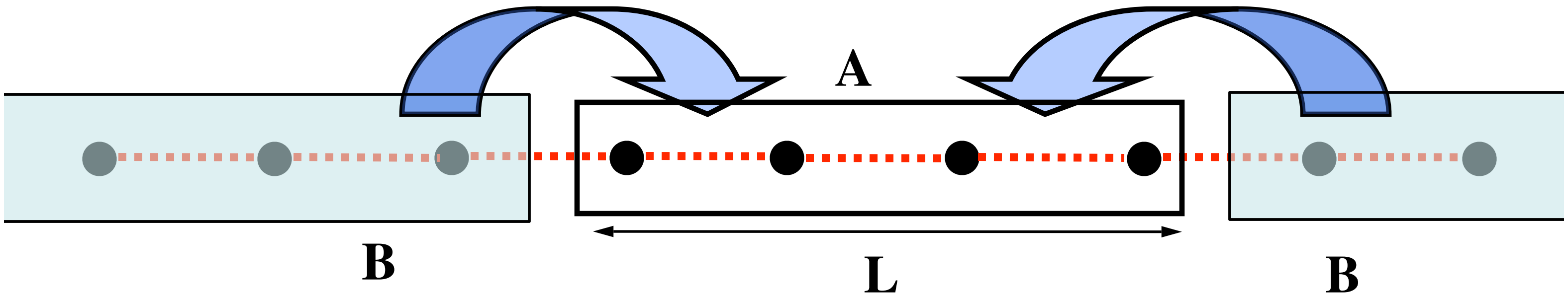

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Spin-1/2 Heisenberg, XXZ	1
Transverse field Ising	$1/2$
Spin-k/2 Heisneberg $SU_k(2)$	$\frac{3k}{k+2}$

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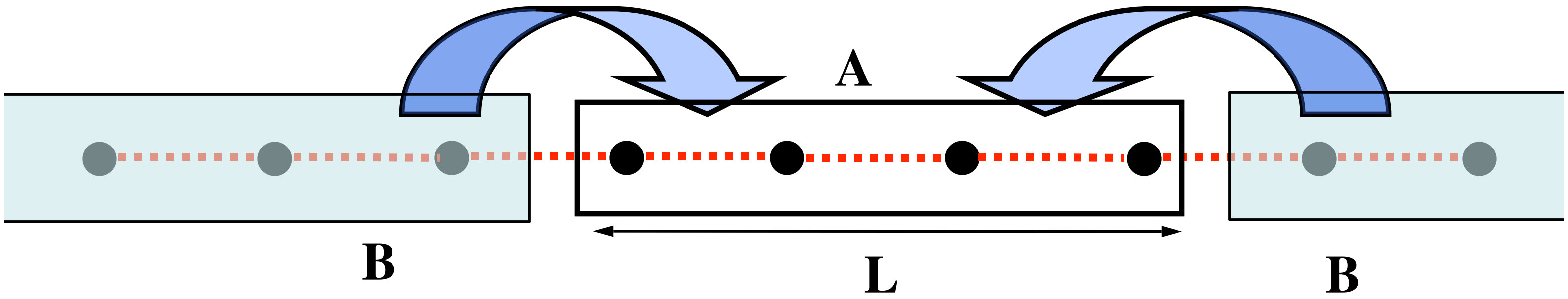

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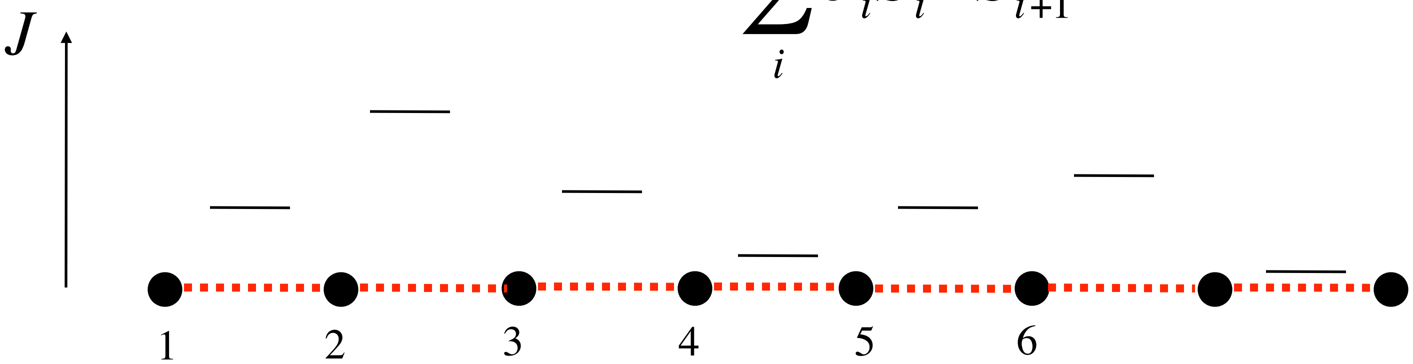

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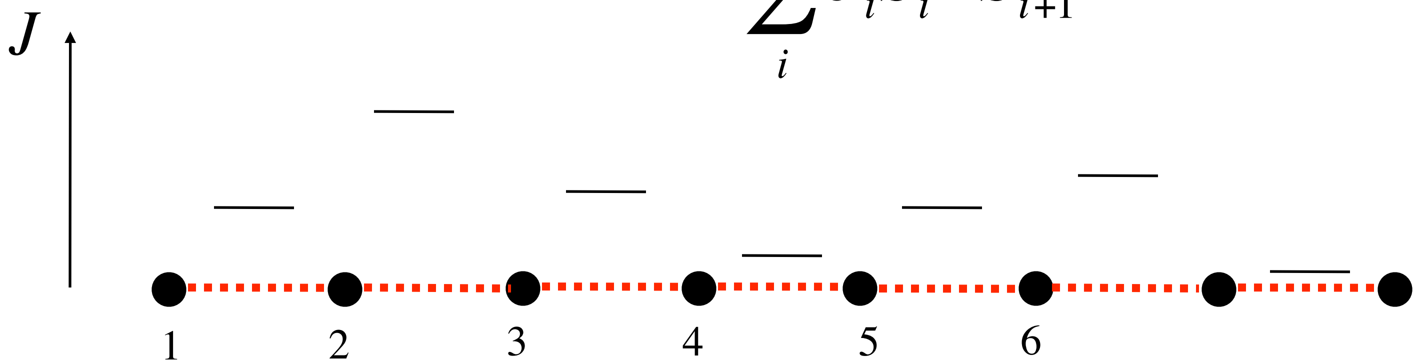

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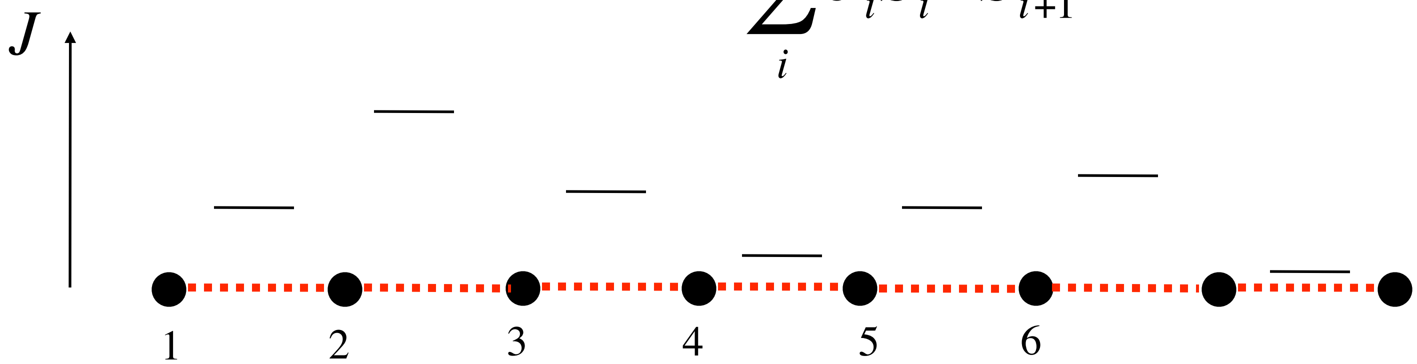

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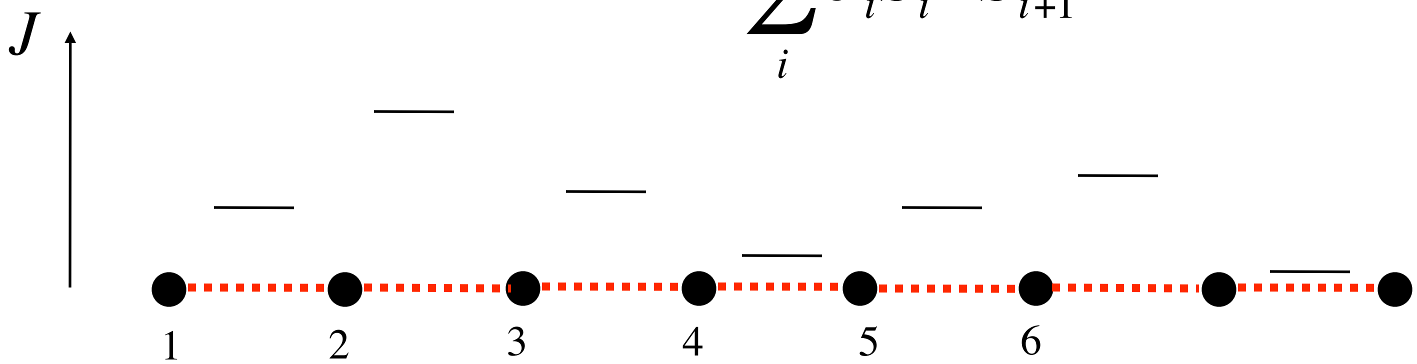

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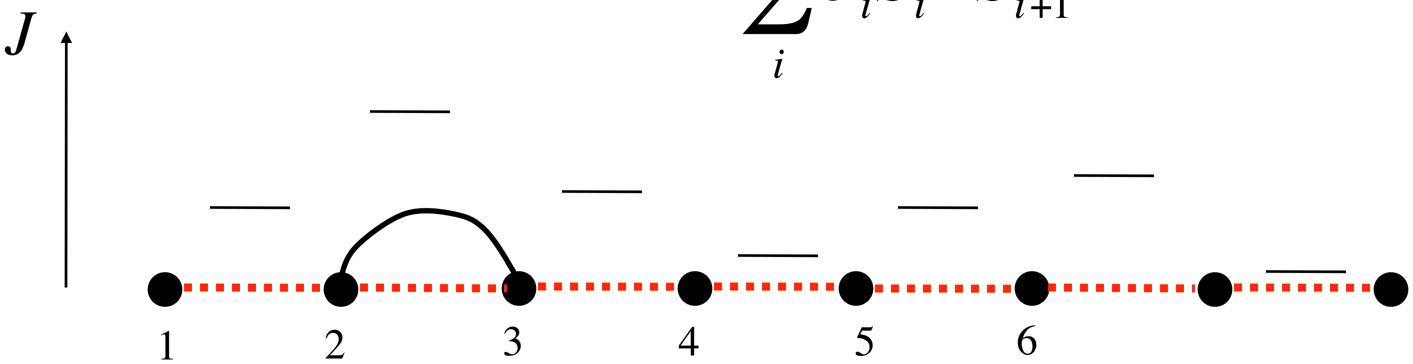

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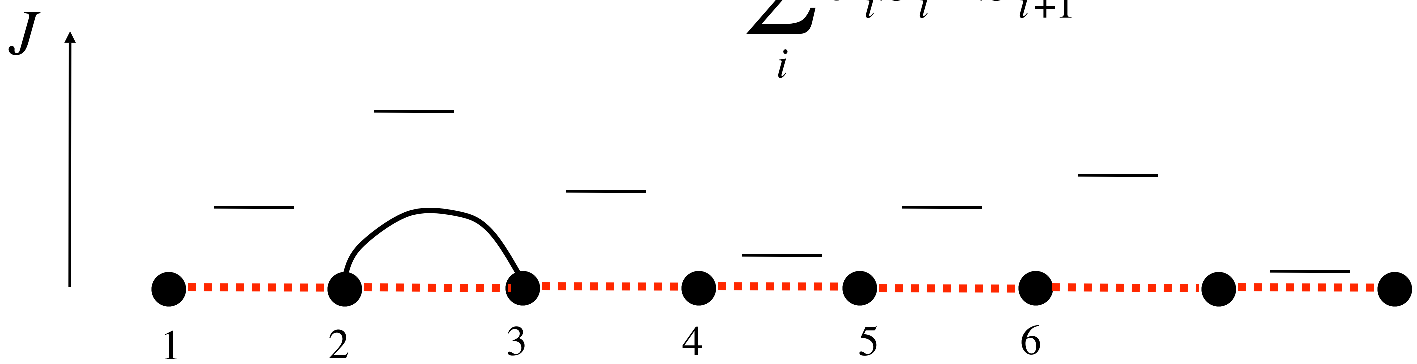

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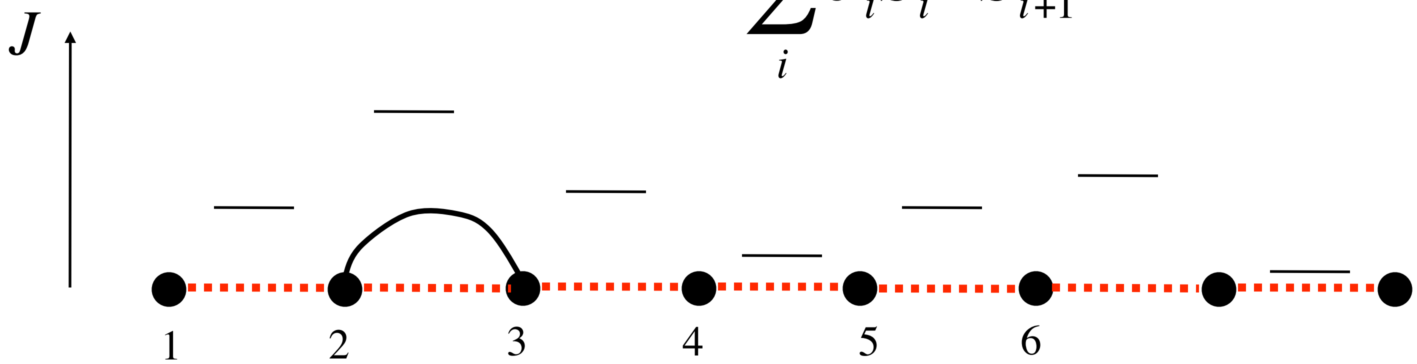

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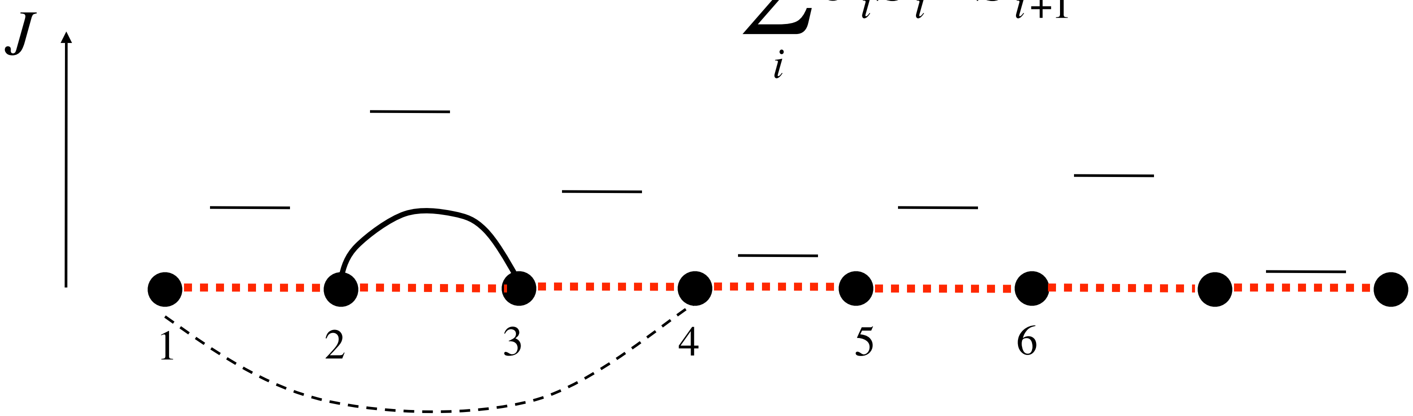

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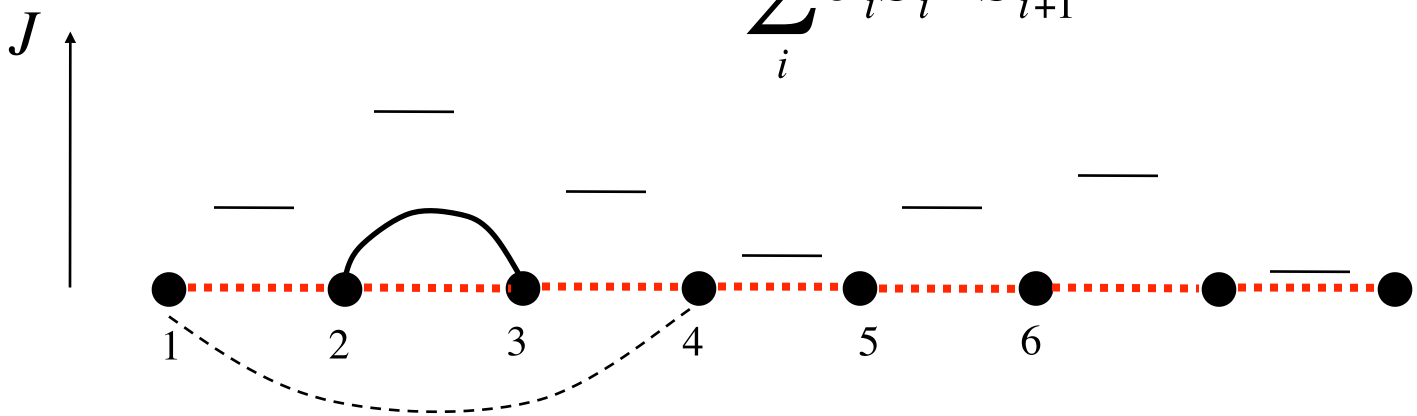

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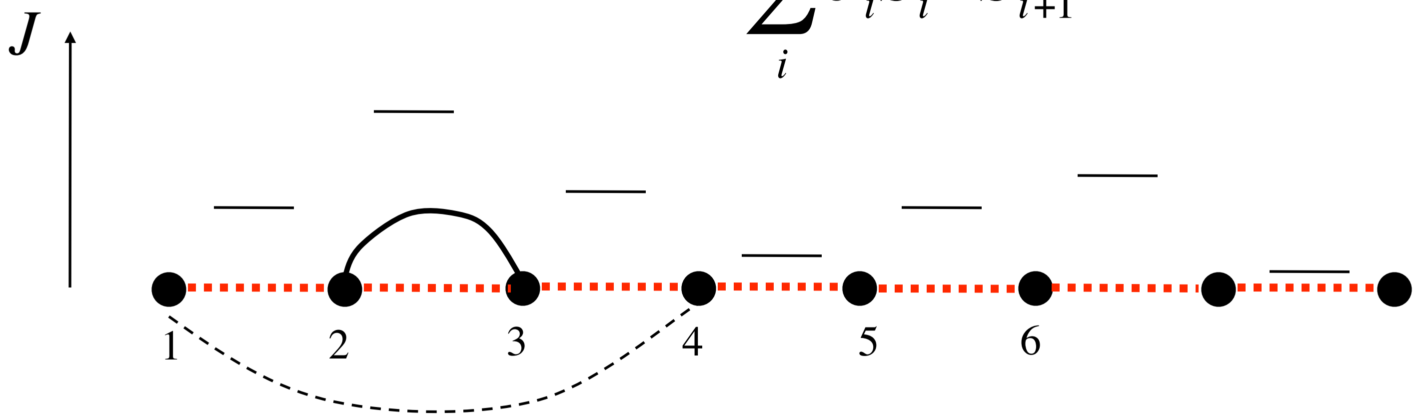

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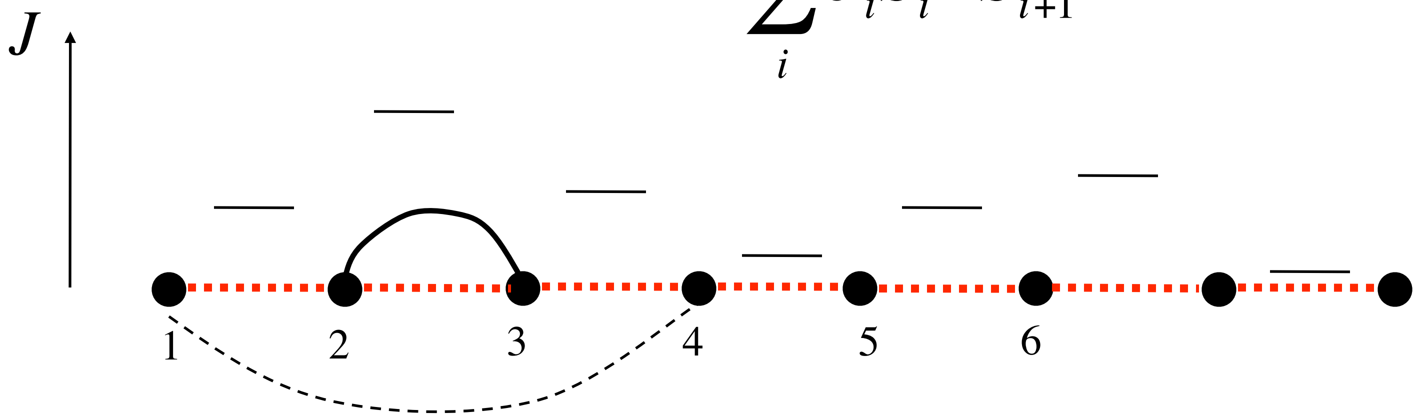

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Ma, Dasgupta, Hu (1979).

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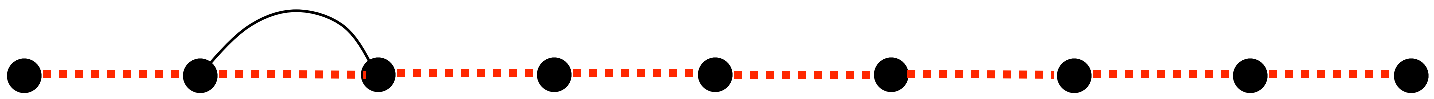

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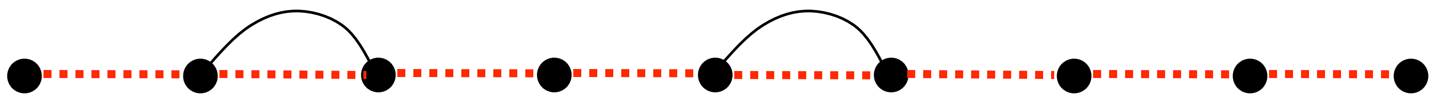

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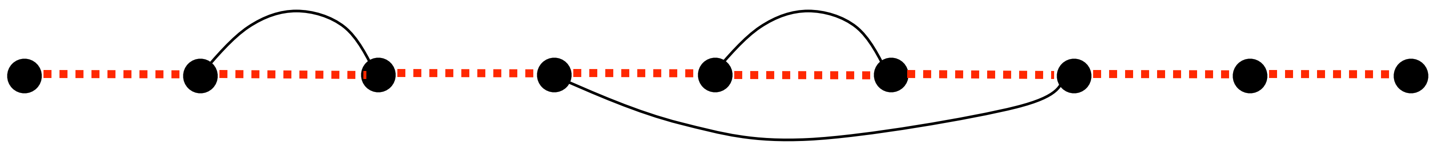

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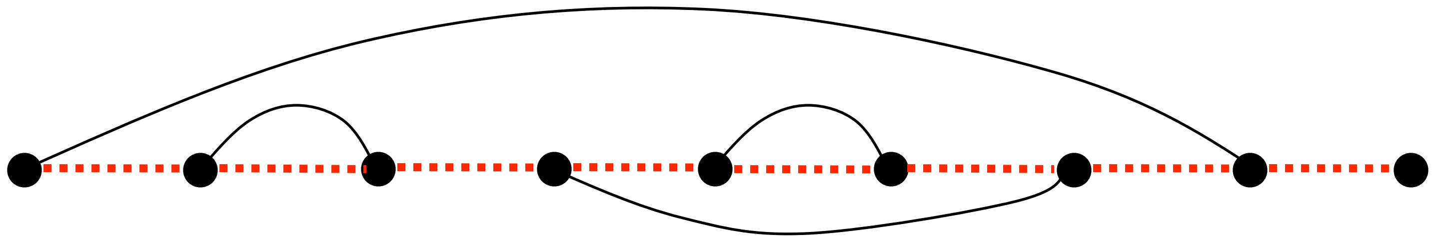

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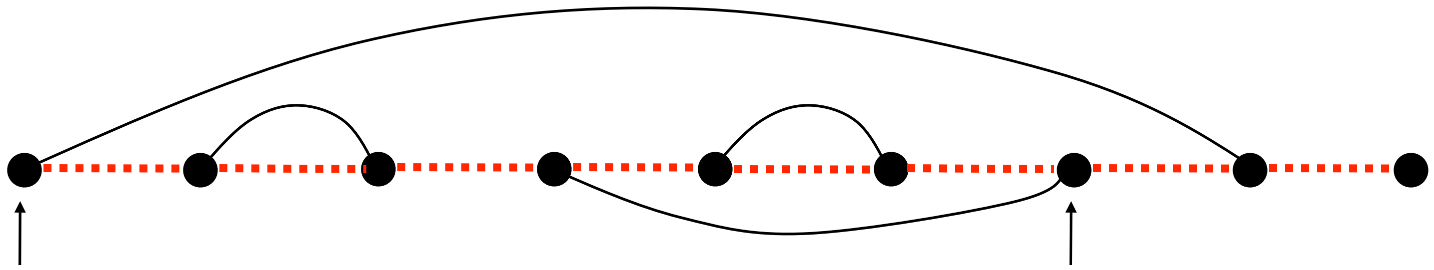

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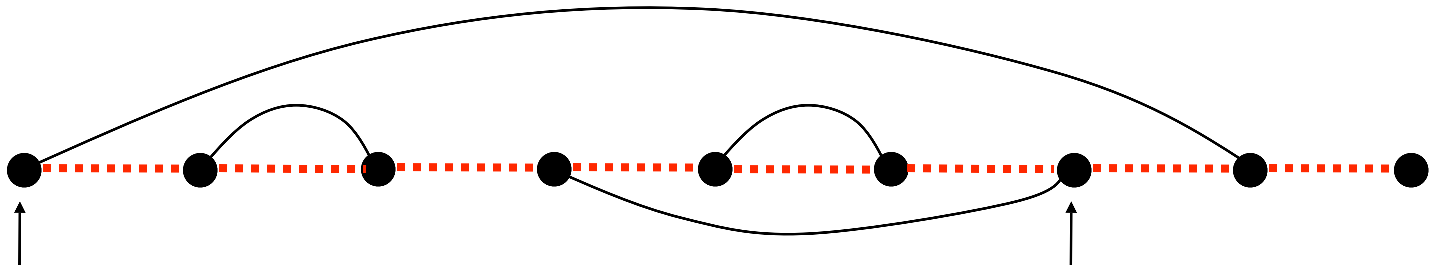

$$\langle \sigma_i^z \sigma_{i+L}^z \rangle \sim \exp\left(-\sqrt{L}\right)$$

(typical)

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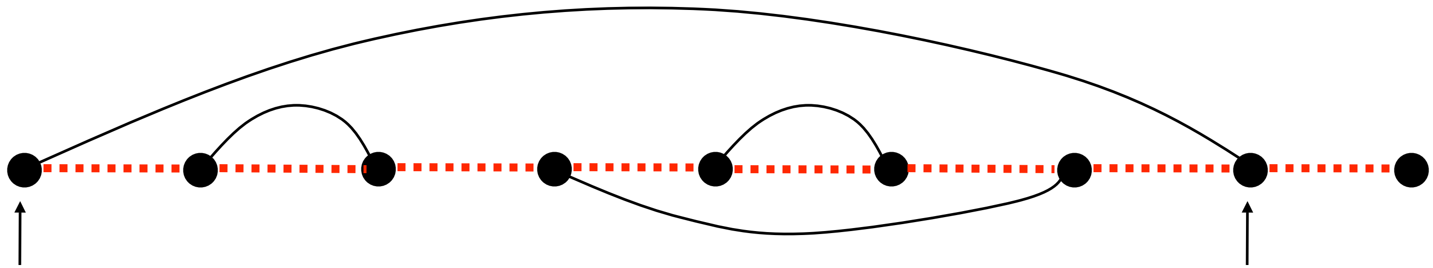

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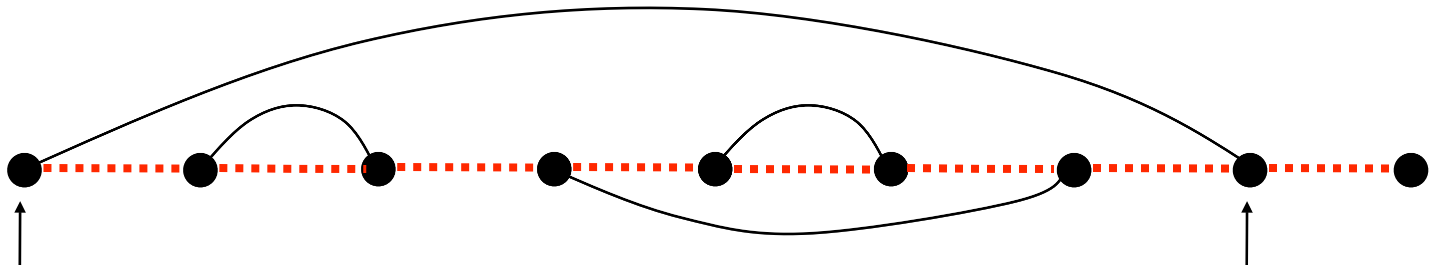

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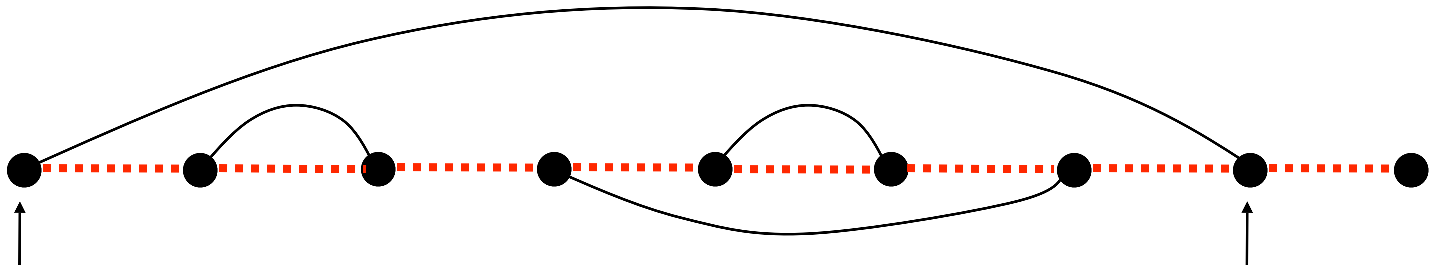

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(typical)

$$\left\langle \sigma_i^z \sigma_{i+L}^z \right\rangle \sim 1$$

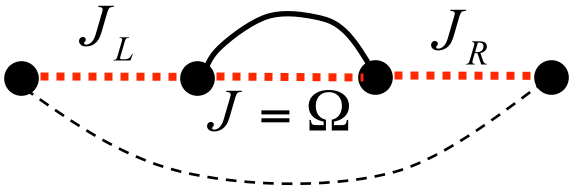
(singlet)

$$\overline{\left\langle \sigma_i^z \sigma_{i+L}^z \right\rangle} \sim p_{singlet} \sim \frac{1}{L} \times \frac{1}{L}$$

Strong randomness RG and flow equations

(D.S. Fisher, 1994)

- Bond decimation: $$J_{eff} = \frac{J_L \times J_R}{\Omega}$$



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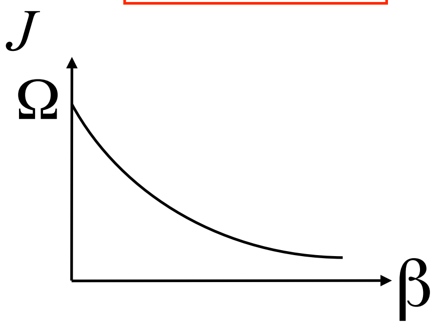

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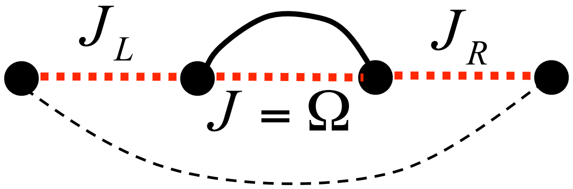

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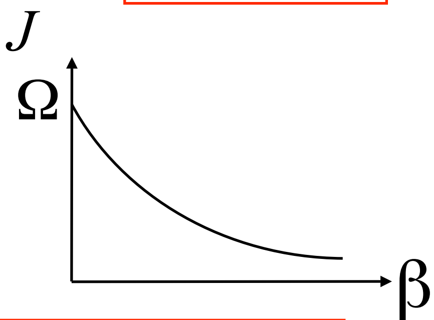

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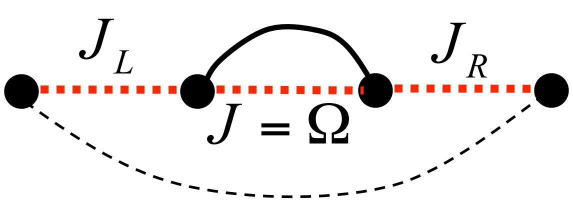

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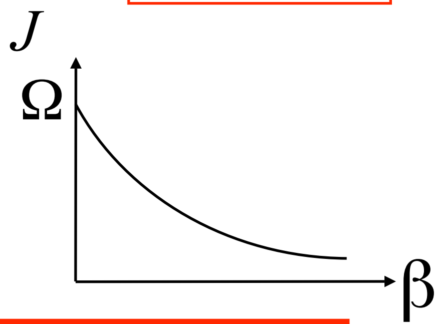

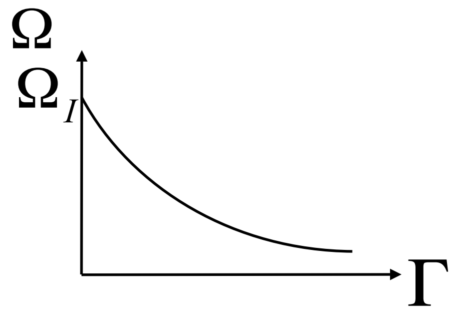

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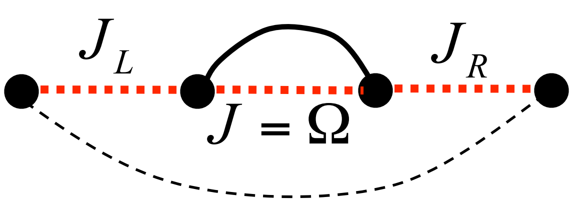

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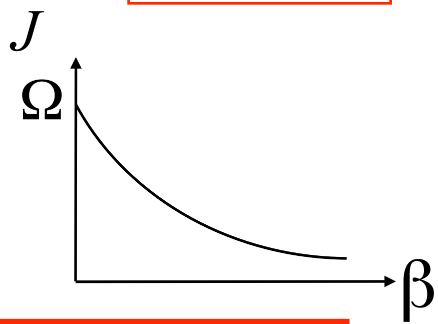

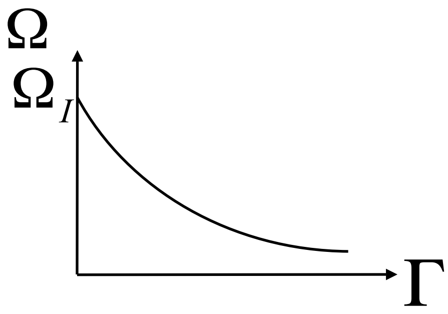

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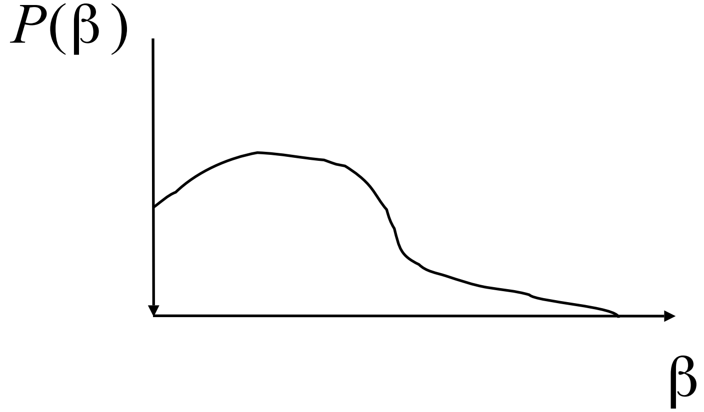

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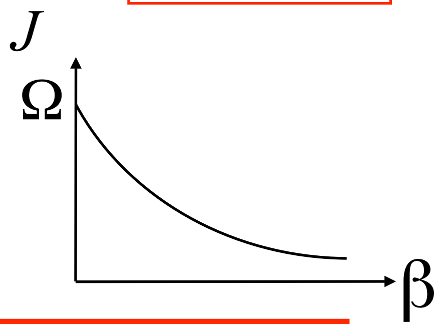

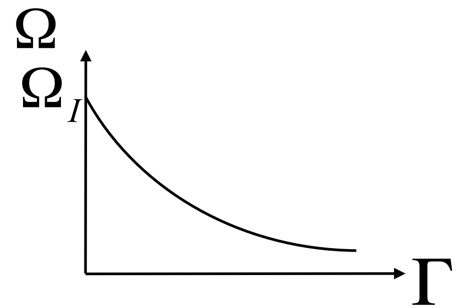

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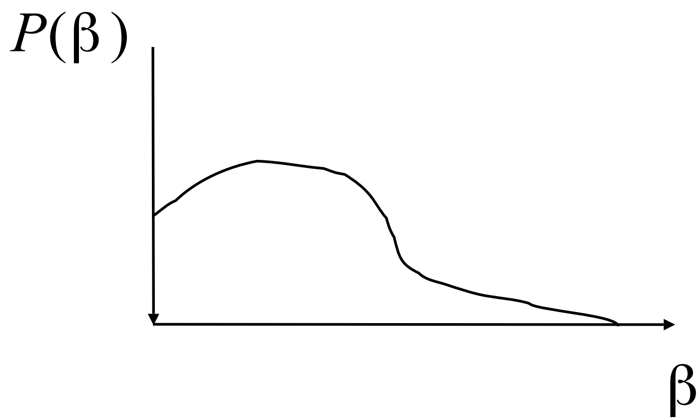

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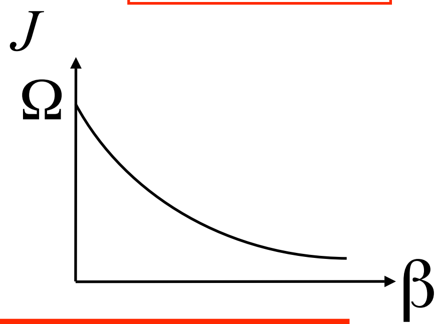

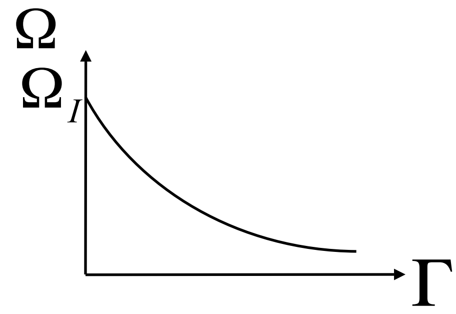

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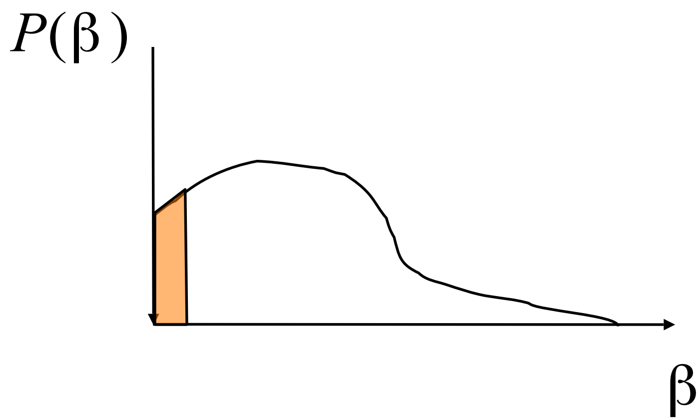

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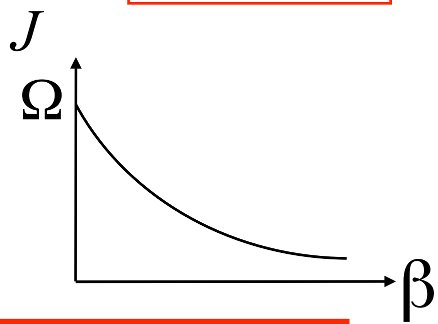

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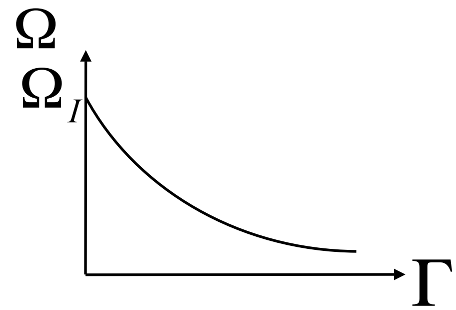

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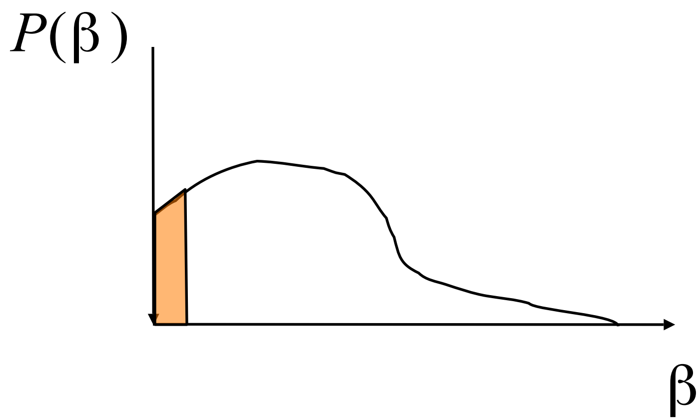

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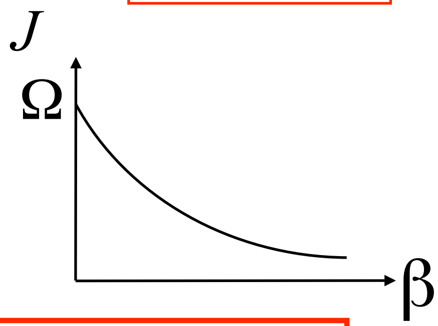

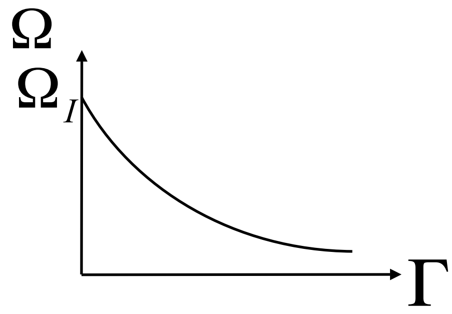

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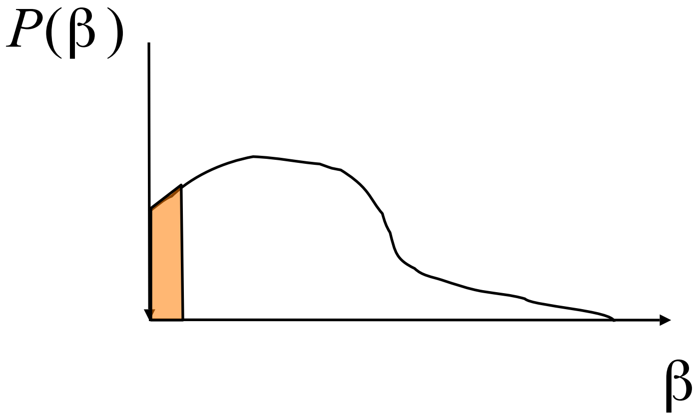

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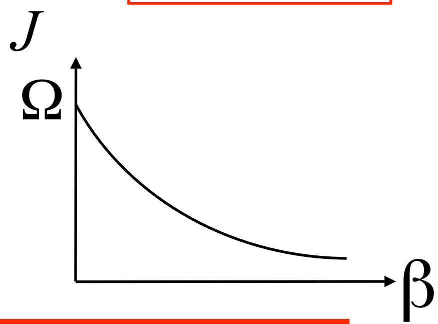

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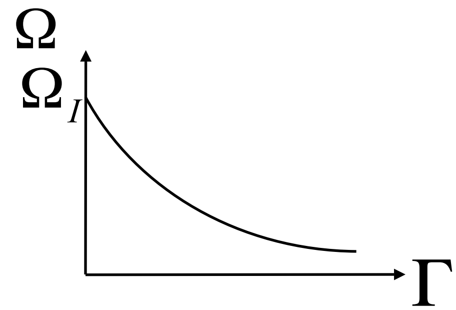

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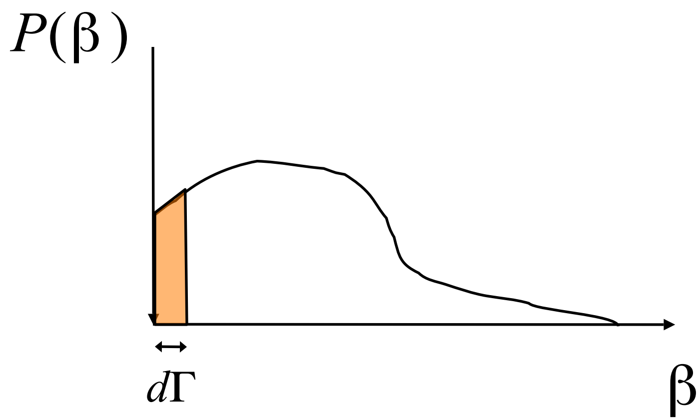

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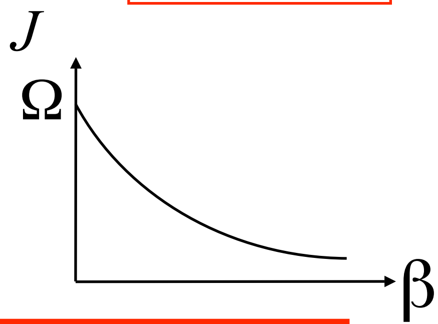

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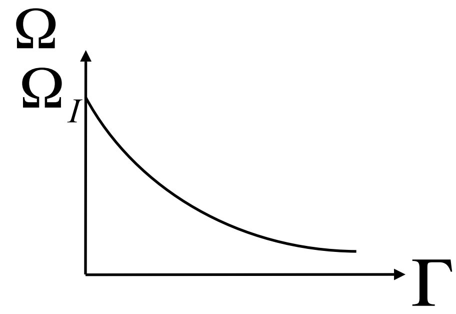

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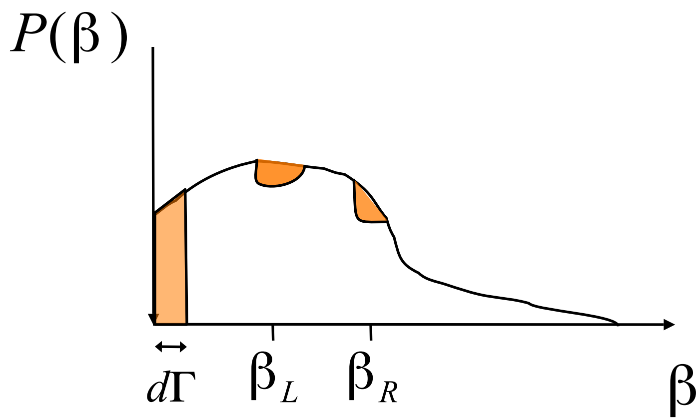

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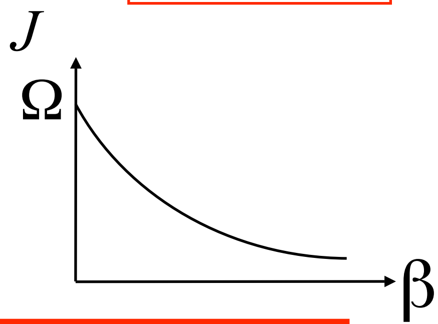

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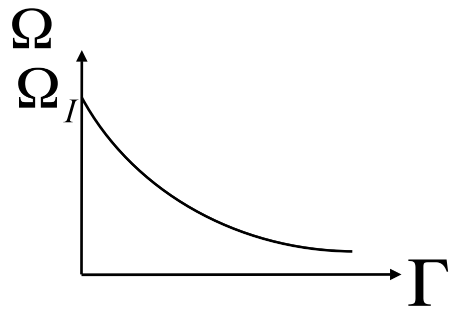

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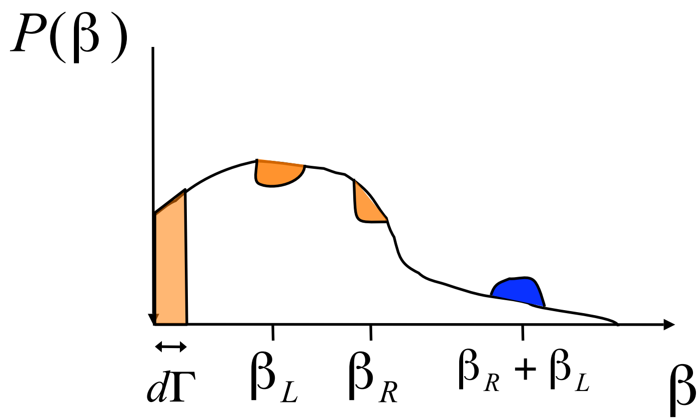

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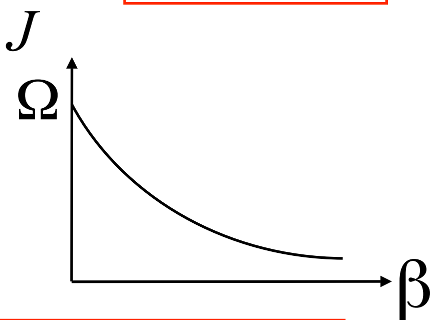

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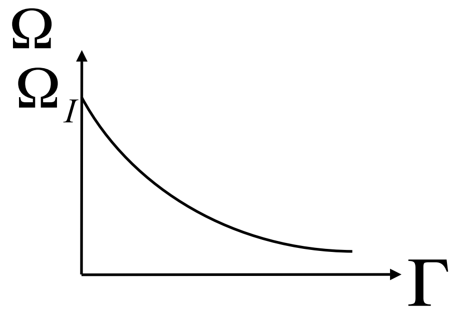

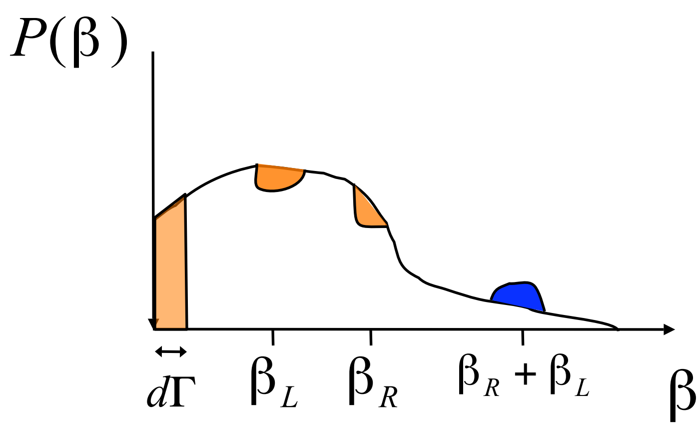

- $P(\beta)$:

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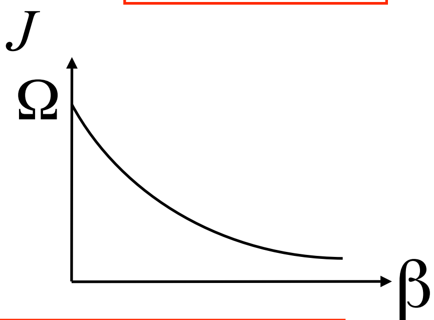

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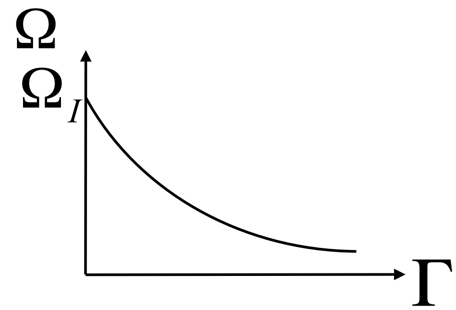

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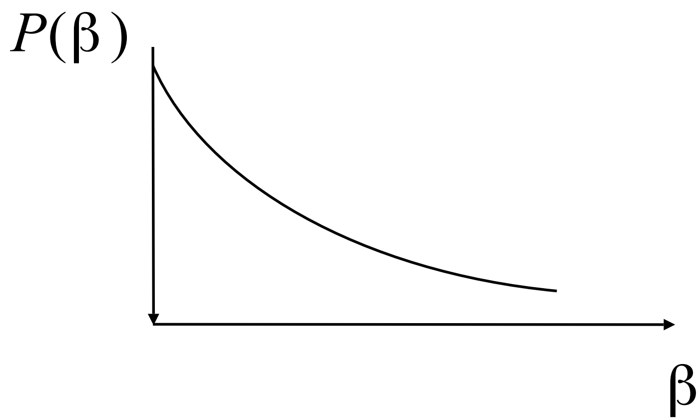

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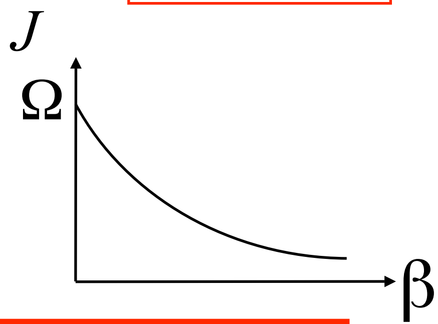

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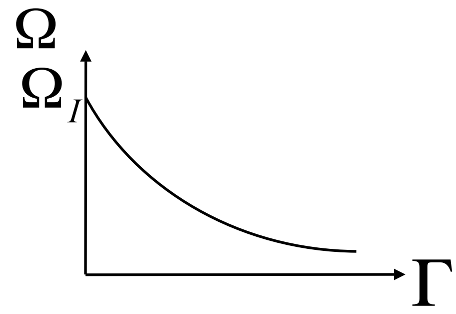

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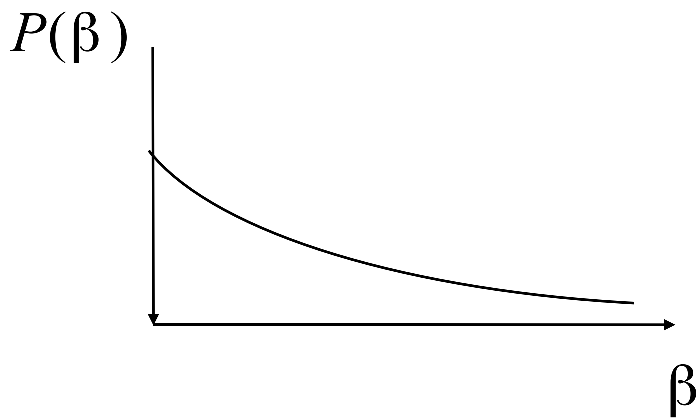

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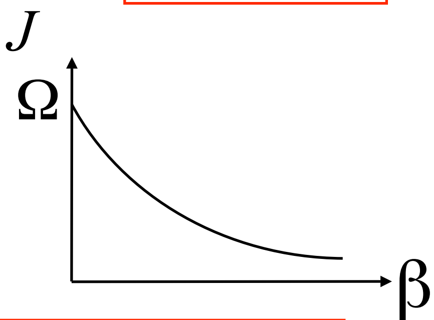

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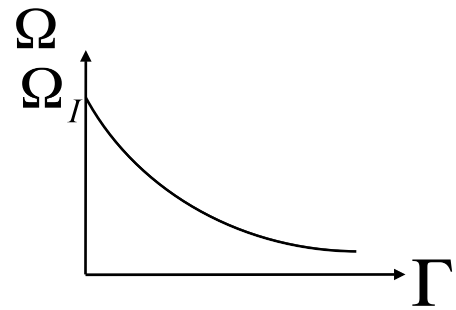

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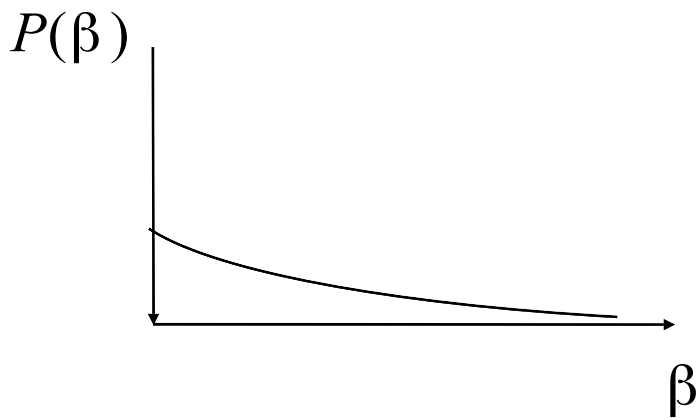

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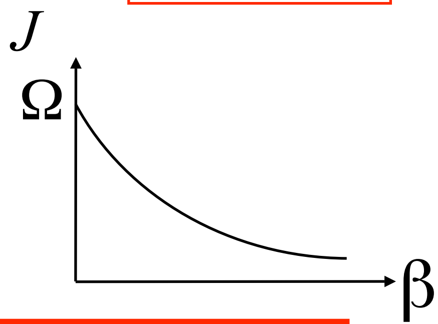
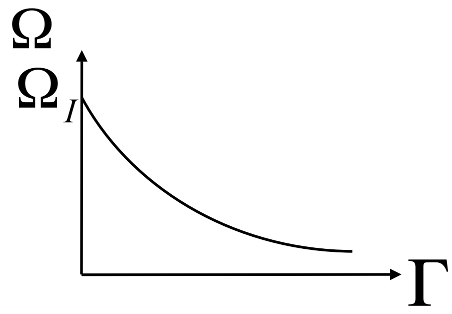
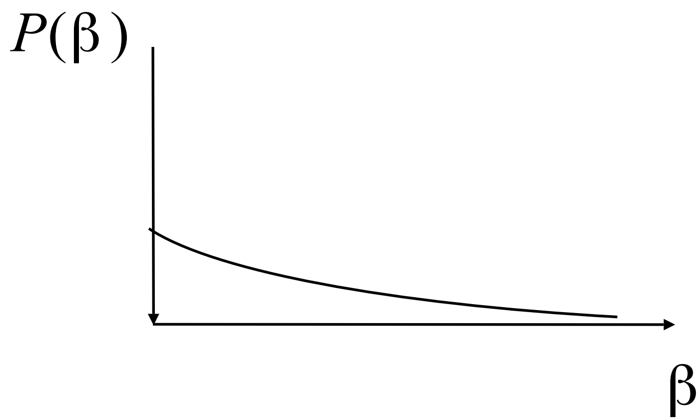




$$\beta_{eff} = \beta_L + \beta_R$$

- Solution:

$$P_\Gamma(\beta) = \frac{1}{\Gamma} e^{-\beta/\Gamma}$$



- $P(\beta)$:

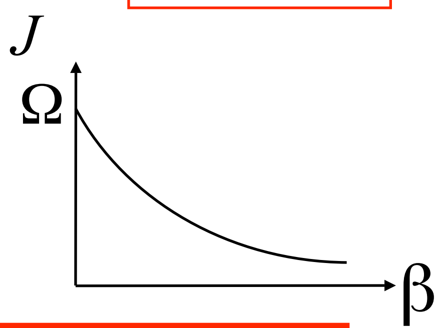
$$\frac{dP(\beta)}{d\Gamma} = \frac{\partial P(\beta)}{\partial \beta} + P(0)\int d\beta_R d\beta_L P(\beta_R) P(\beta_L)\delta(\beta_R + \beta_L - \beta)$$

Strong randomness RG and flow equations

(D.S. Fisher, 1994)

Define :

$$\beta_i = \ln \frac{\Omega}{J_i}$$



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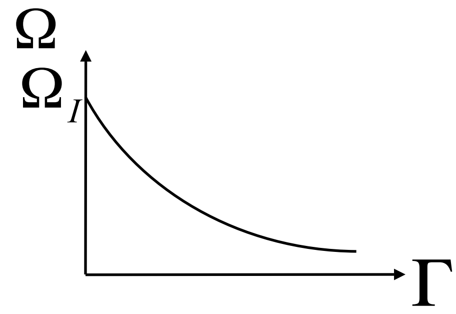

$$\Omega \rightarrow \Omega - d\Omega$$

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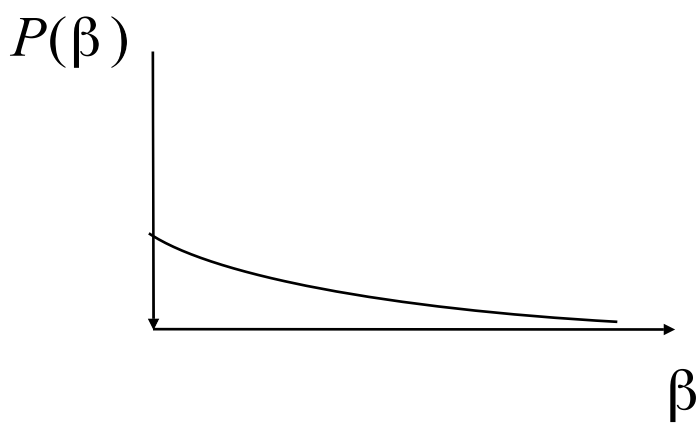
Universal!



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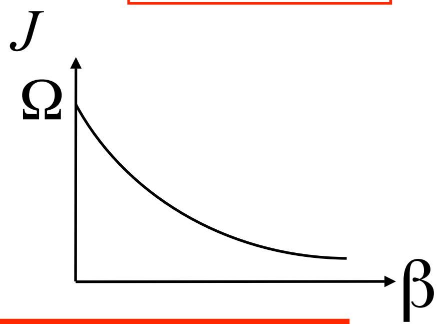

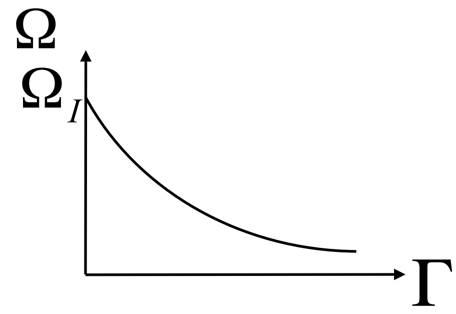

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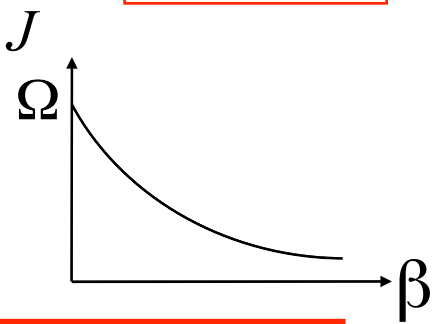

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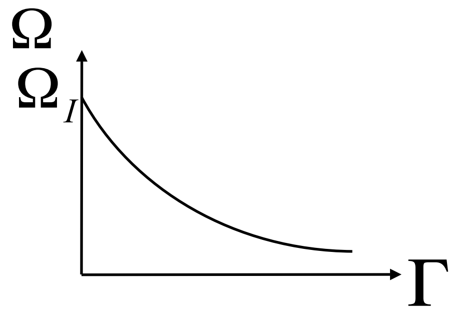

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Universal!

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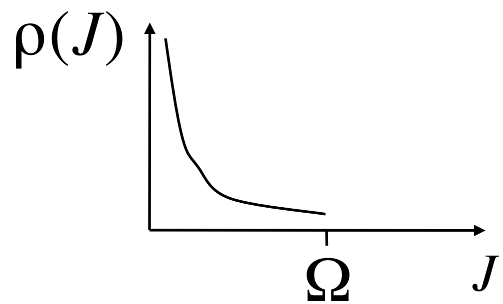

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Infinite randomness scaling and entanglement

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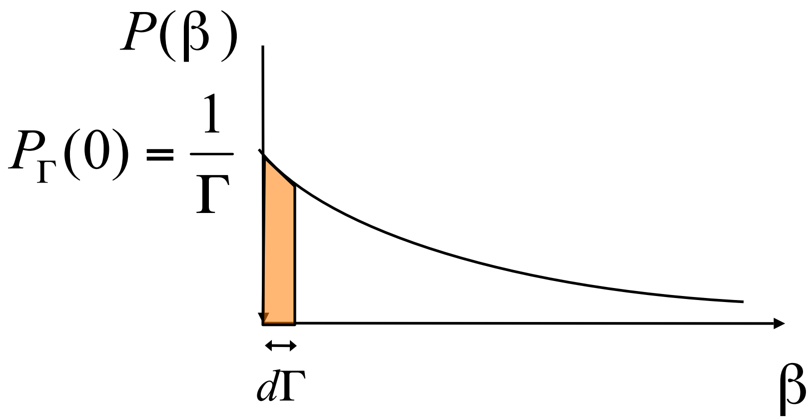

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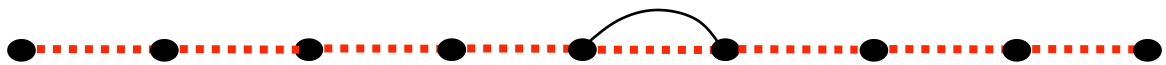

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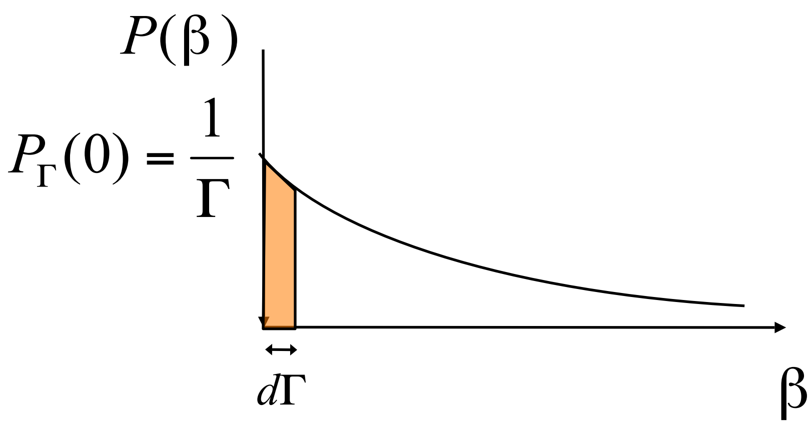

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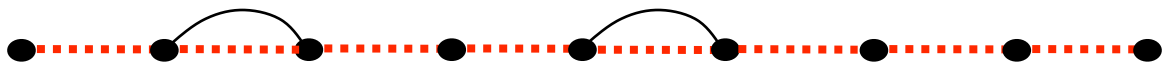

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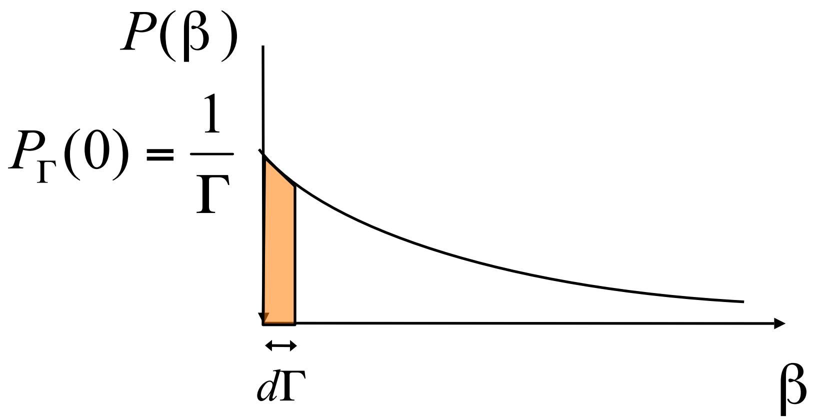

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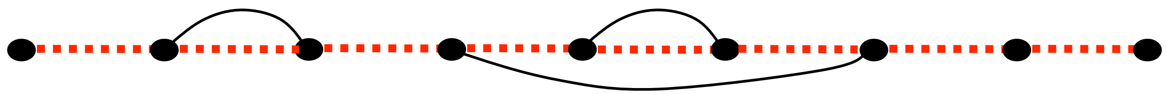

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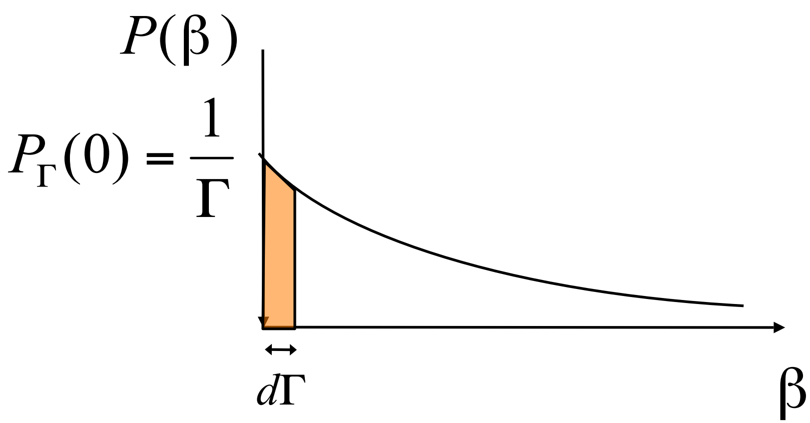

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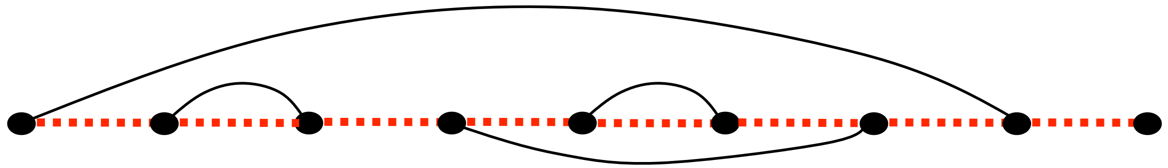

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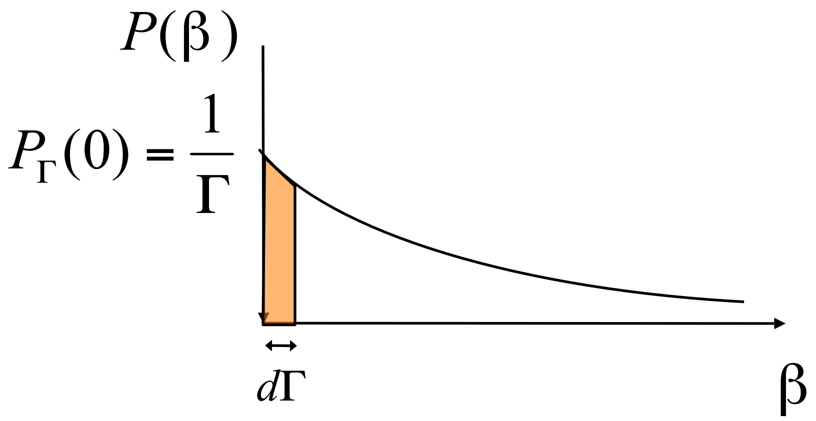

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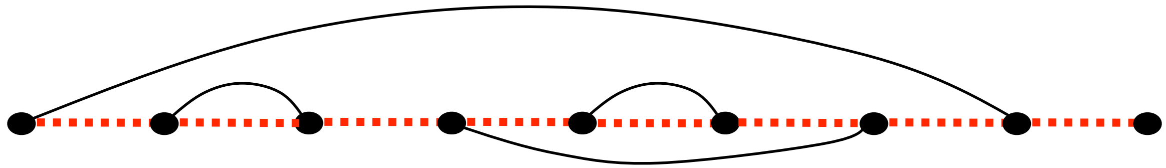

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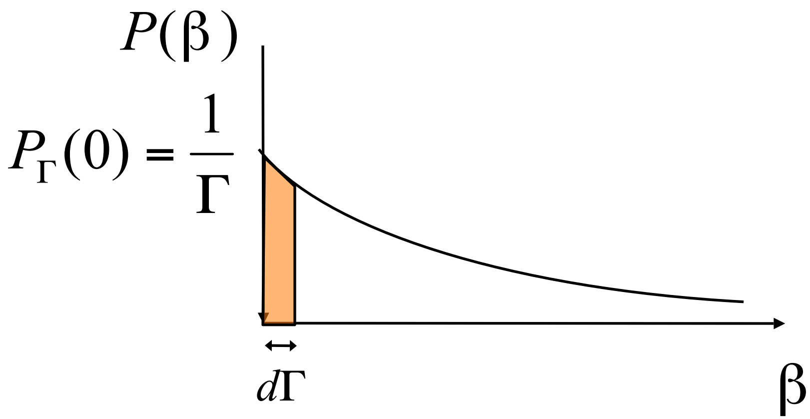
$$dn = -2 \times n \times P(0) d\Gamma = -2n \frac{d\Gamma}{\Gamma}$$



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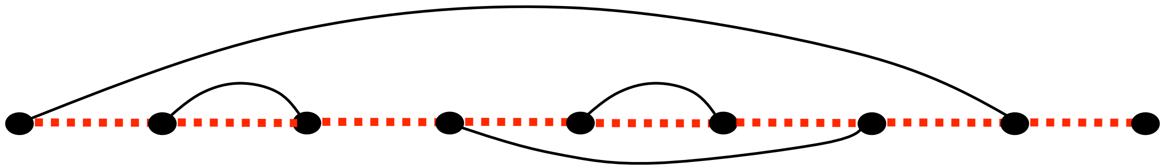

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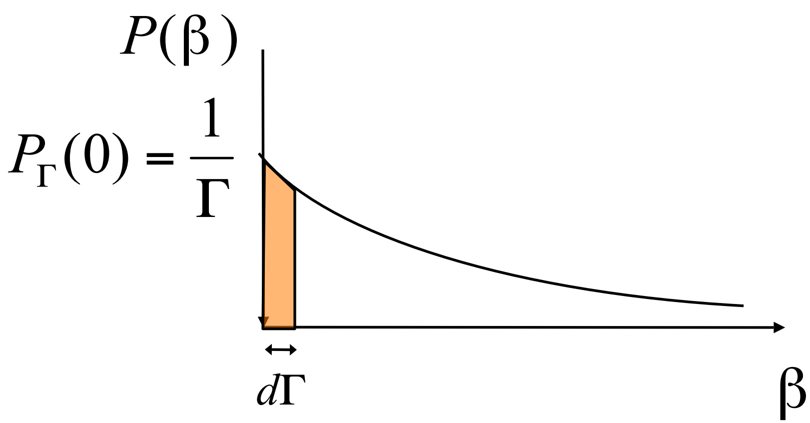

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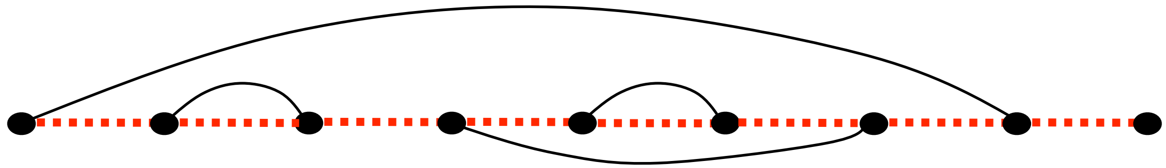

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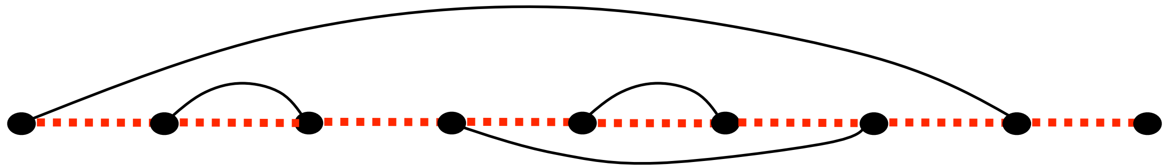

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$$L \sim \Gamma^2 \longrightarrow \ln\frac{1}{E} \sim L^{1/2}$$

Infinite-randomness FP's:
Random Singlet Phase (and universality class)

(D.S. Fisher (1994), Bhatt, Lee (1982))

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Entanglement in the random singlet phase

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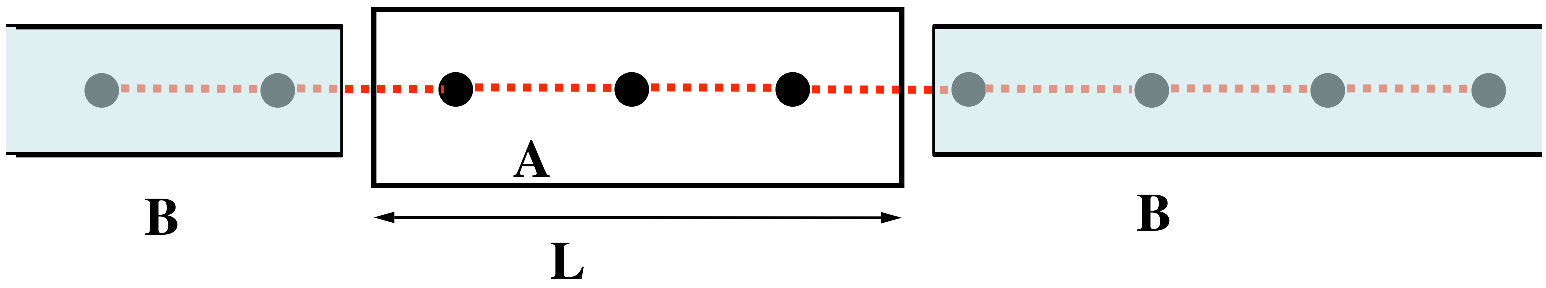

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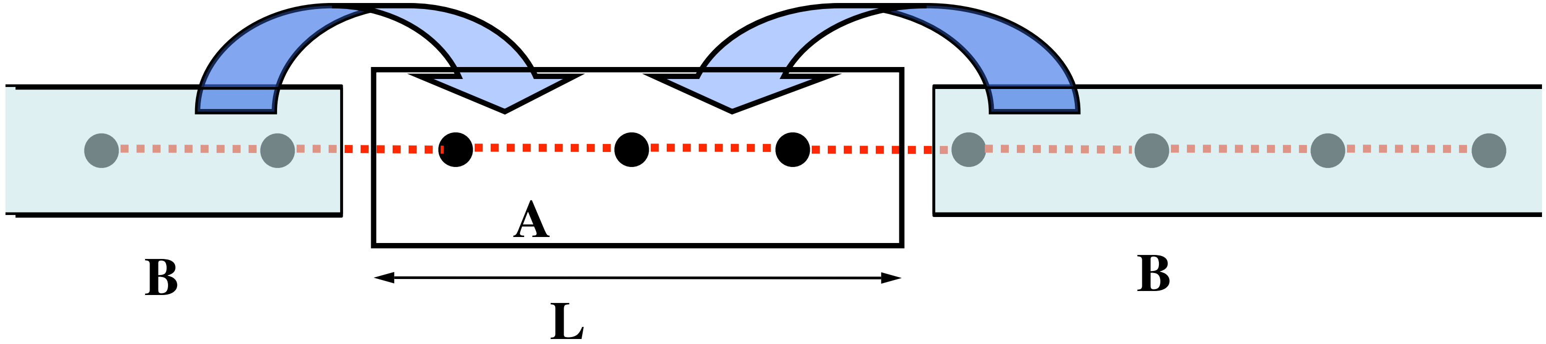

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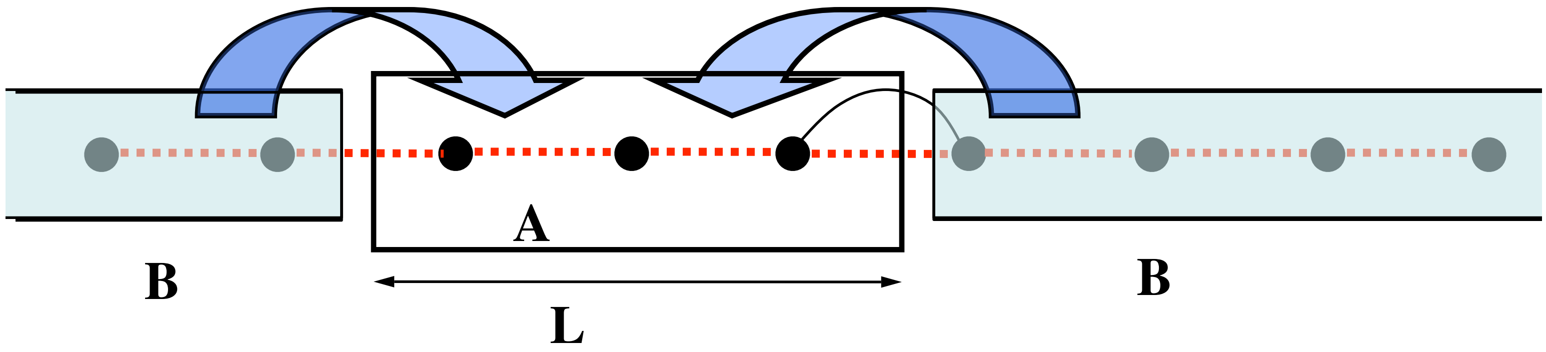

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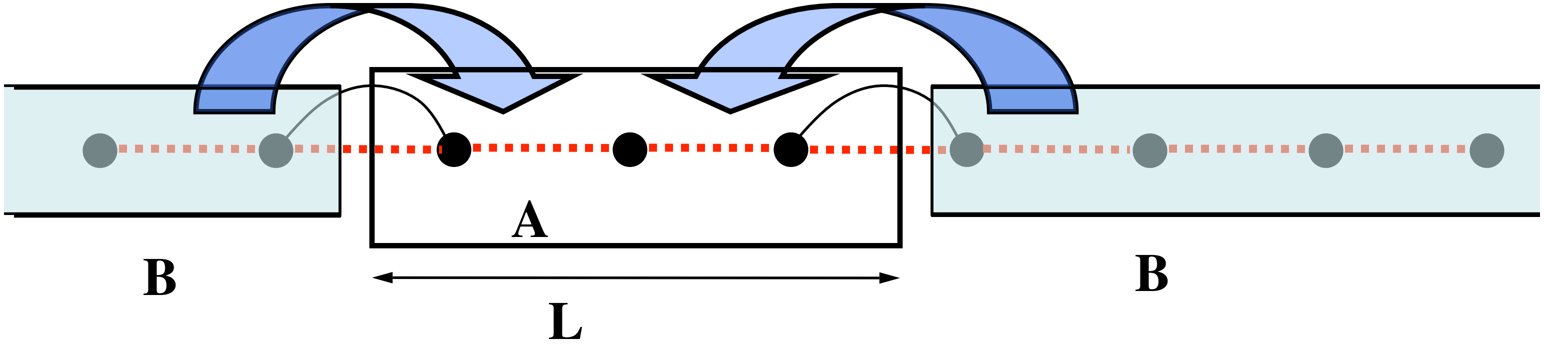

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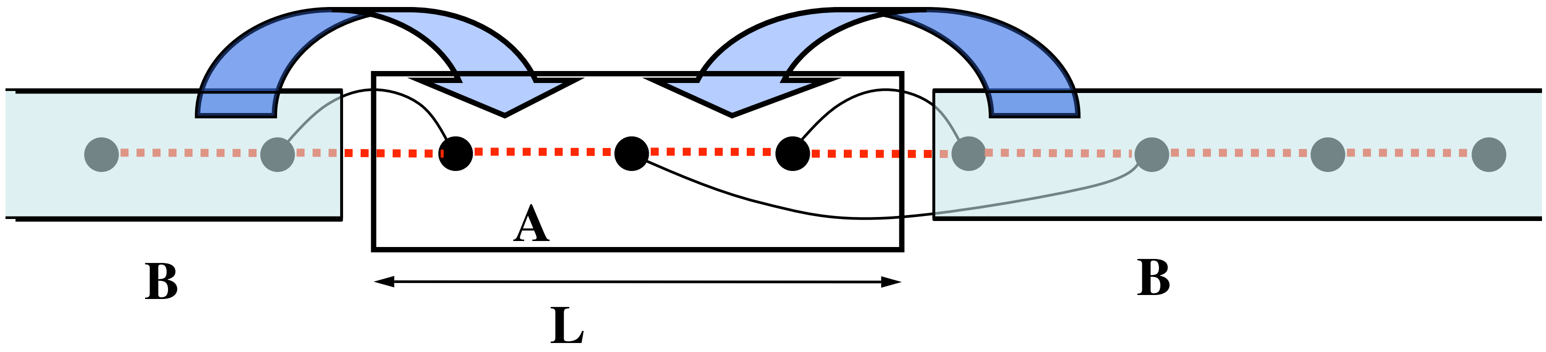

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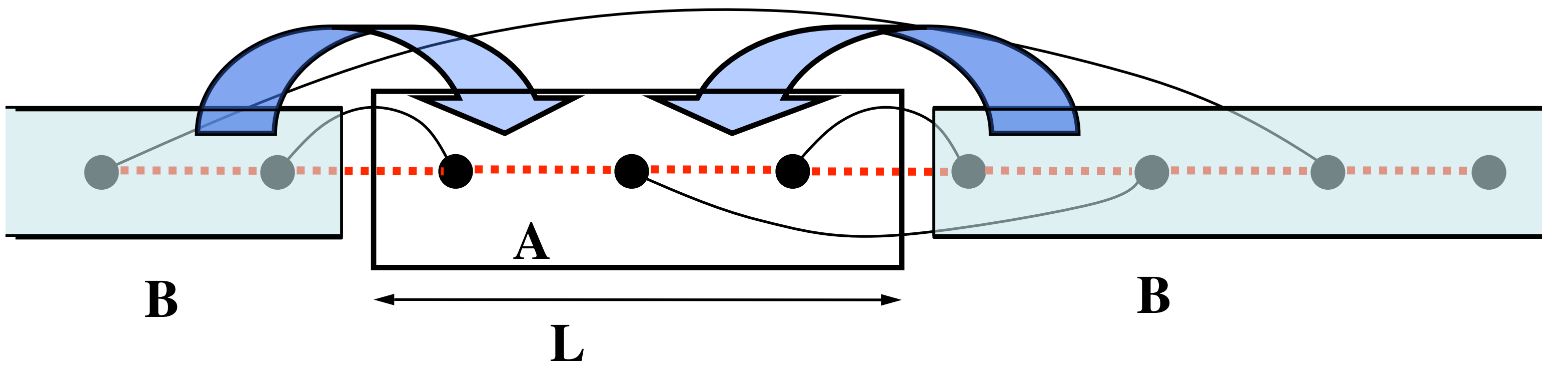

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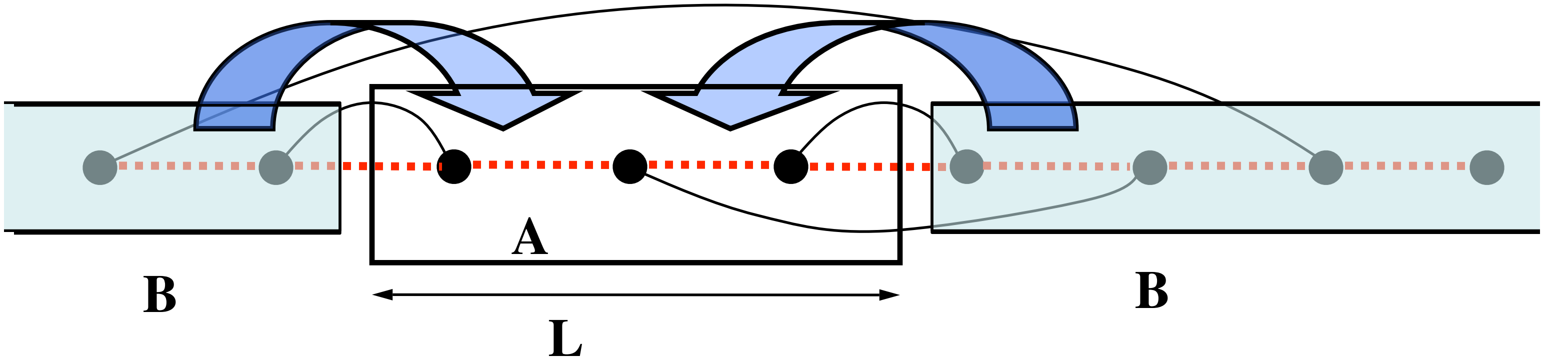
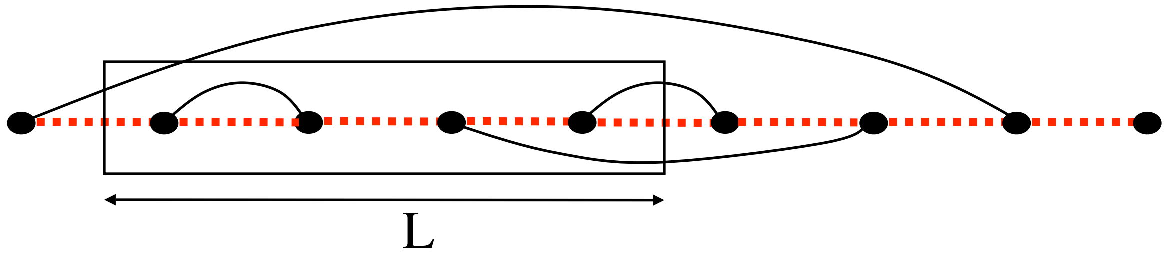


• Every singlet connecting A to B → entanglement entropy 1.

$$E_L = -Tr_A \rho_A \log_2 \rho_A = N_{singlet}$$

Where $N_{singlet}$ is the number of singlets entering region A.

Infinite randomness scaling and entanglement



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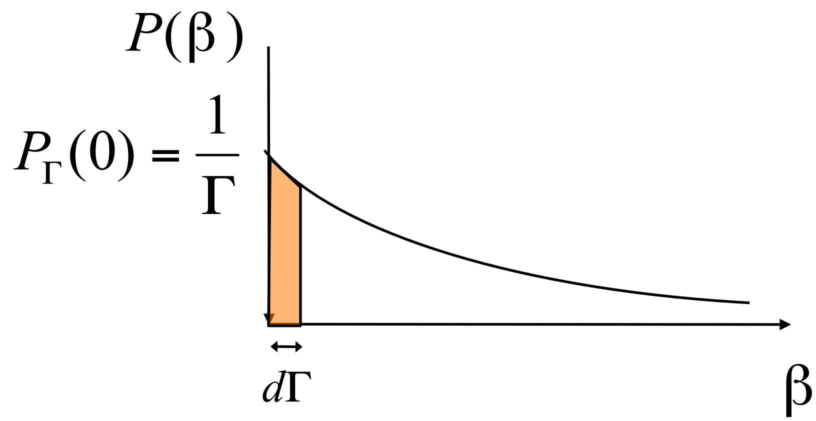

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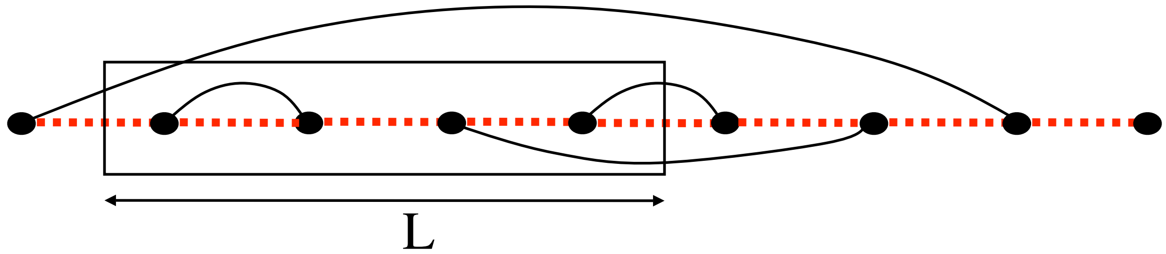

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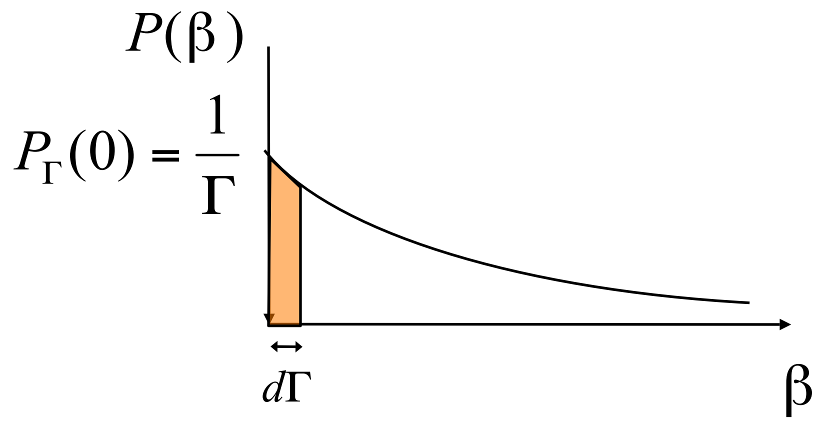

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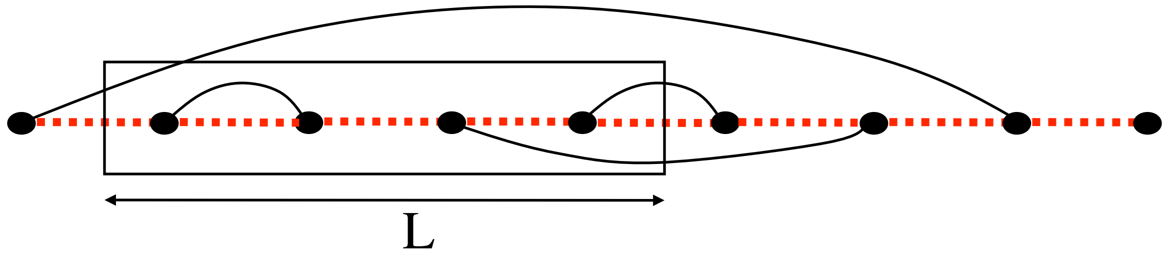

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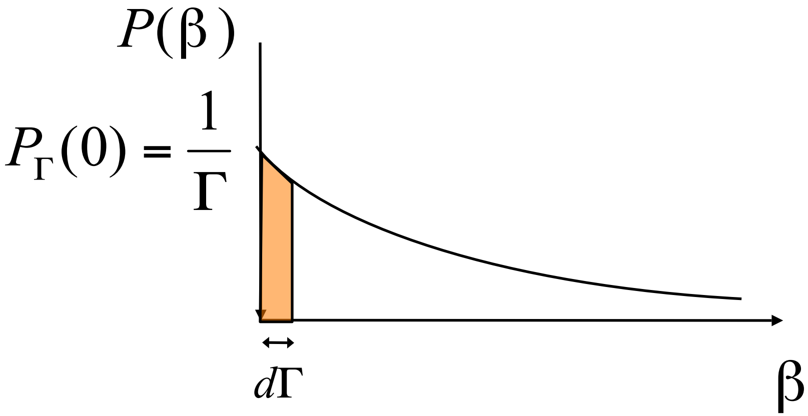

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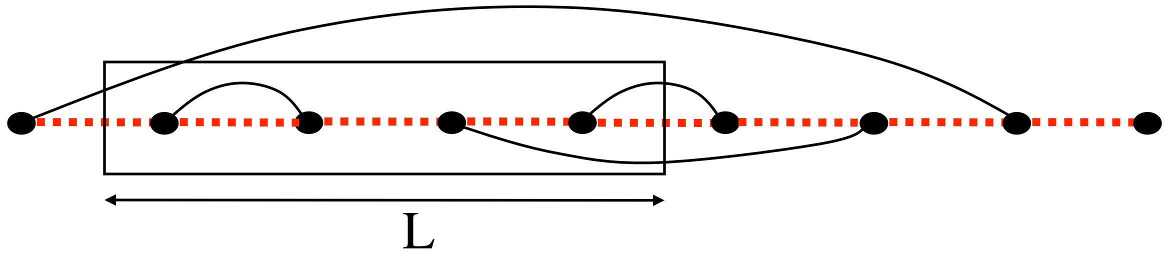

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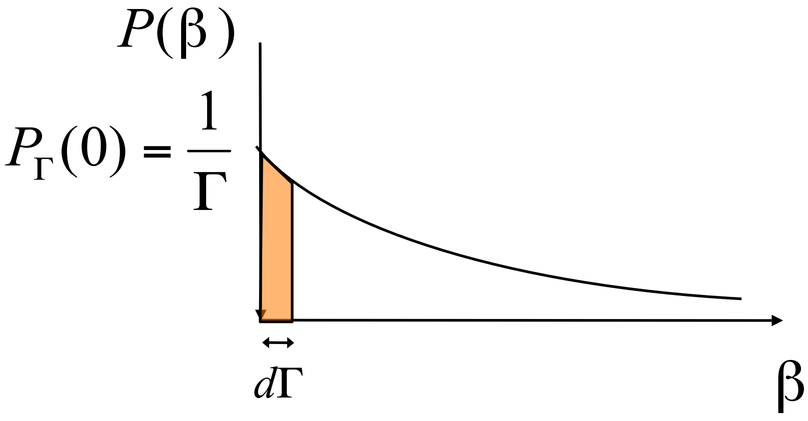

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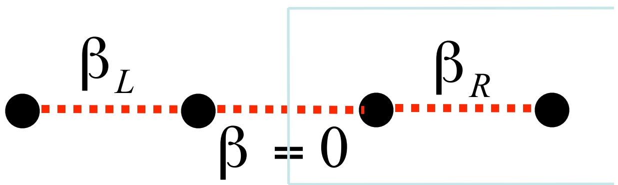
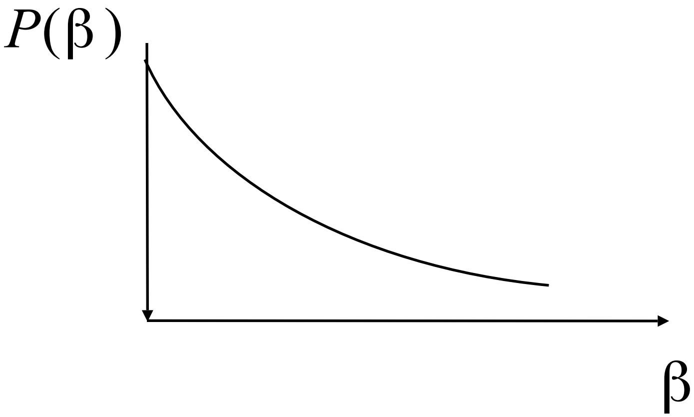
- Entanglement entropy: $E_L = \ln L$

Entropy revisited





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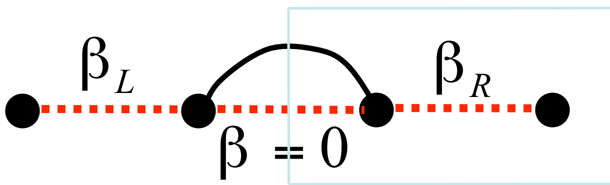

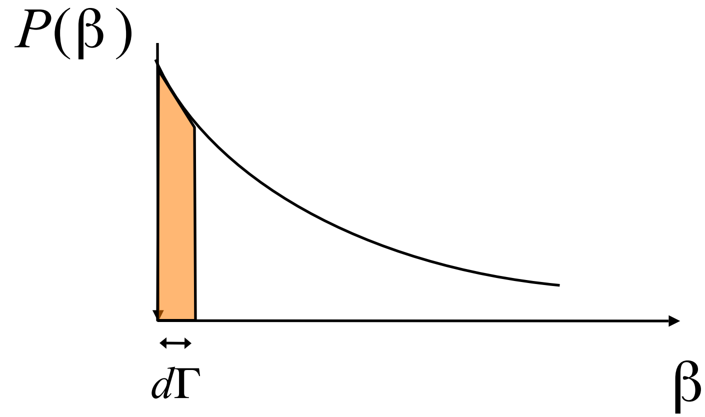

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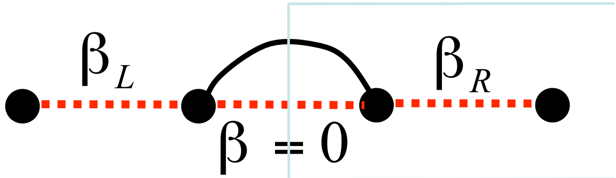

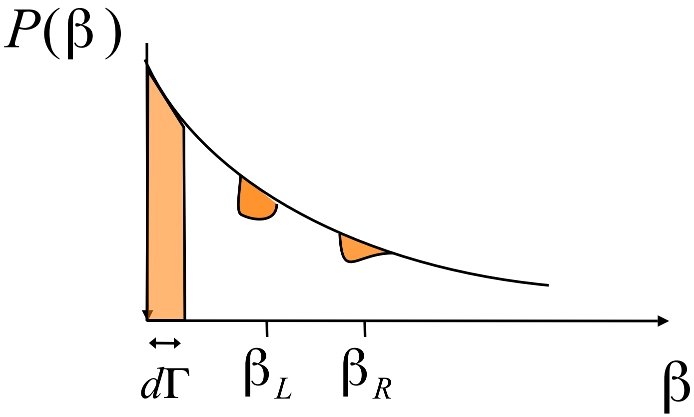

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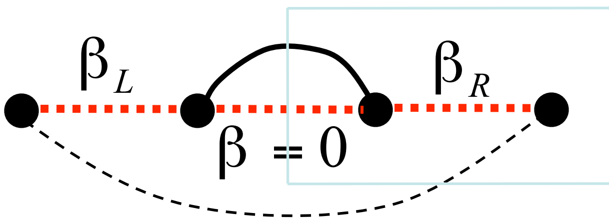

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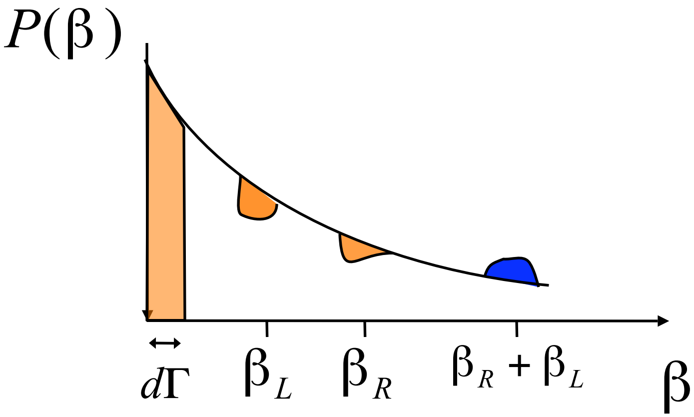

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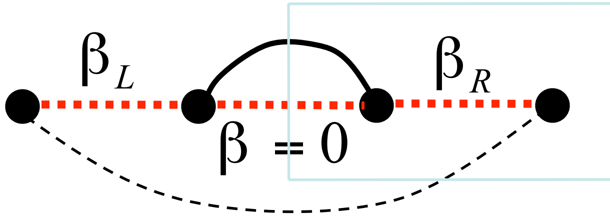

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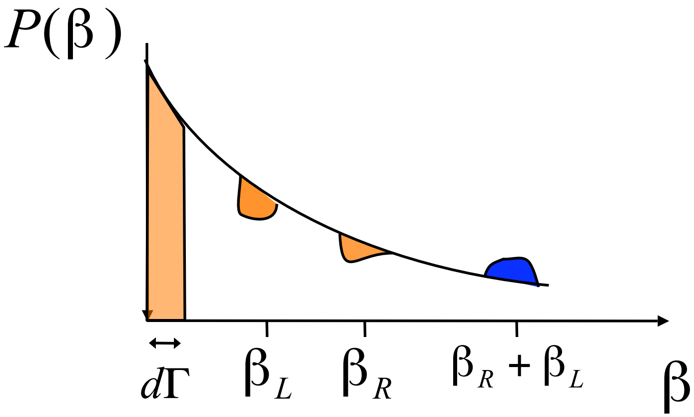

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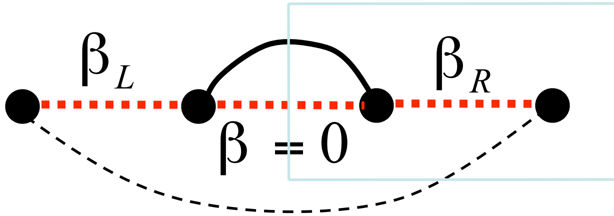
Entropy revisited



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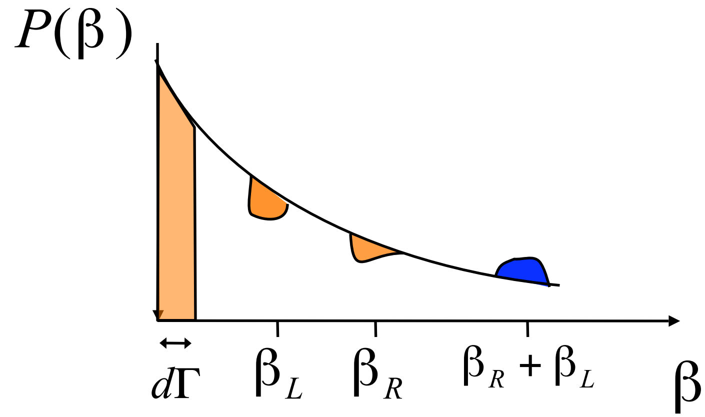

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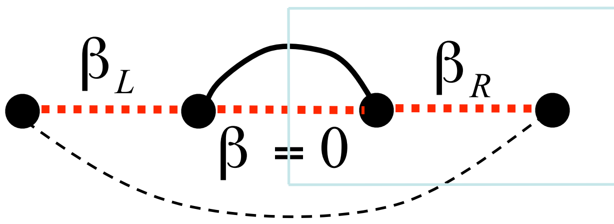

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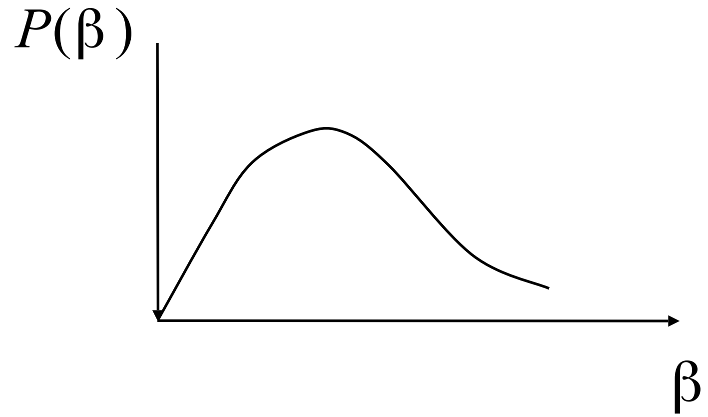

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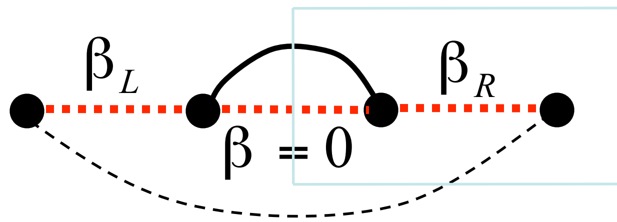

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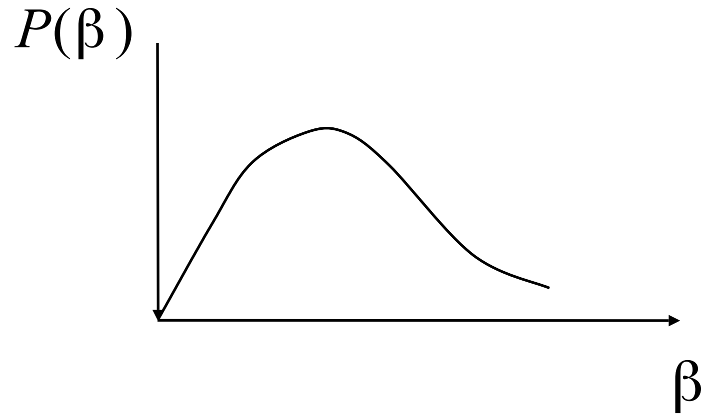

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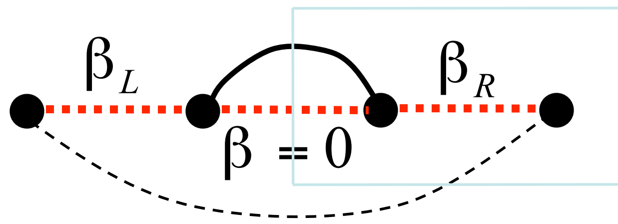

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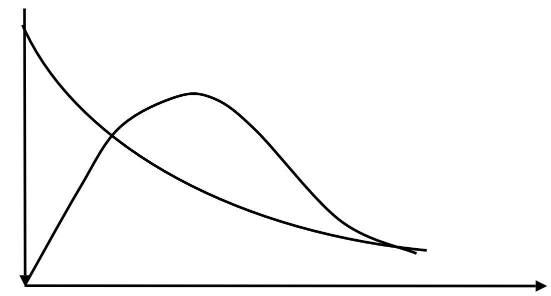

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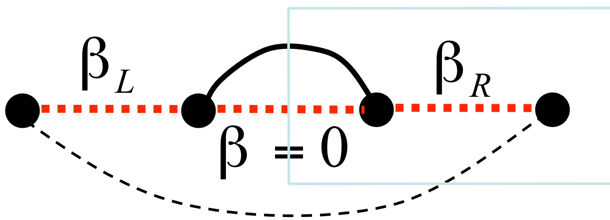

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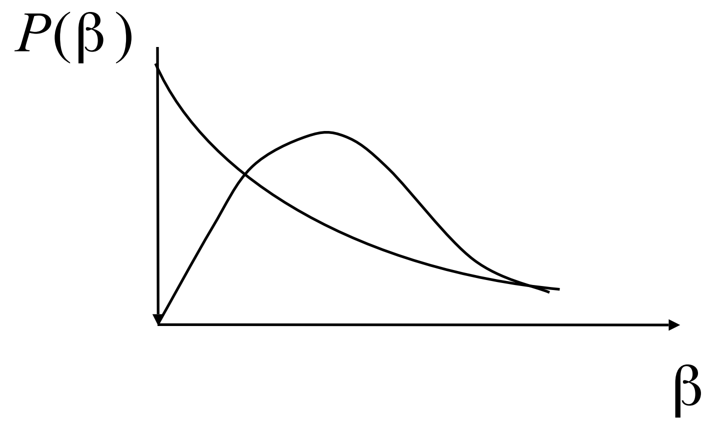

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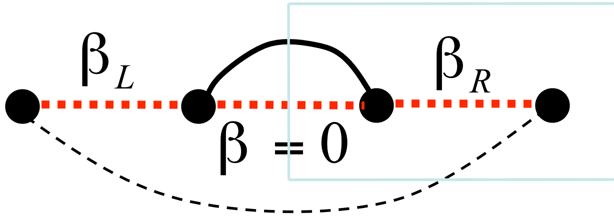

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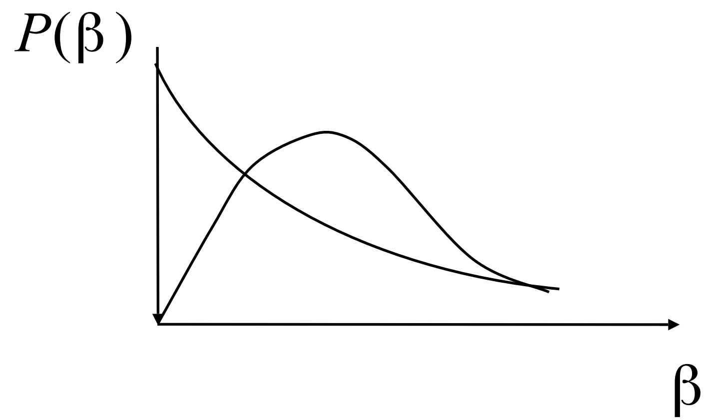

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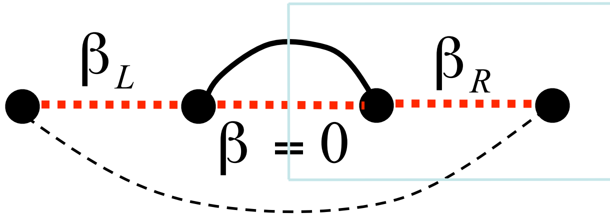

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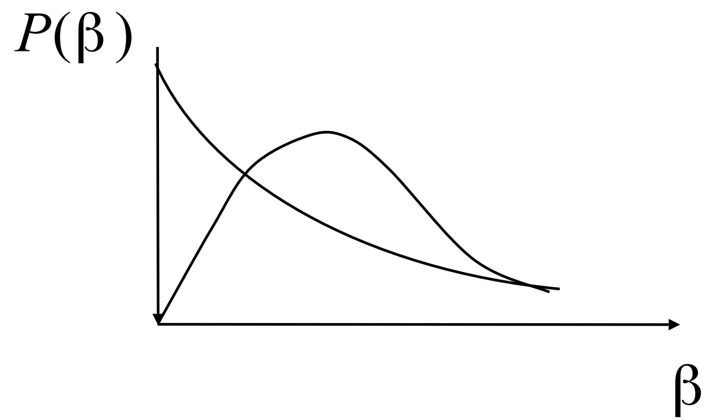

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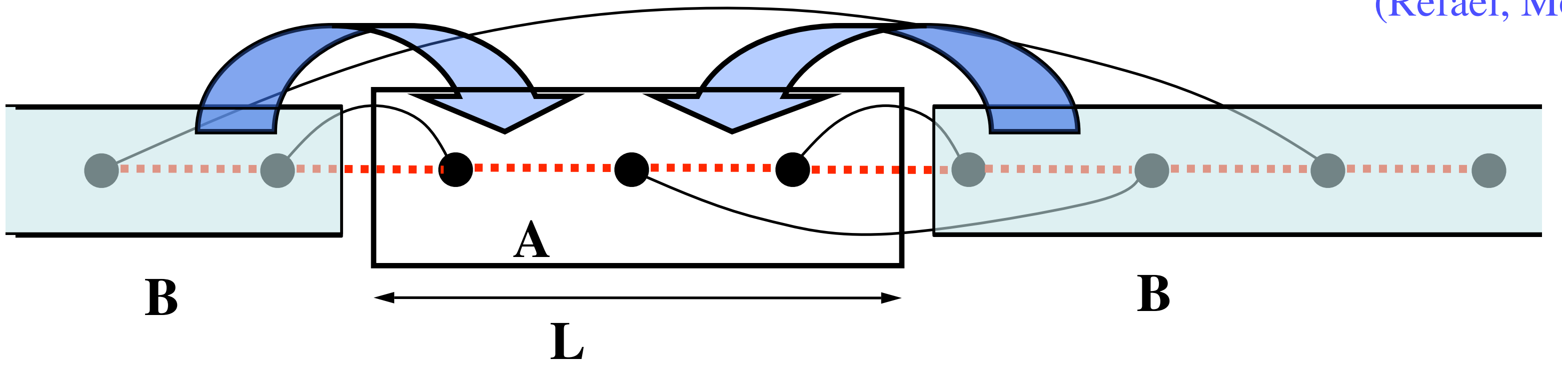
$$E_L = \frac{1}{3} \ln L \; = \frac{1}{3} \ln 2 \times \log_2 L$$

Entanglement in the random singlet phase

(Refael, Moore, 2004)



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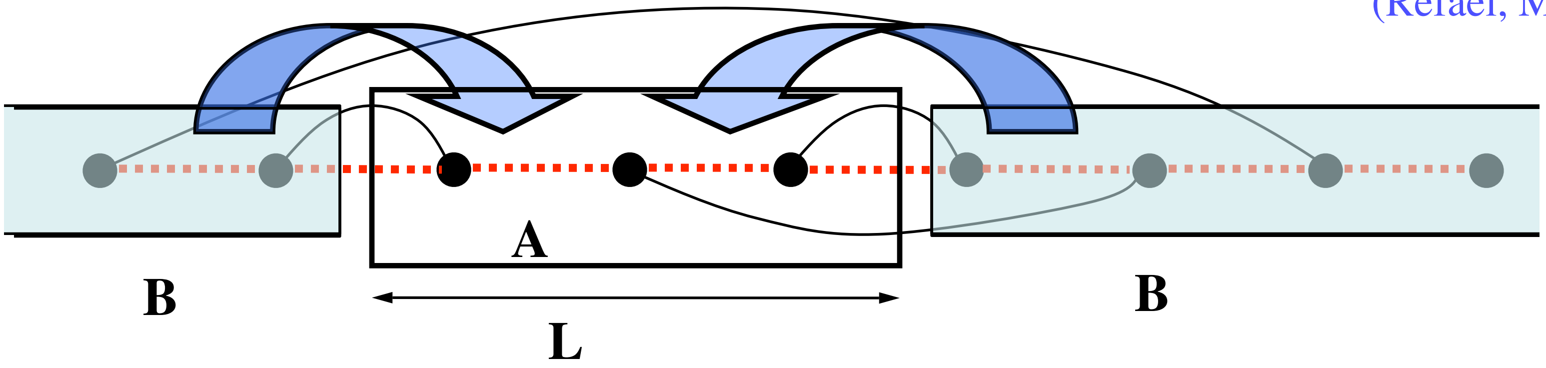

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$$E_L = 2\frac{E_{singlet}}{\Delta l_{singlet}} \times \Delta l_L$$

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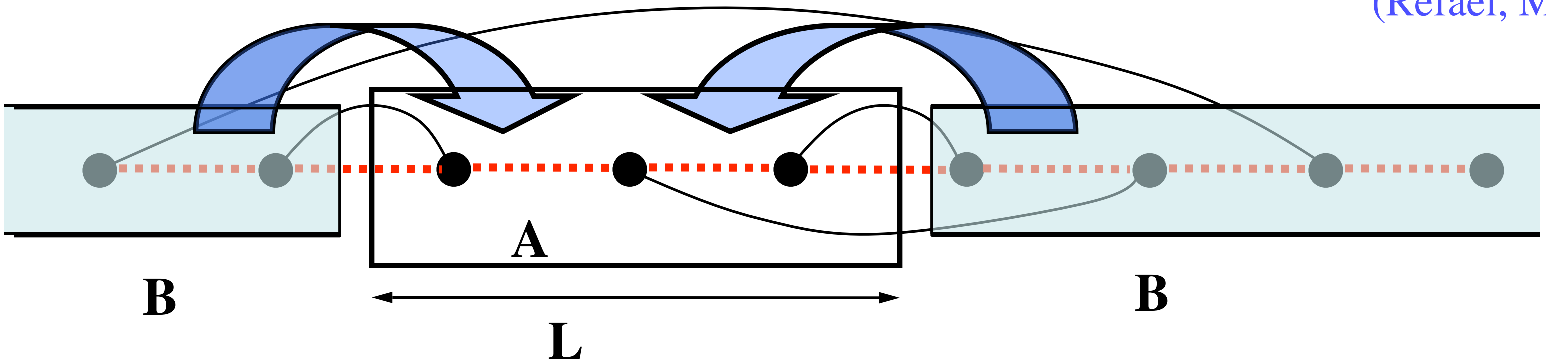

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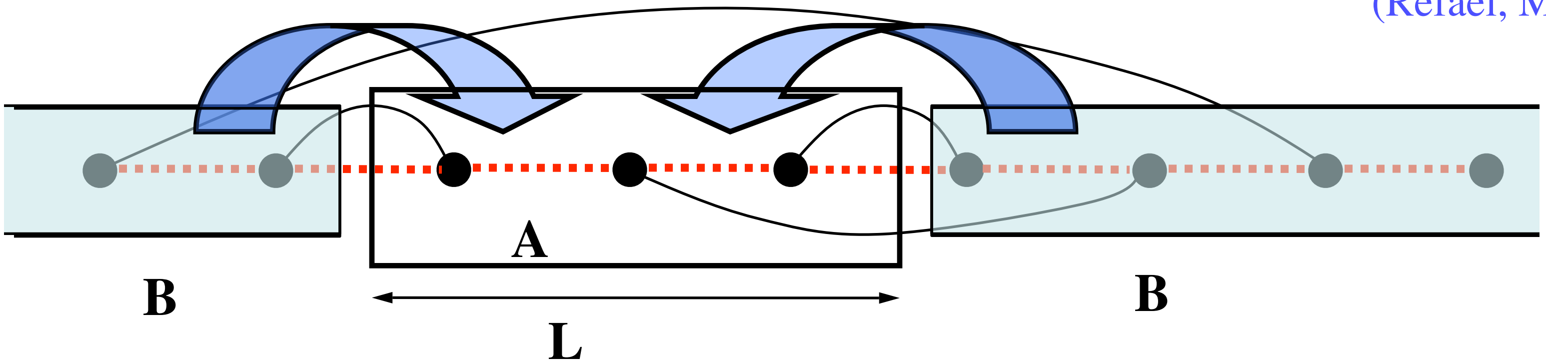

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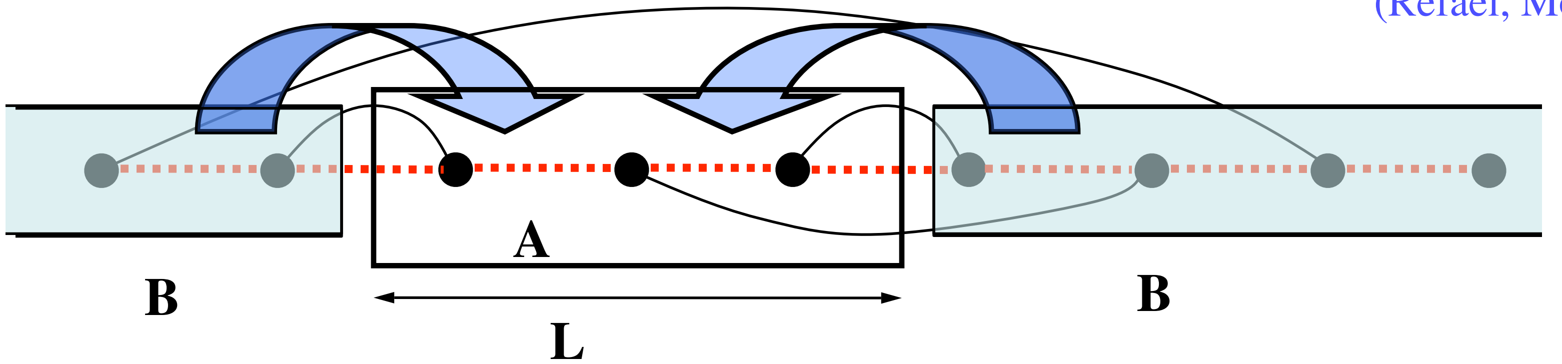

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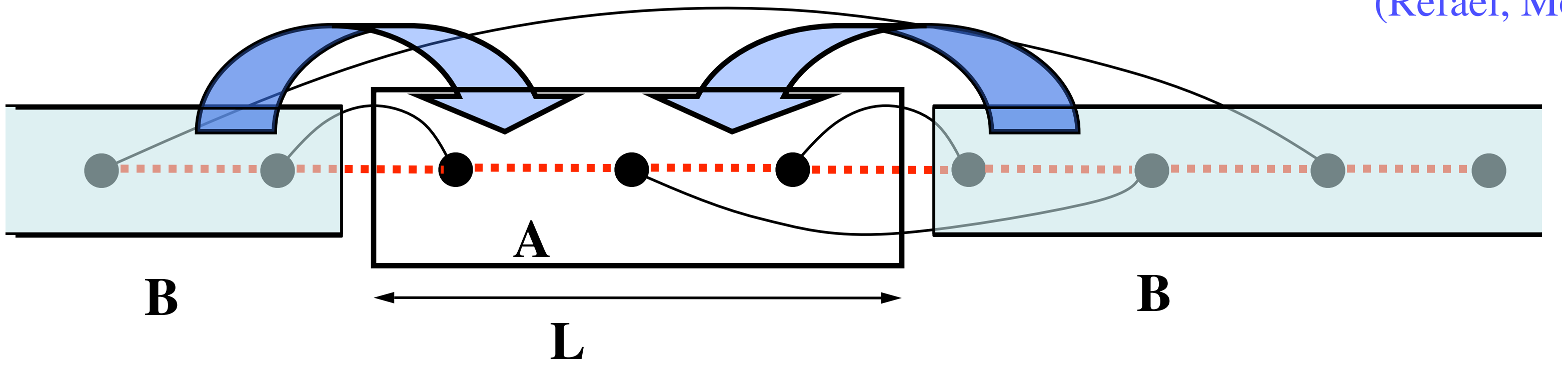

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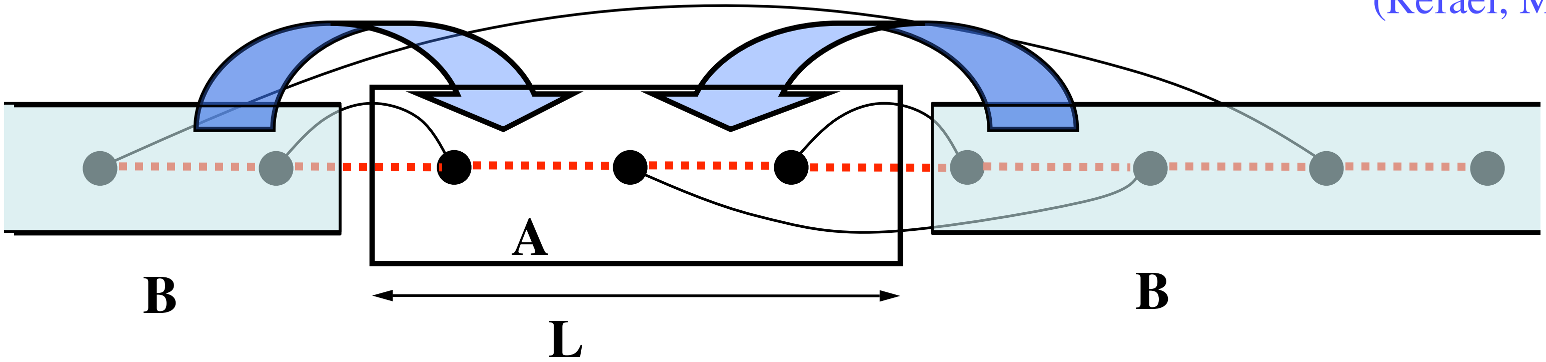

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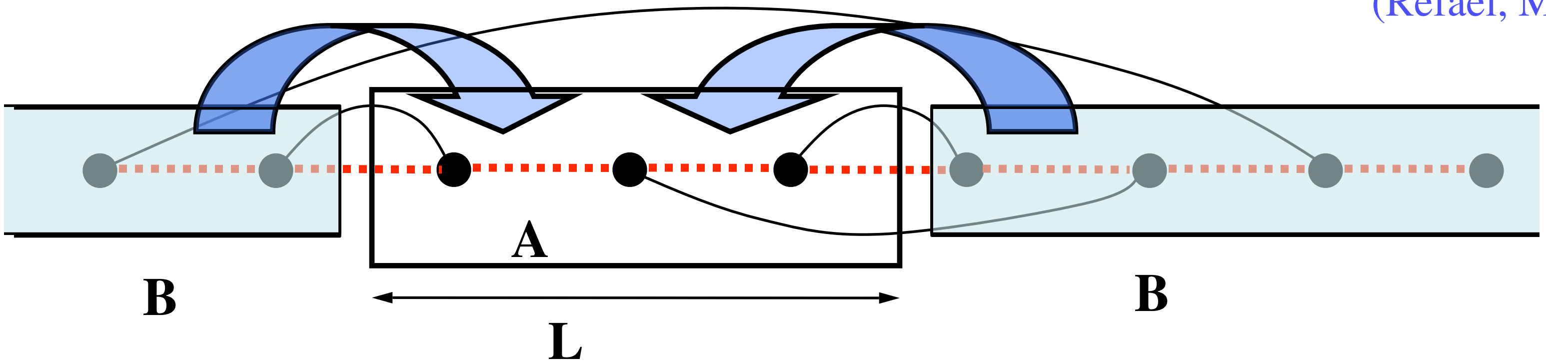

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• **Effective central charge:**

$$c_{random} = 1 \times \ln 2 \qquad \left(c_{pure} = 1 \right)$$

Numerical verification

Laflorencie (2005): XXZ easy-plane model

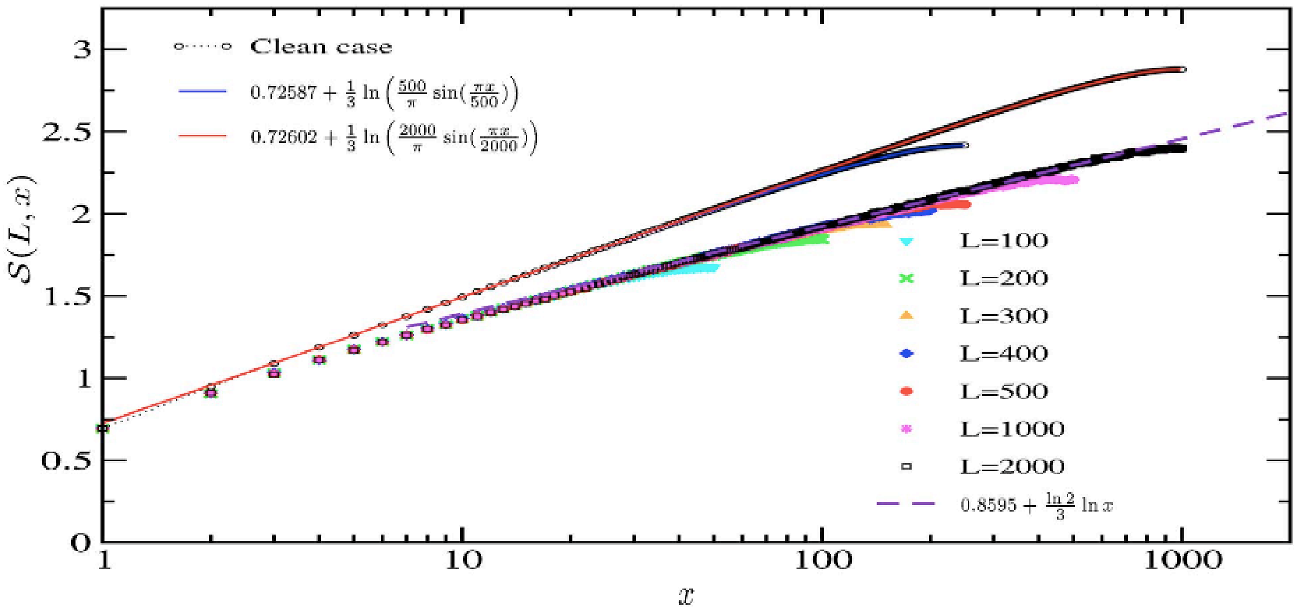

FIG. 2. (Color online) Entanglement entropy of a subsystem of size x embedded in a closed ring of size L, shown vs x in a log-linear plot. Numerical results obtained by exact diagonalizations performed at the XX point. For clean nonrandom systems with $L=500$ and $L=2000$ (open circles), $S(x)$ is perfectly described by Eq. (3) (red and blue curves). The data for random systems have been averaged over 10^4 samples for $L=500$, 1000, 2000, and 2×10^4 samples for $100 \leqslant L \leqslant 400$. The expression $0.8595+(\ln 2/3)\ln x$ (dashed line) fits the data in the regime where finite size effects are absent.

H. Tran and N. Bonesteel: Confirmed the Heisenberg model result

Pure vs. random – effective central charge

$$E_{AB} = -Tr_A \, \rho_A \log_2 \rho_A = \frac{c}{3} \log_2 L$$

Holzhey, Larsen, Wilczek (1994).

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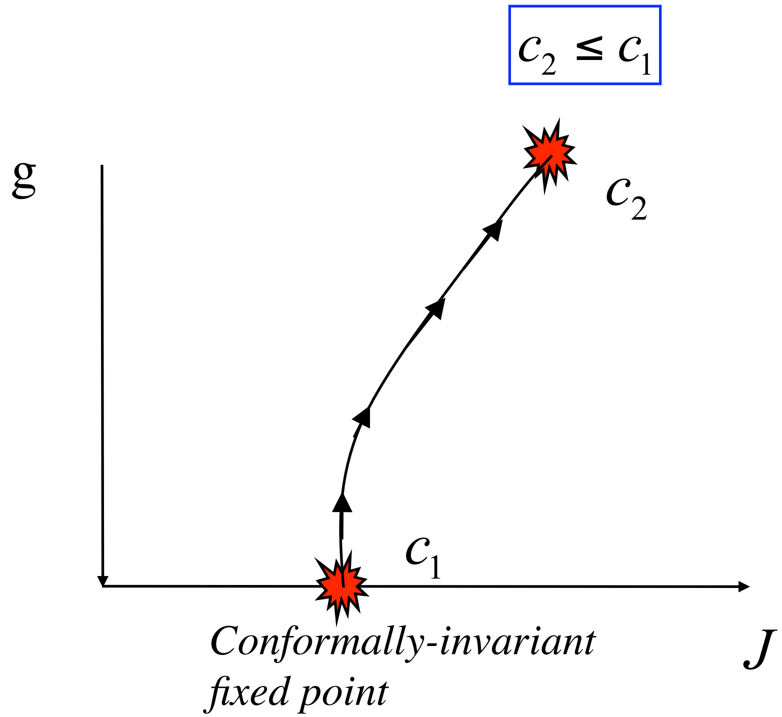
Generally in random singlet phase: $c_{random} = \ln D$

Generalized c-theorem?

- The central charge never increases along RG flows. (Zamolodchikov, 1986)



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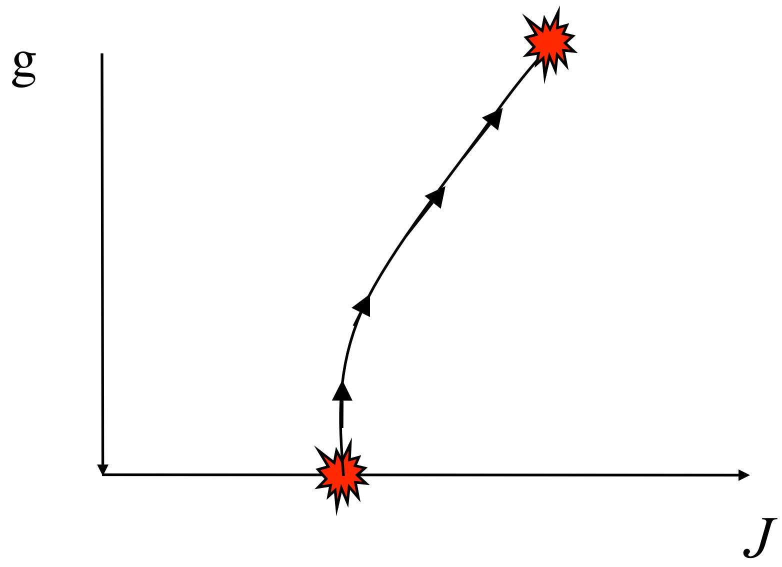

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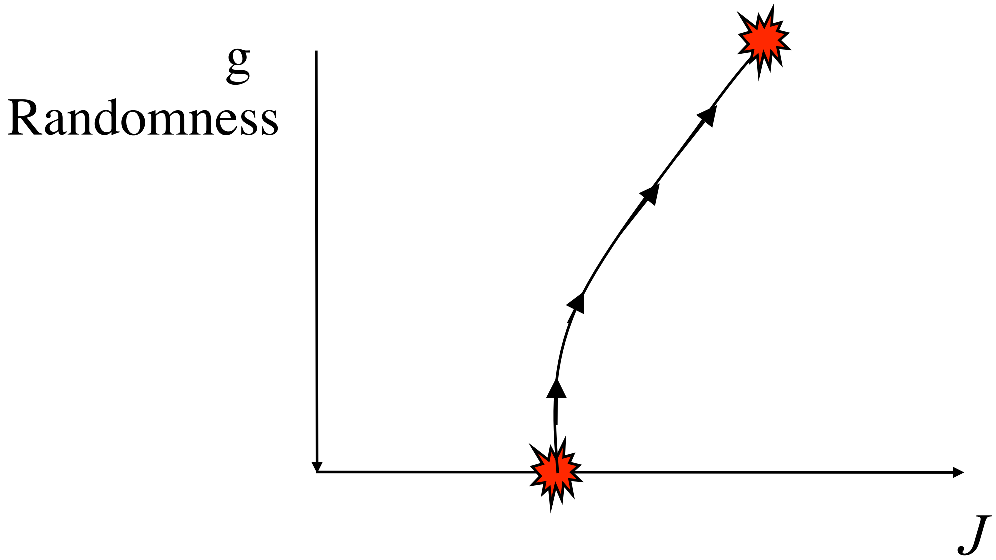

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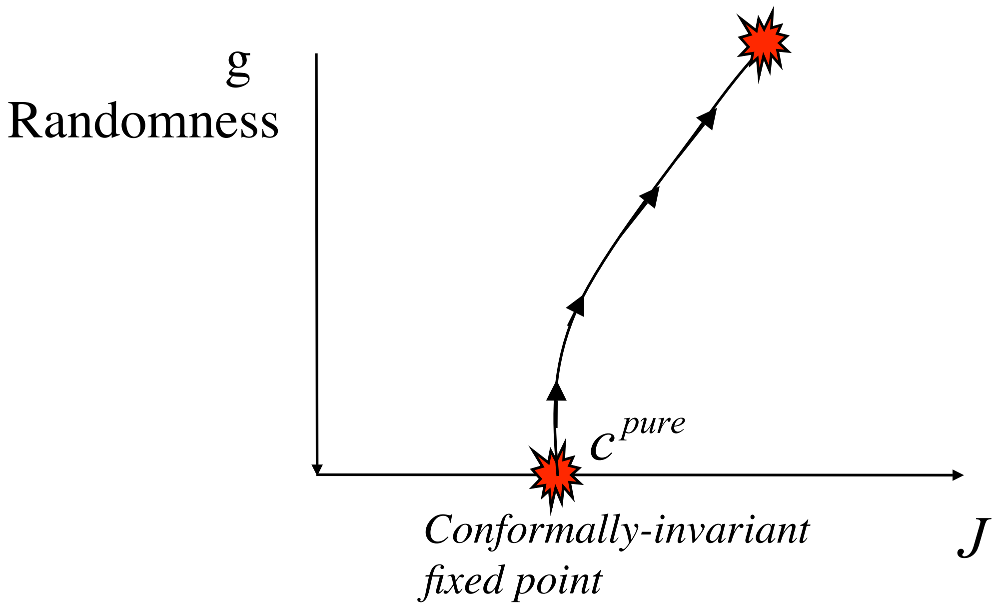

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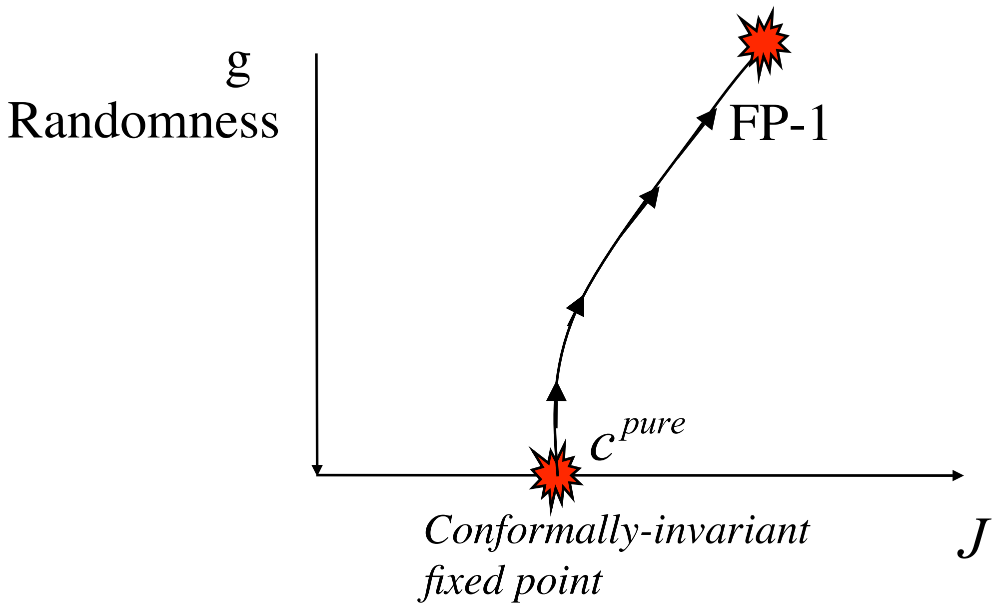

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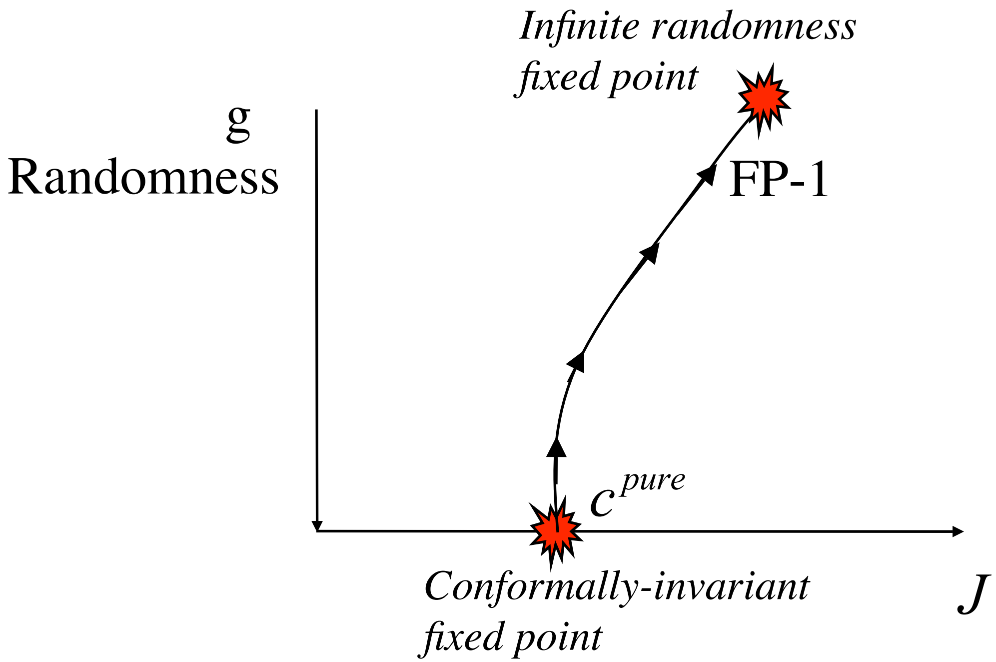

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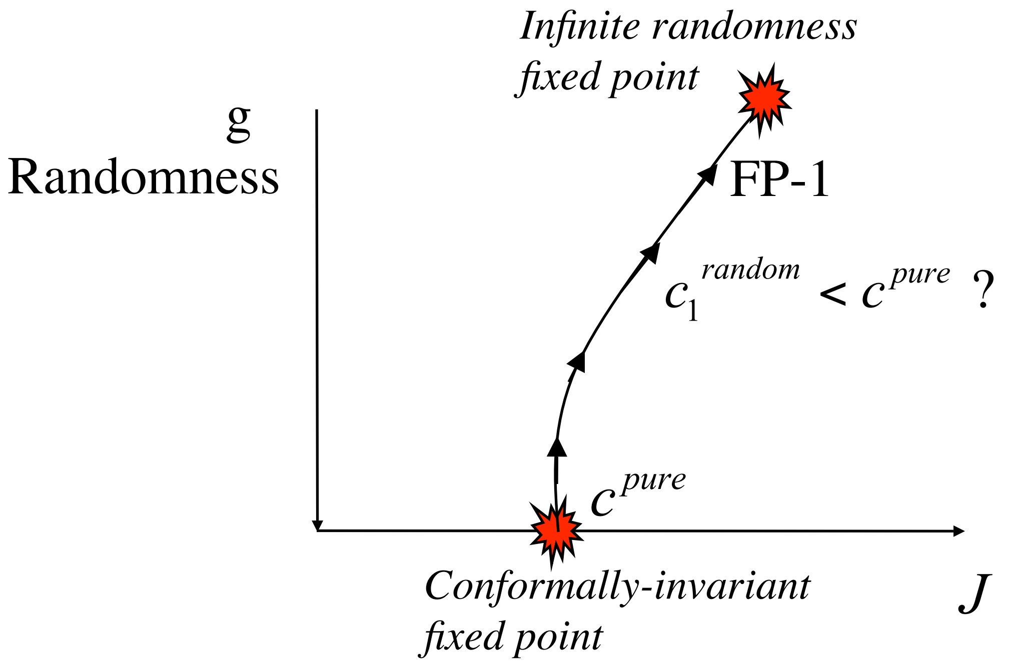

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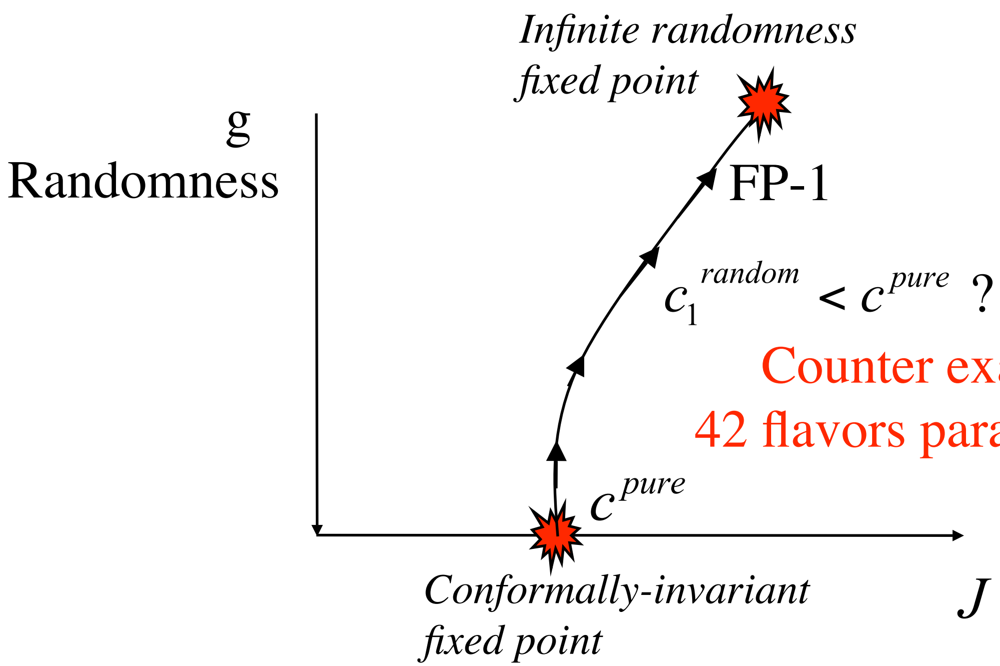

Counter example:
42 flavors parafermionic potts model
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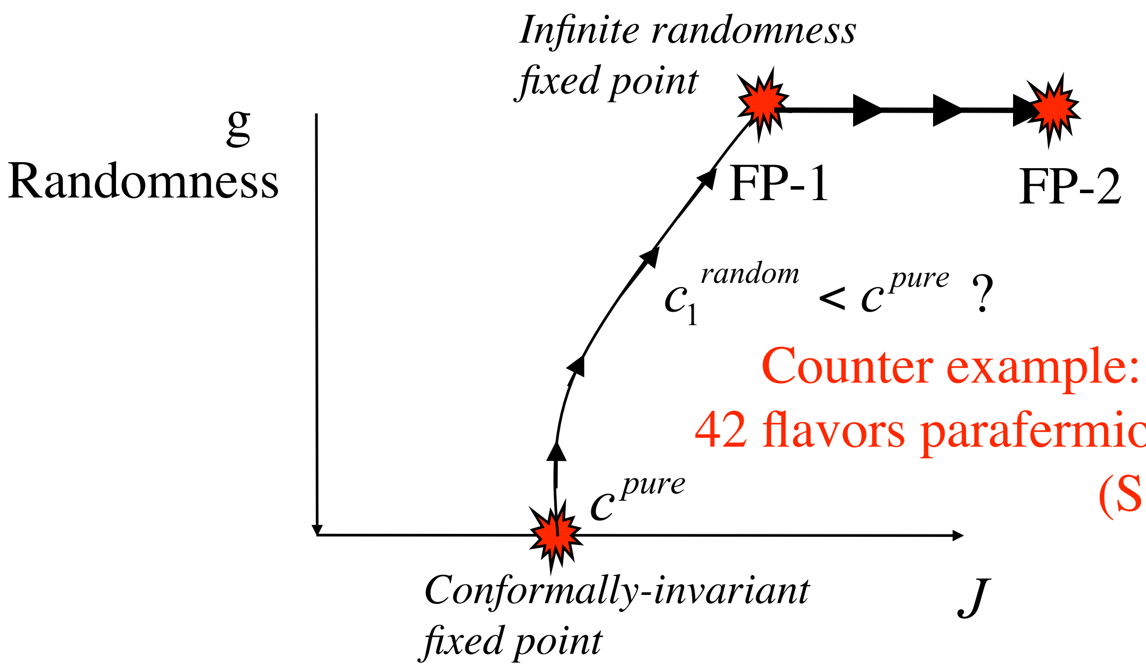

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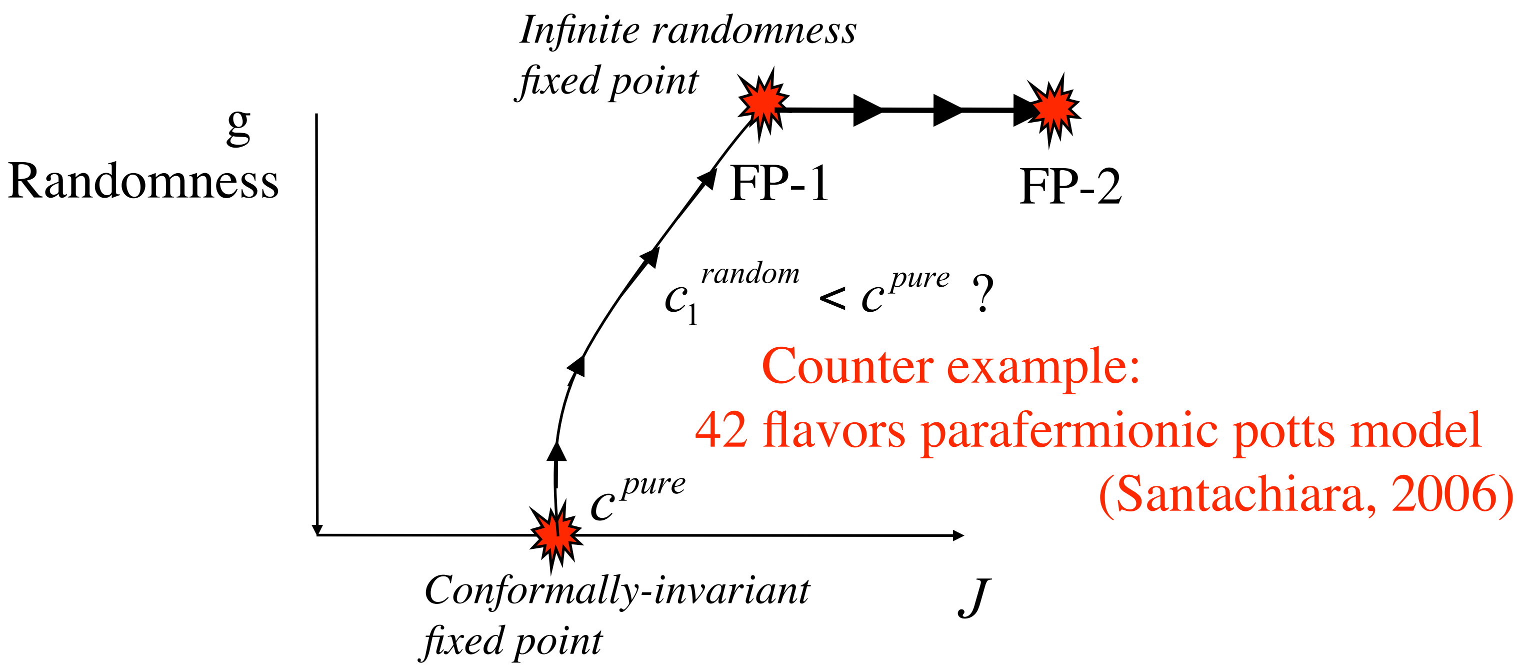

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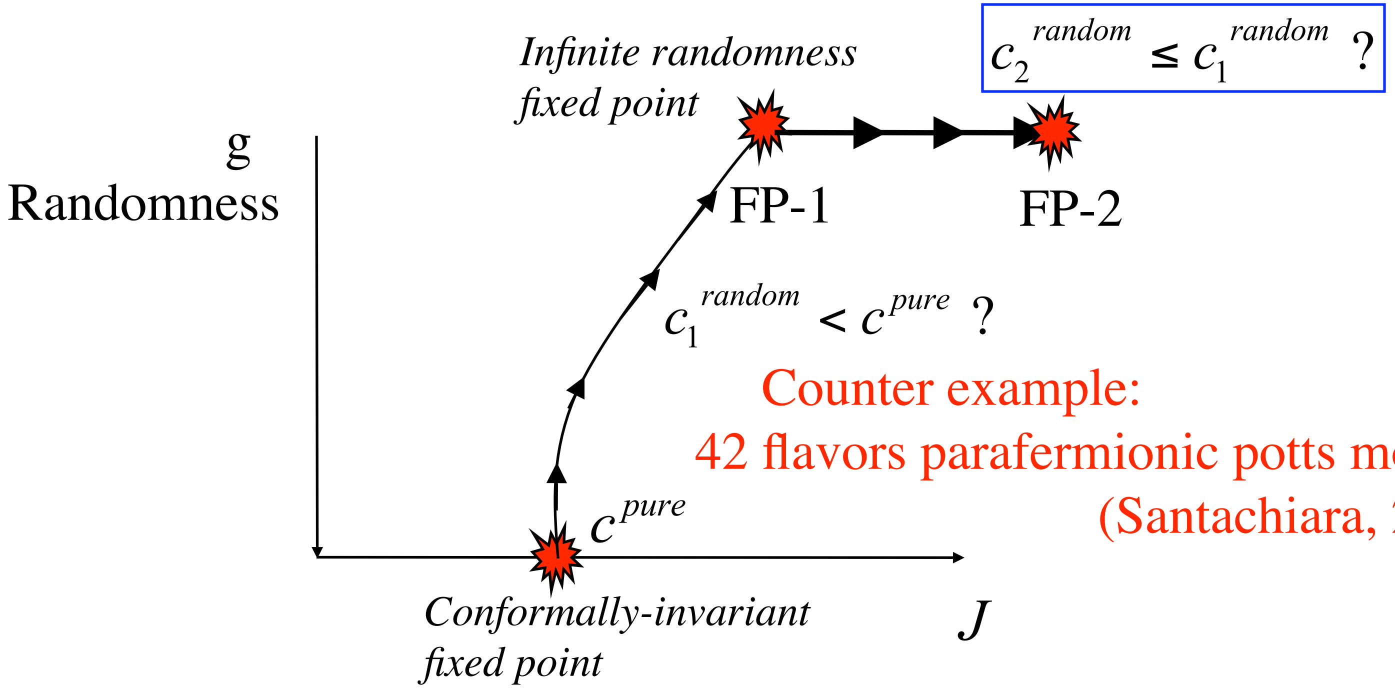

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?	Spin-k/2 Heisneberg $SU_k(2)$	$\frac{3k}{k+2}$	?	

Random Spin-1:
Haldane-RS critical point

Spin-1 Phase Diagram

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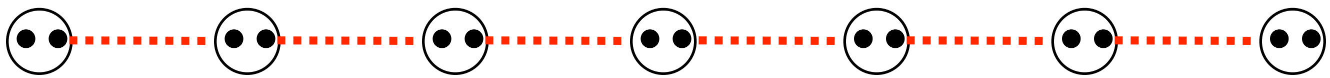

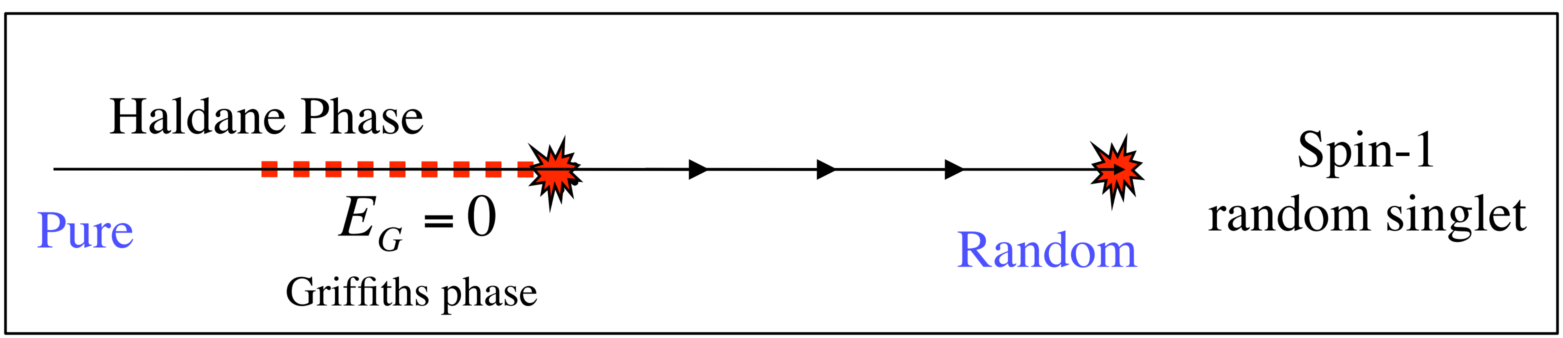

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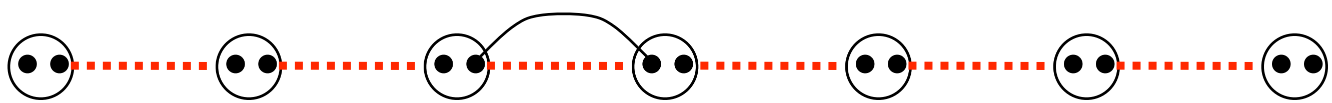

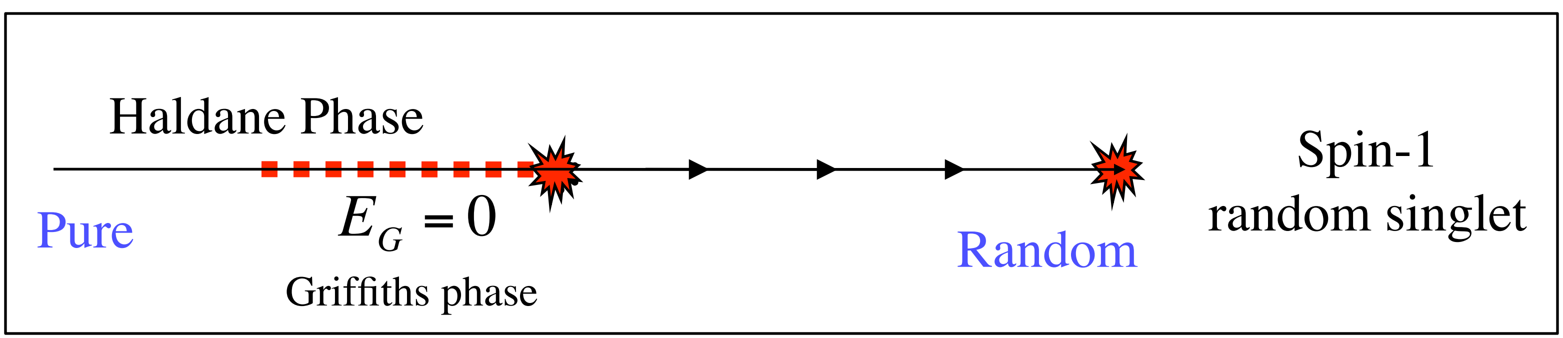

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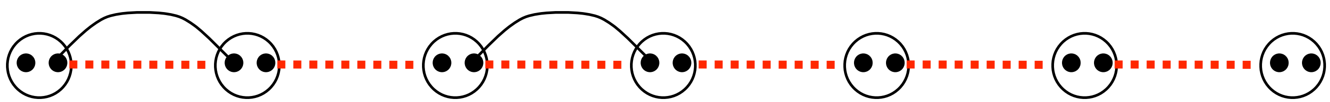

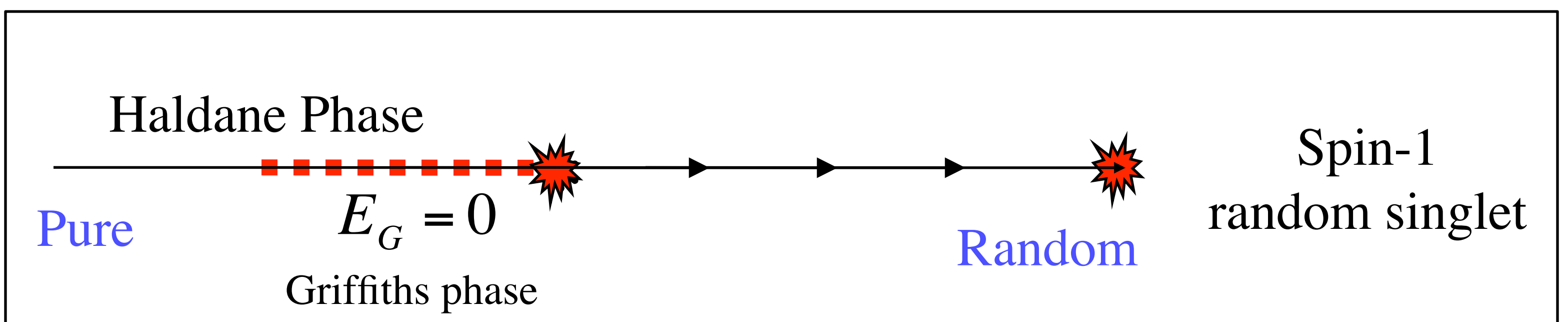

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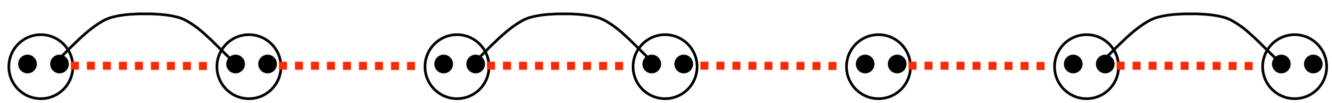

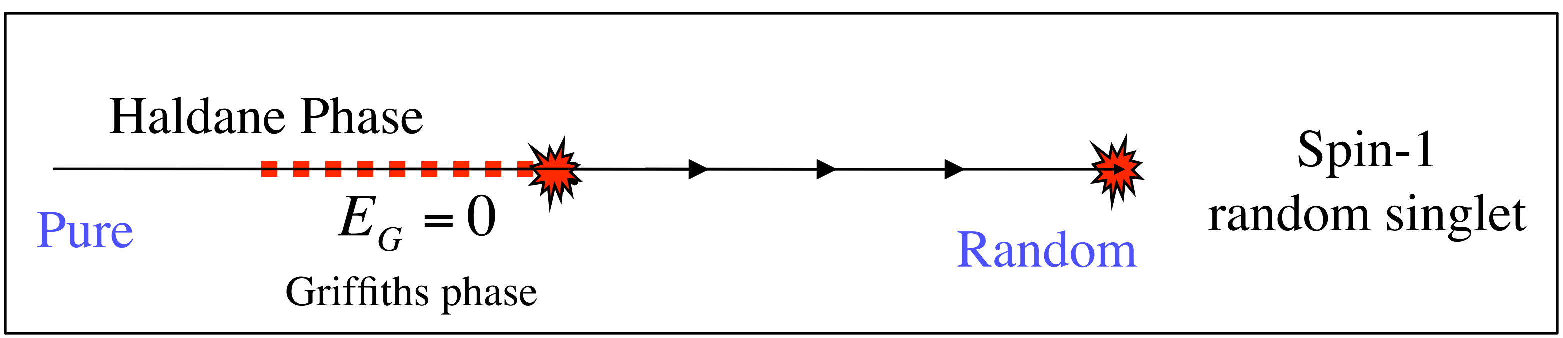

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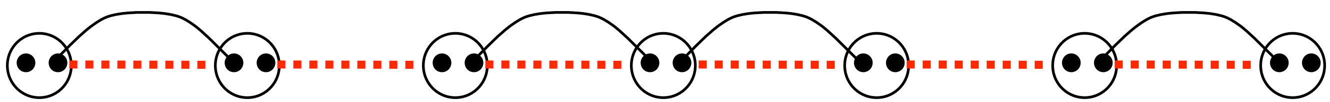

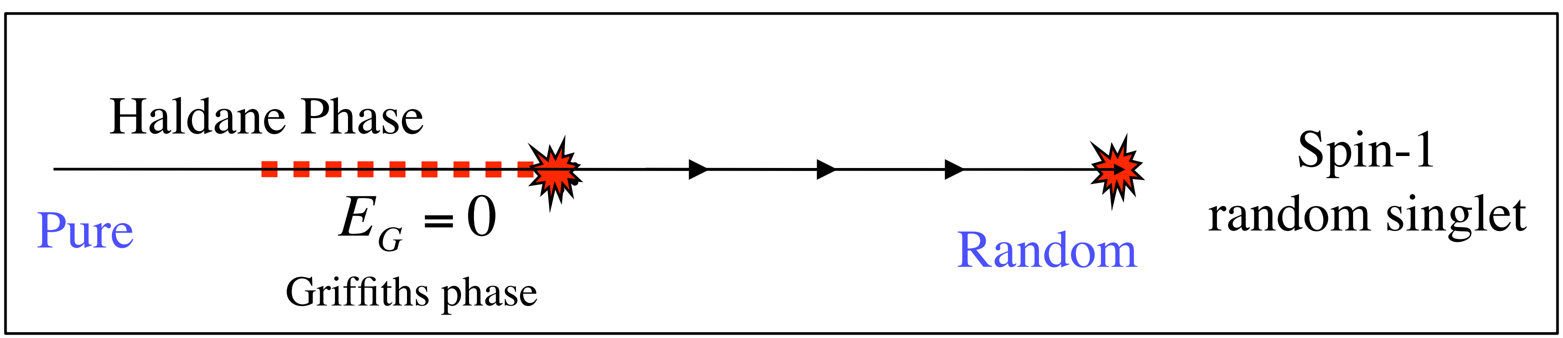

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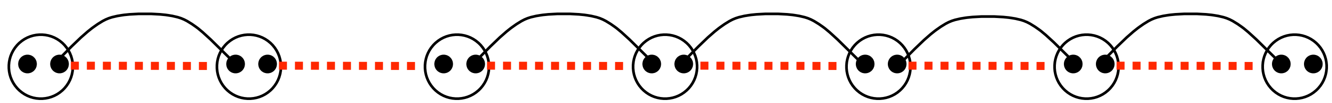

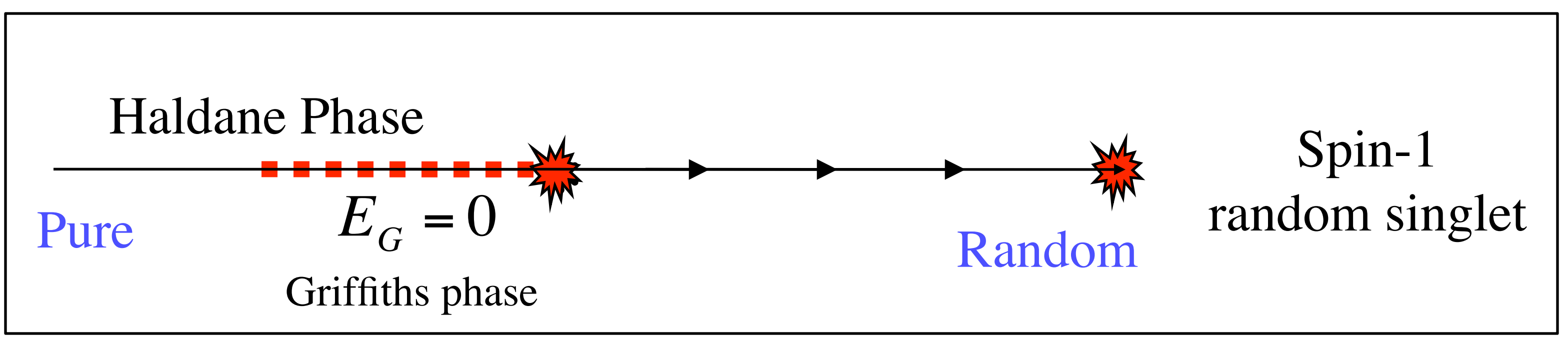

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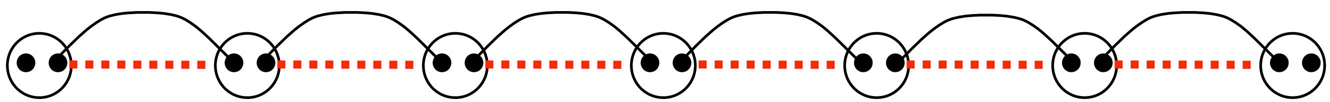

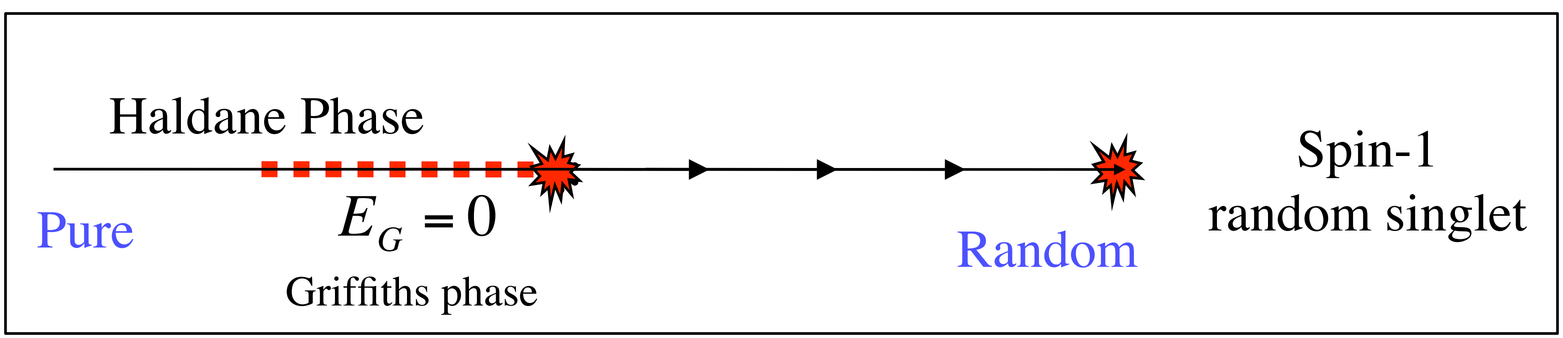

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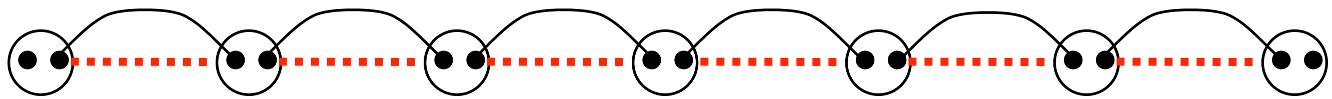

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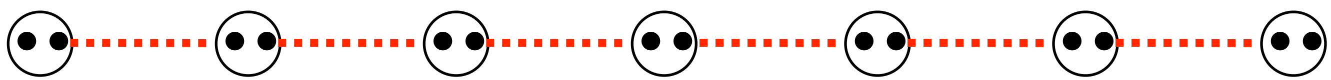

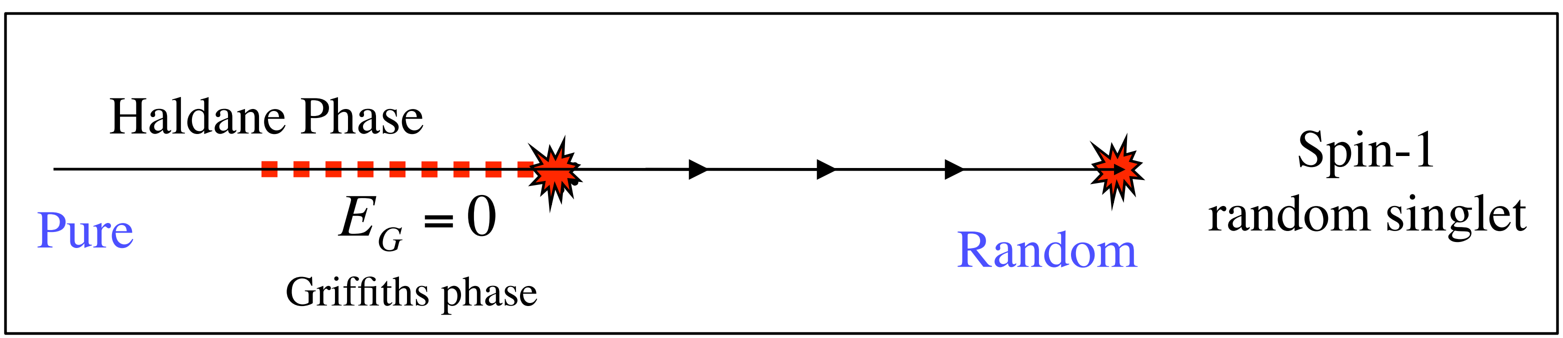

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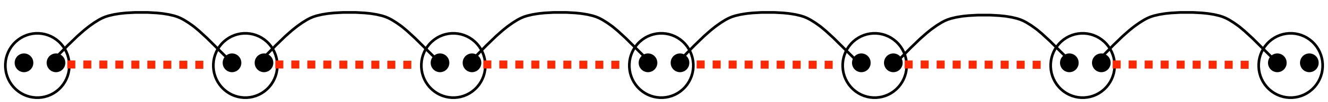

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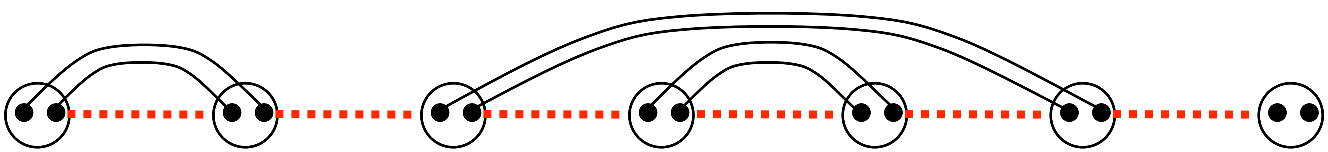

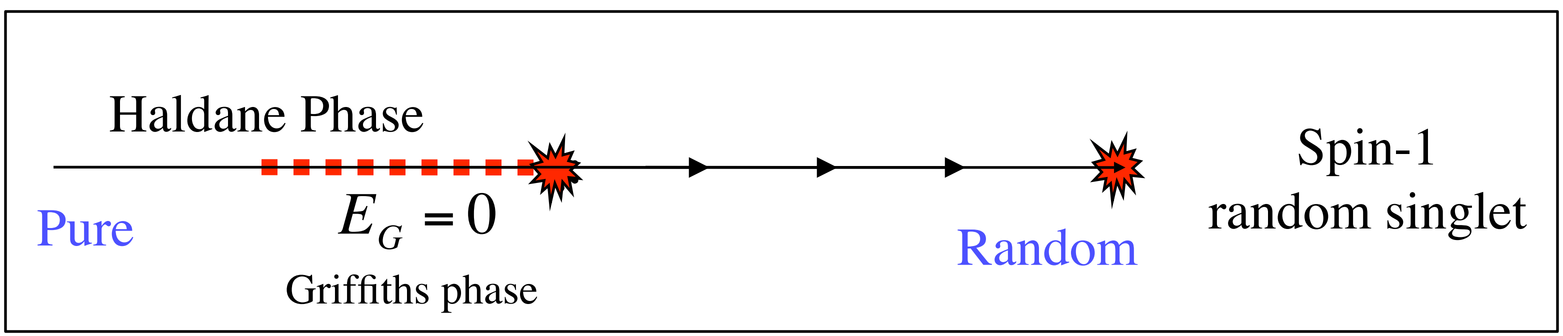

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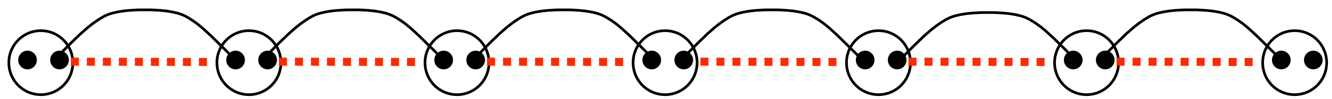

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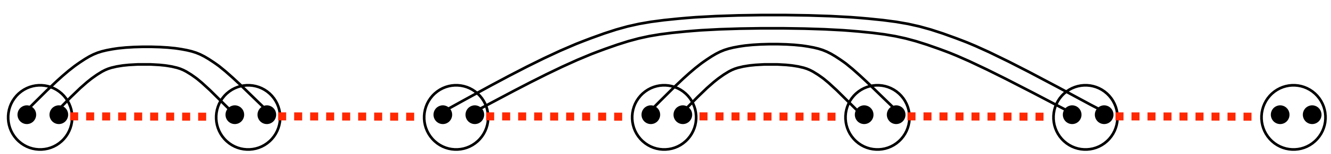

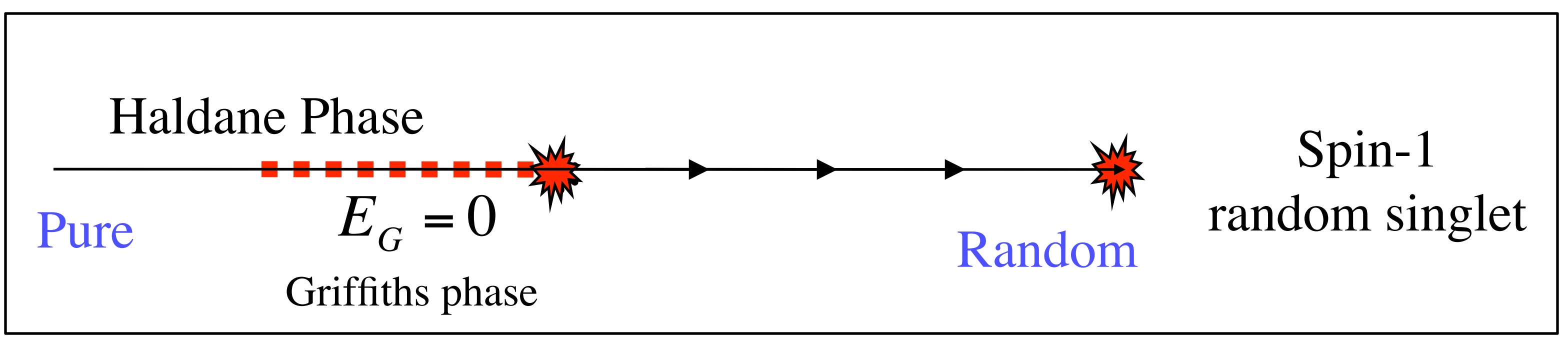

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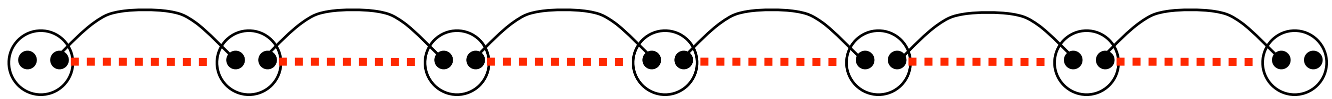

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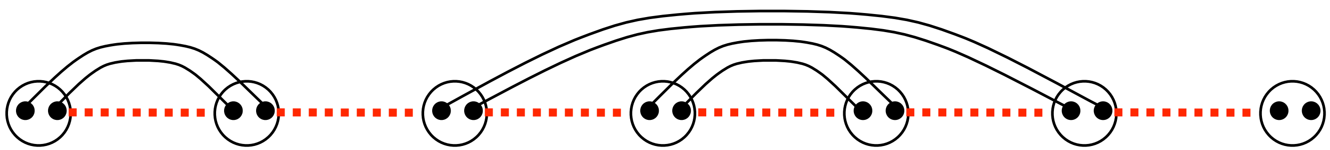

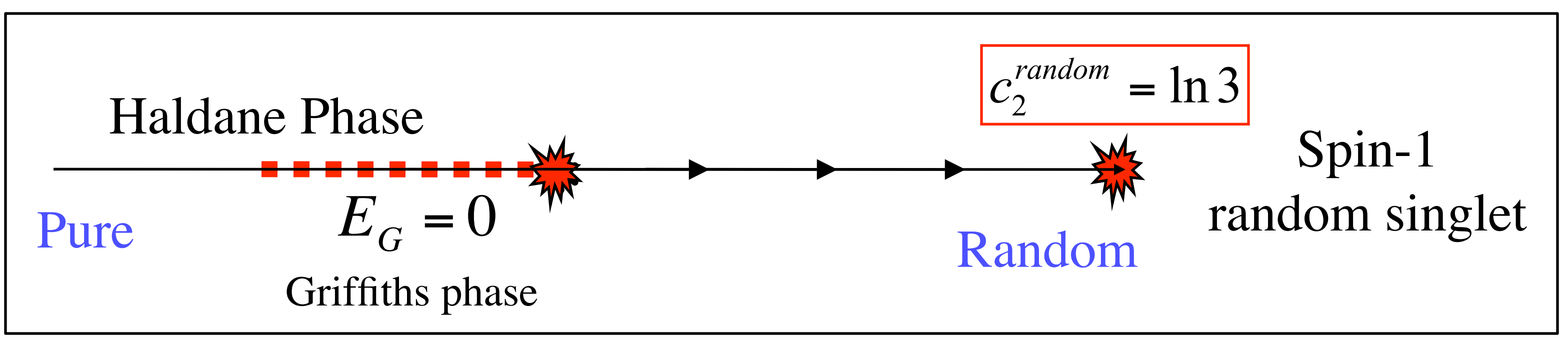

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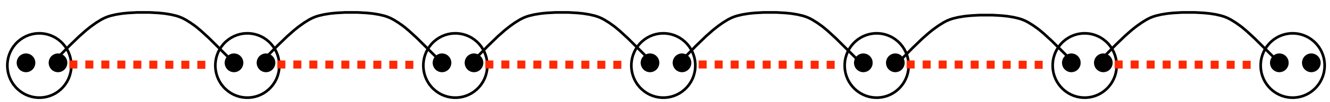

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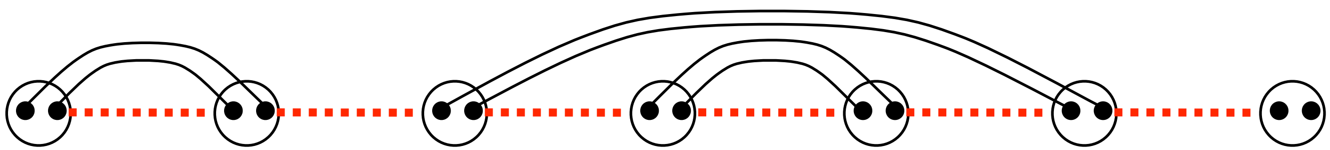

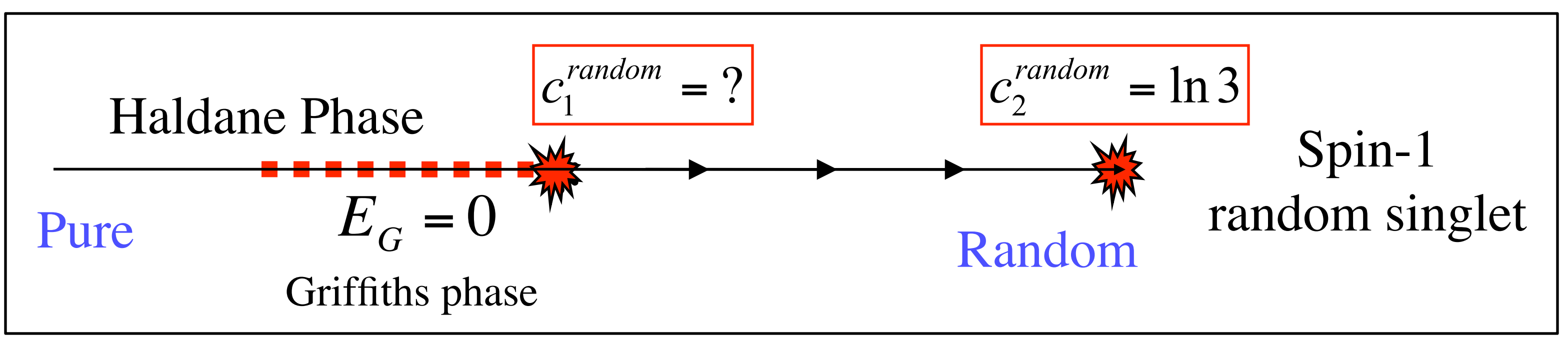

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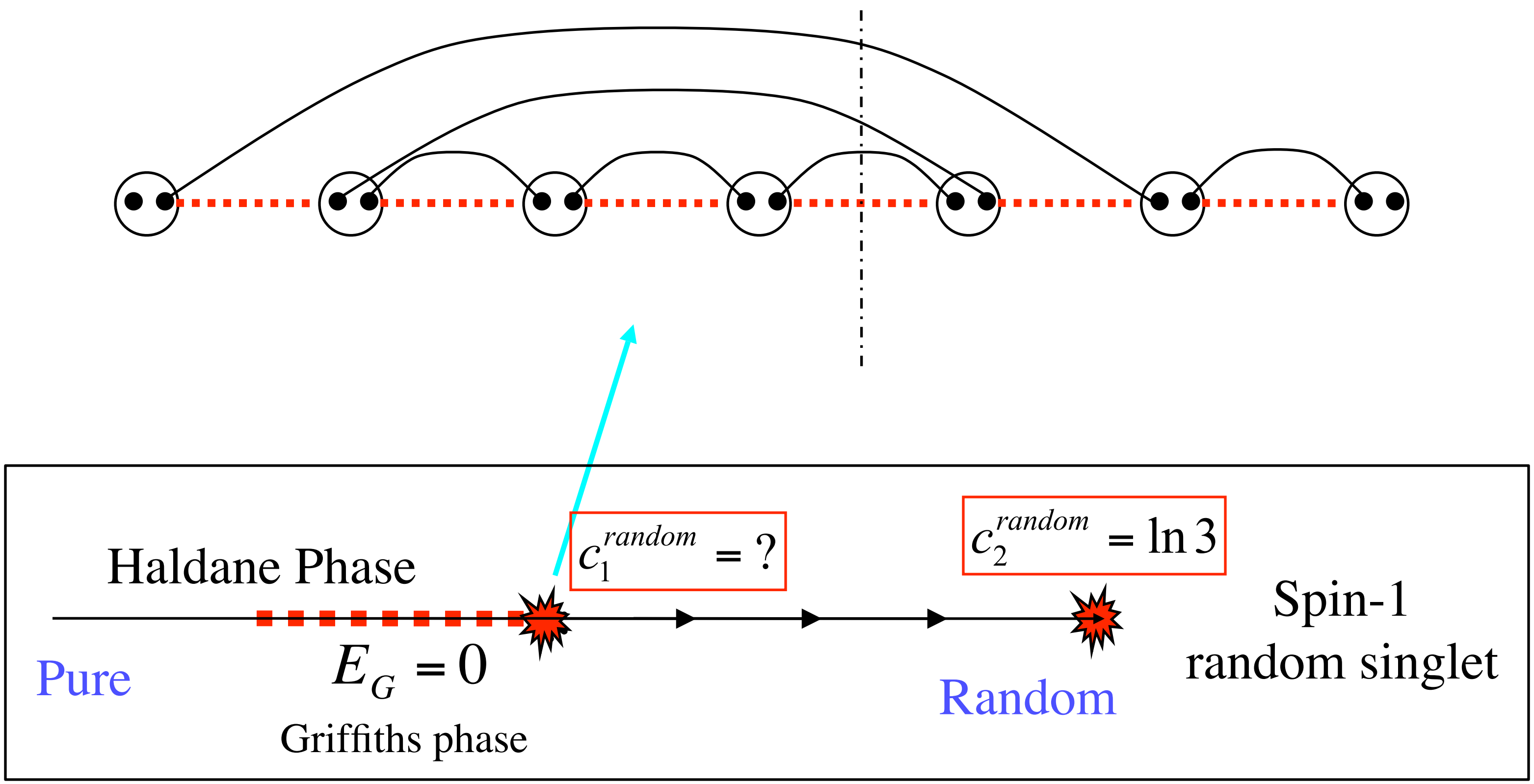
Spin-1 "SU_2 (2)" critical point

- Complex spin configurations:



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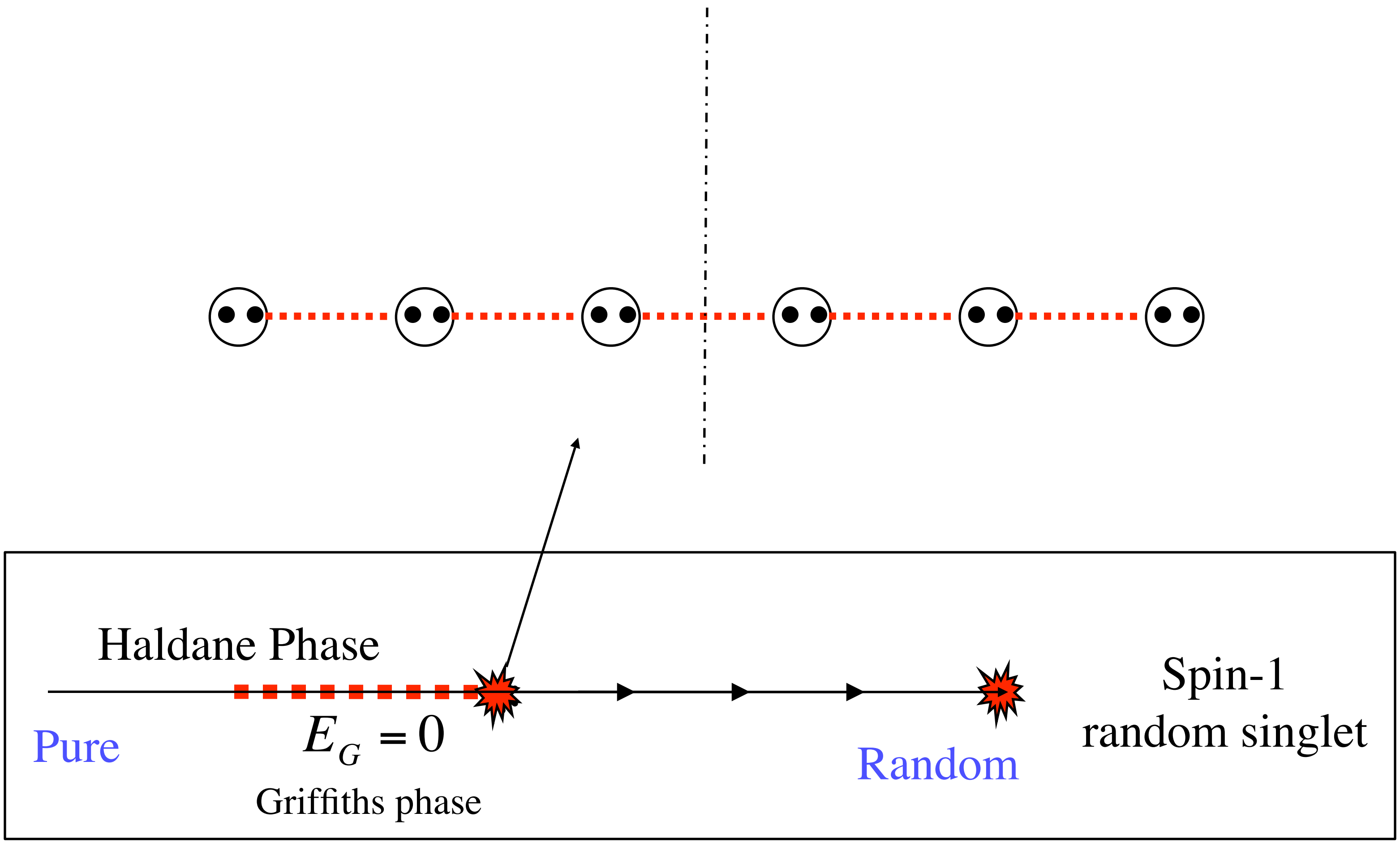

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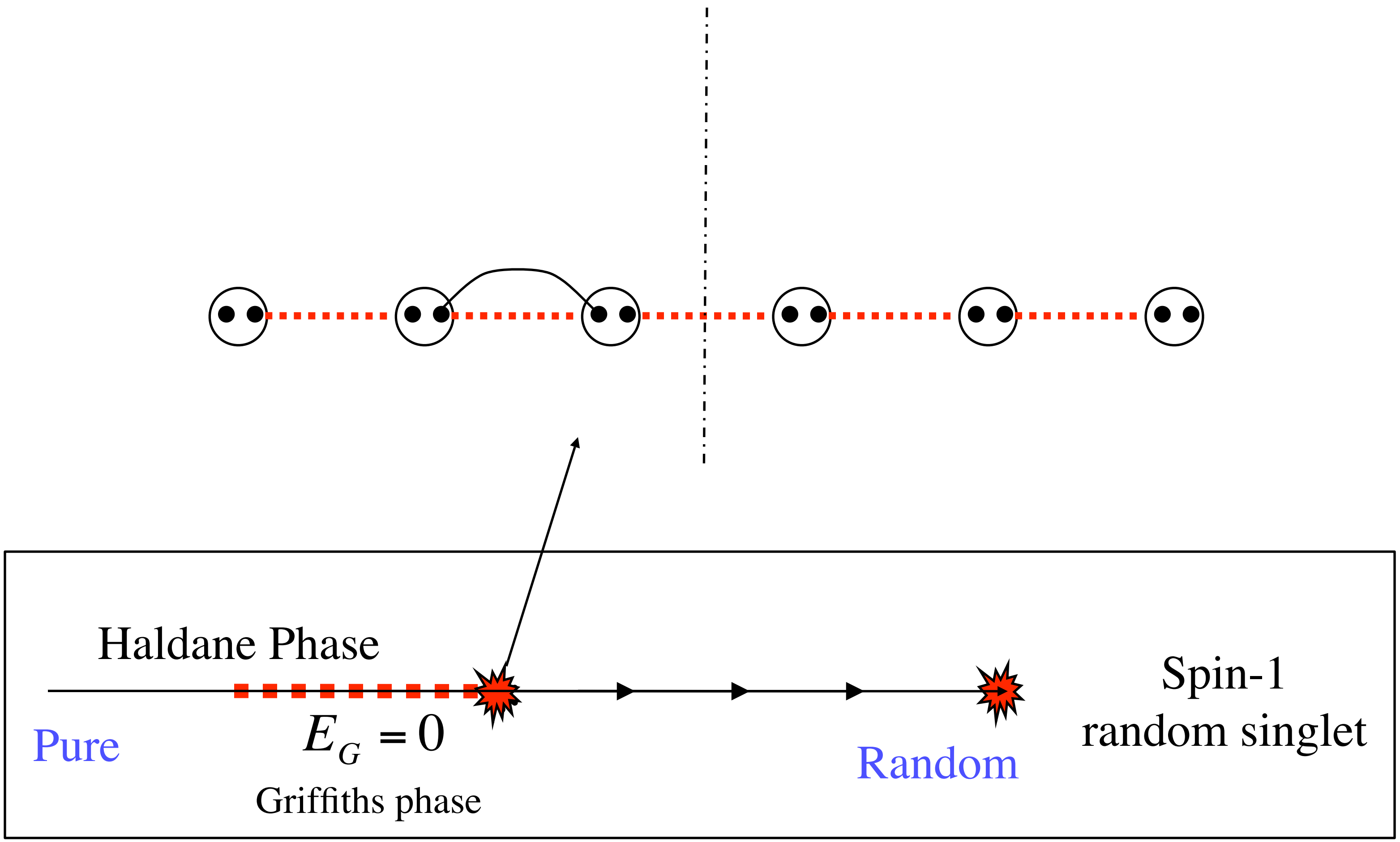

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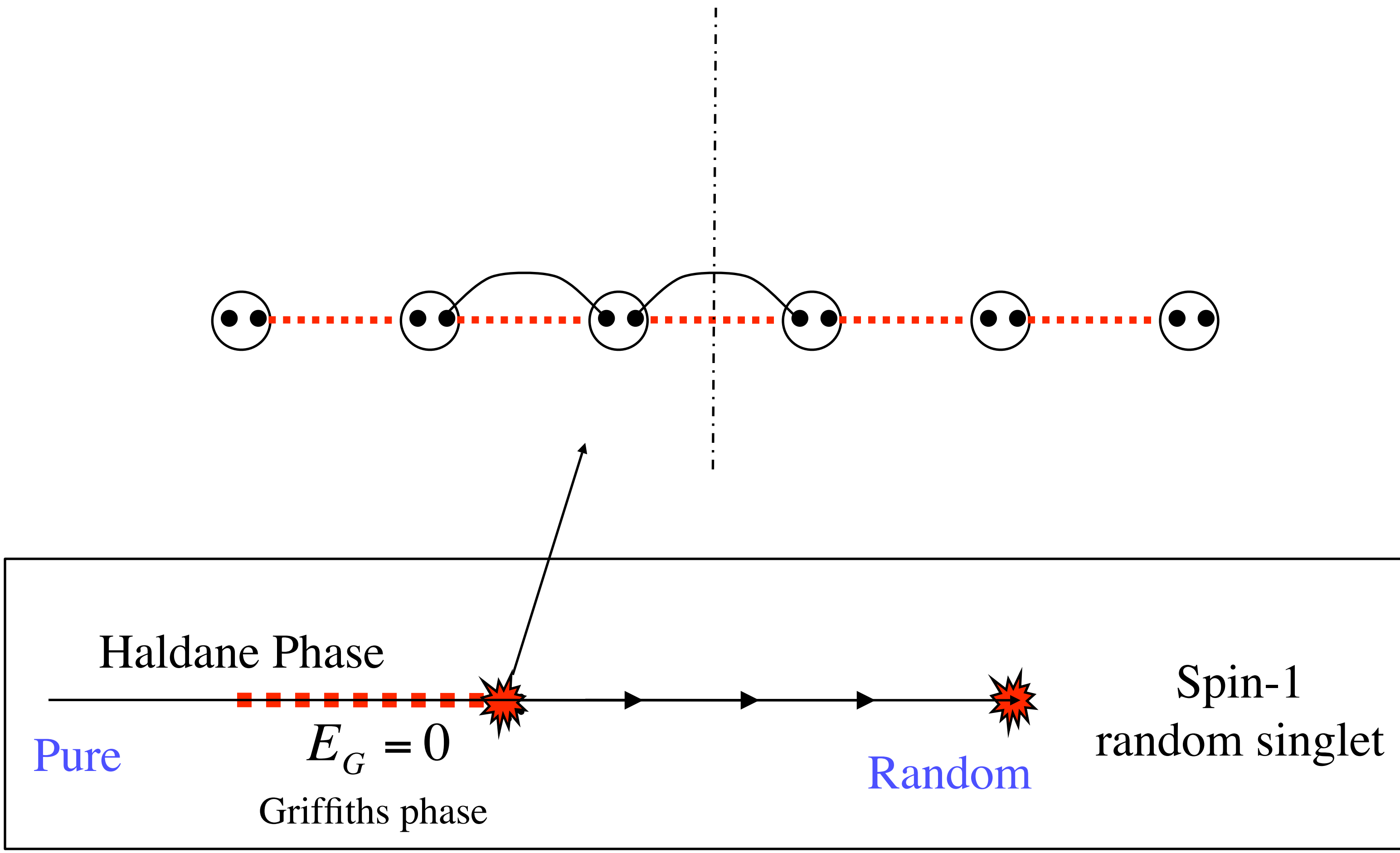

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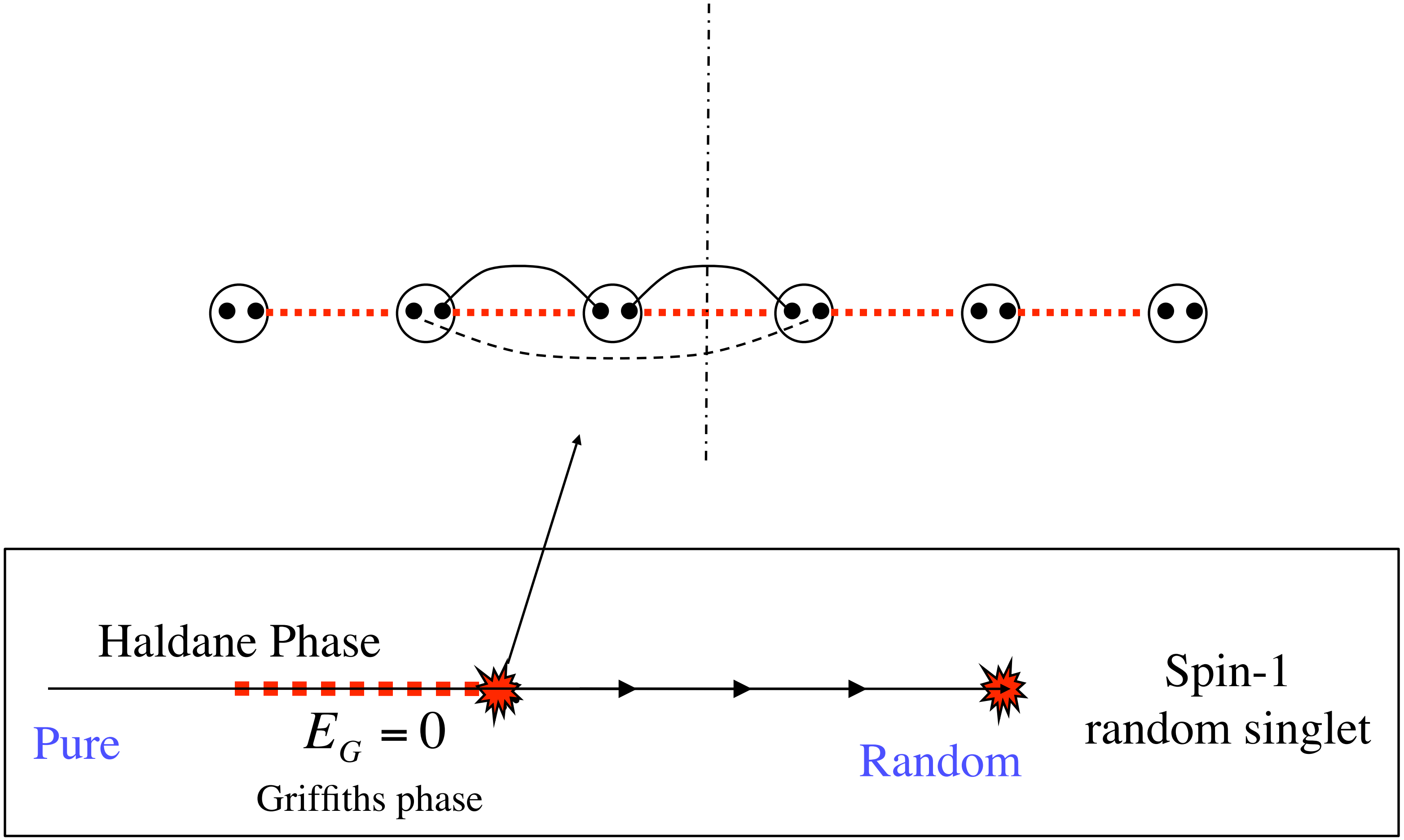

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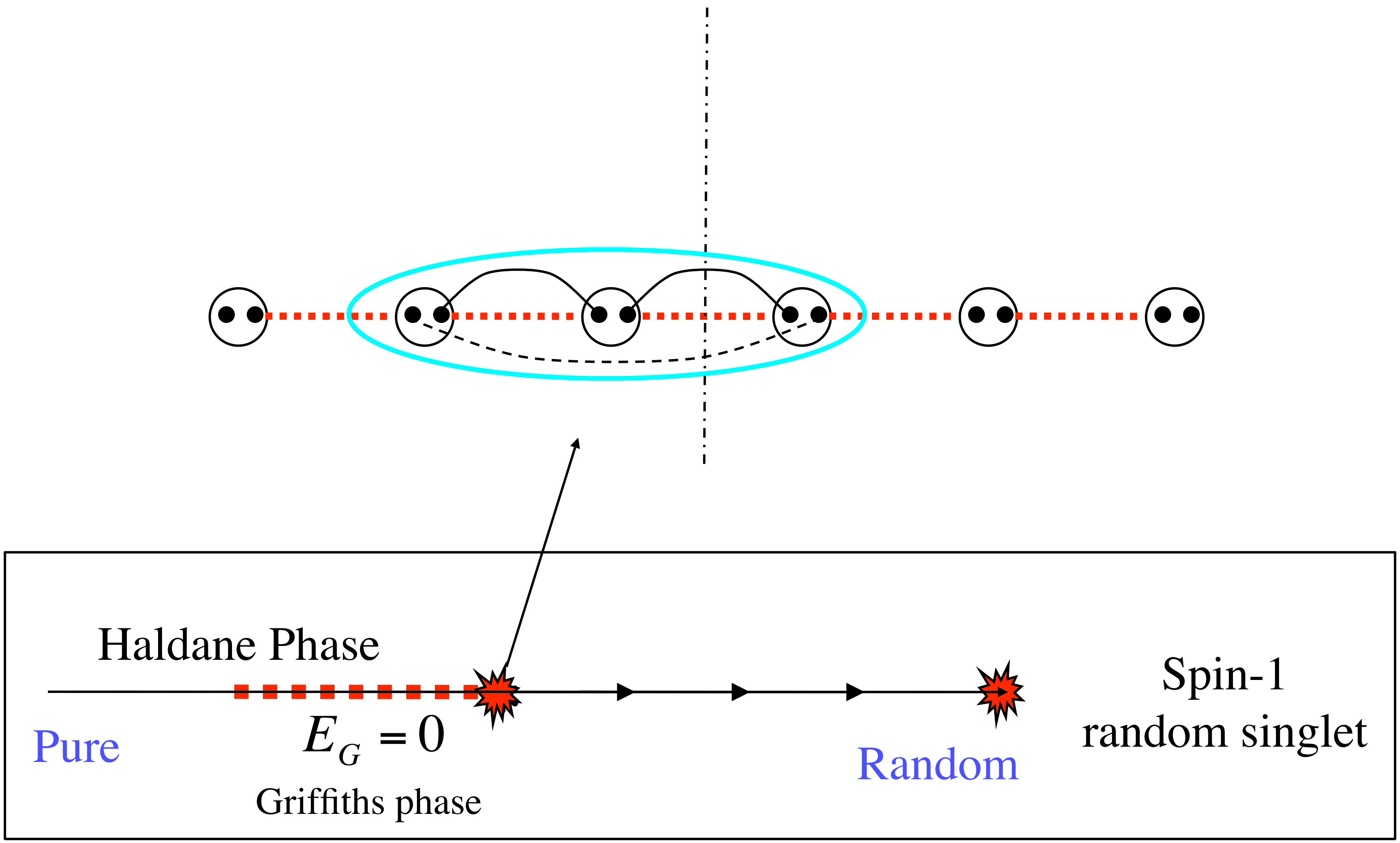

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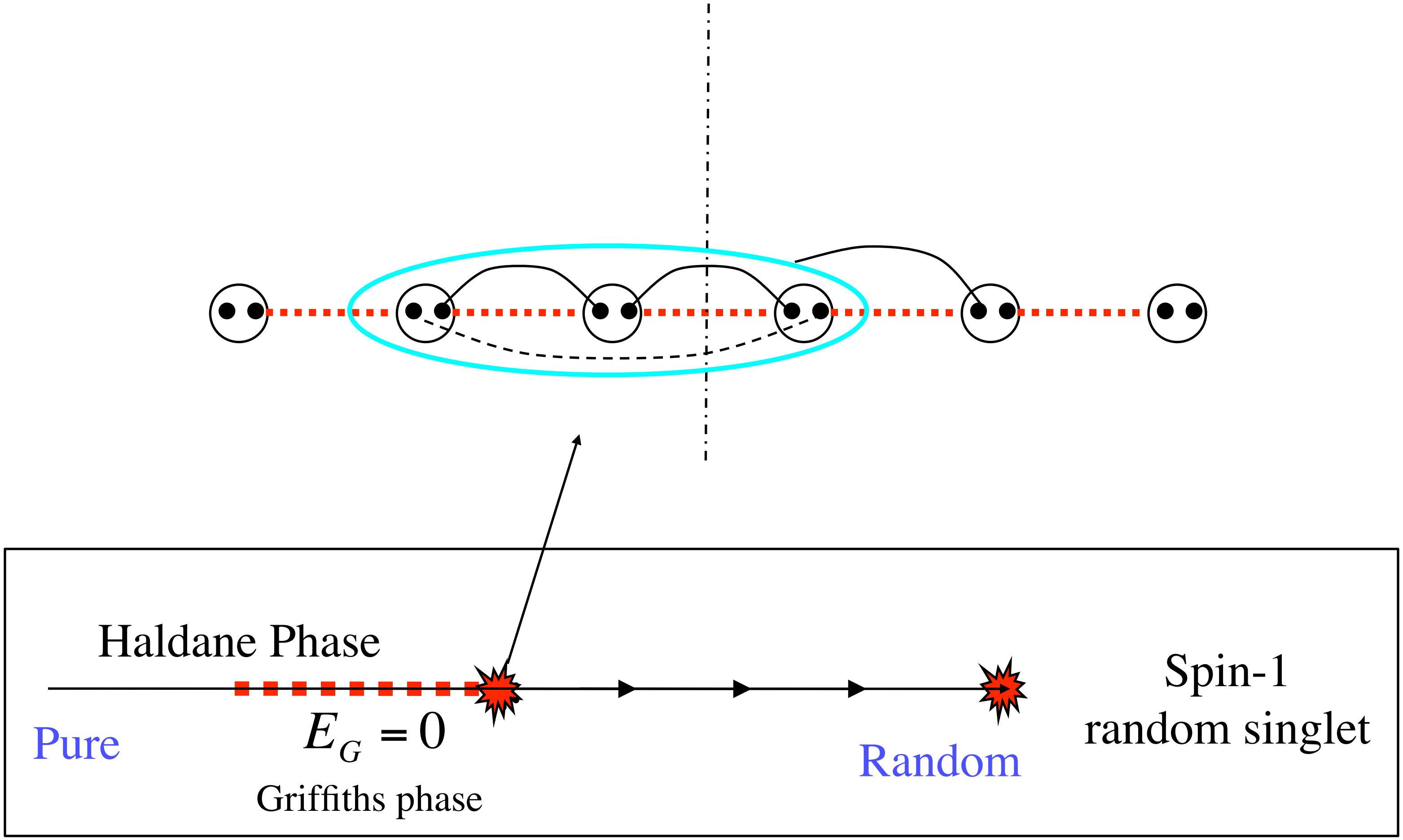

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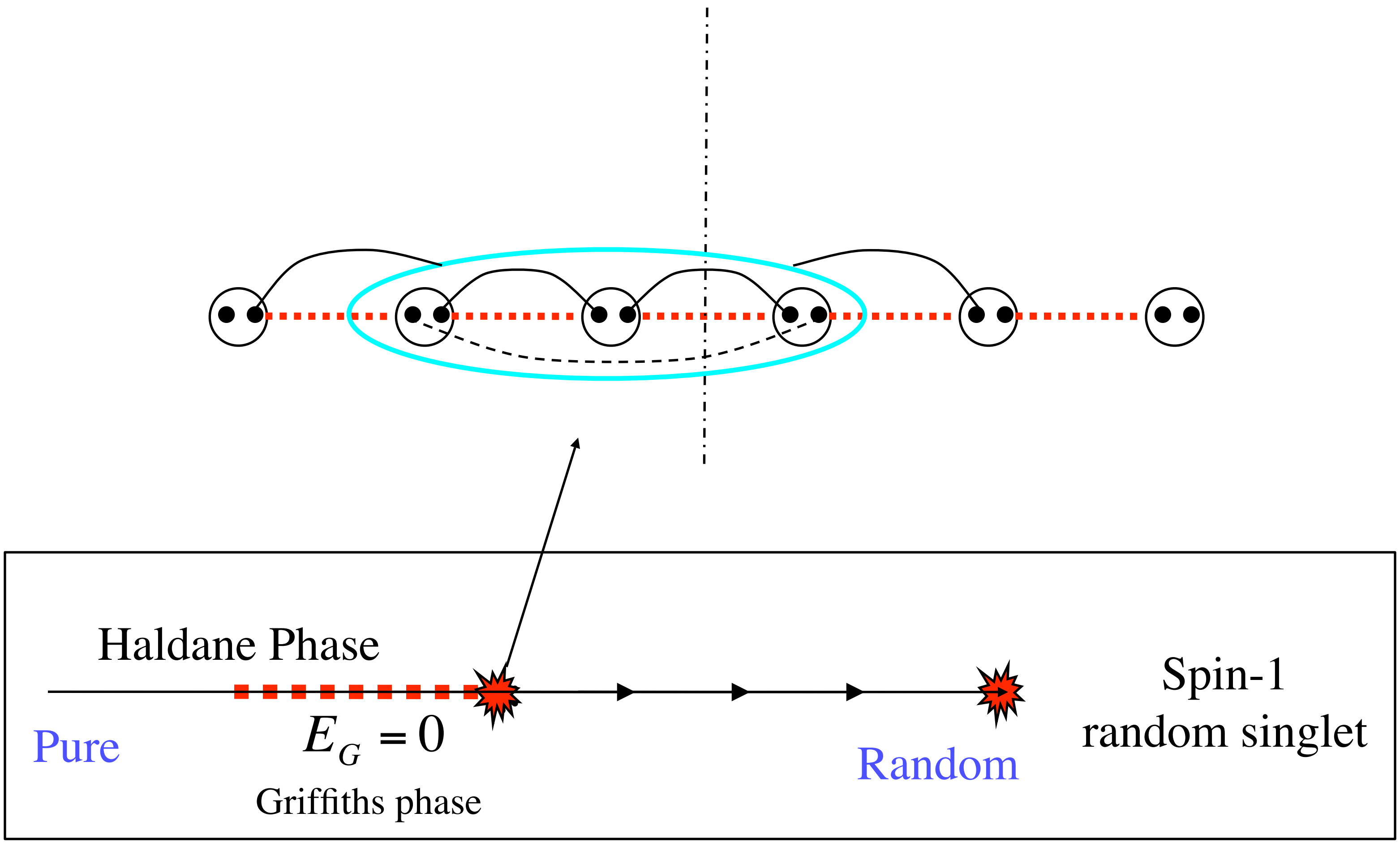

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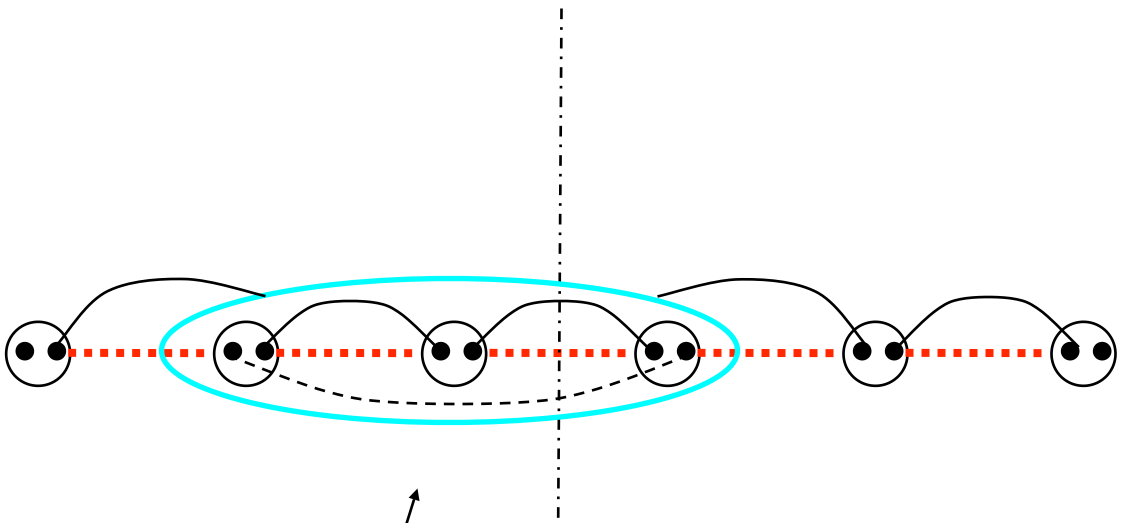

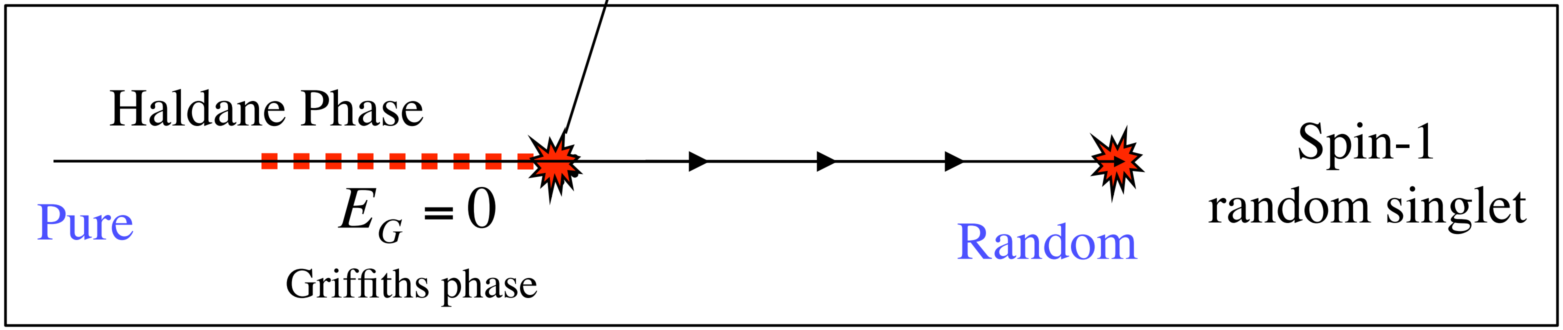

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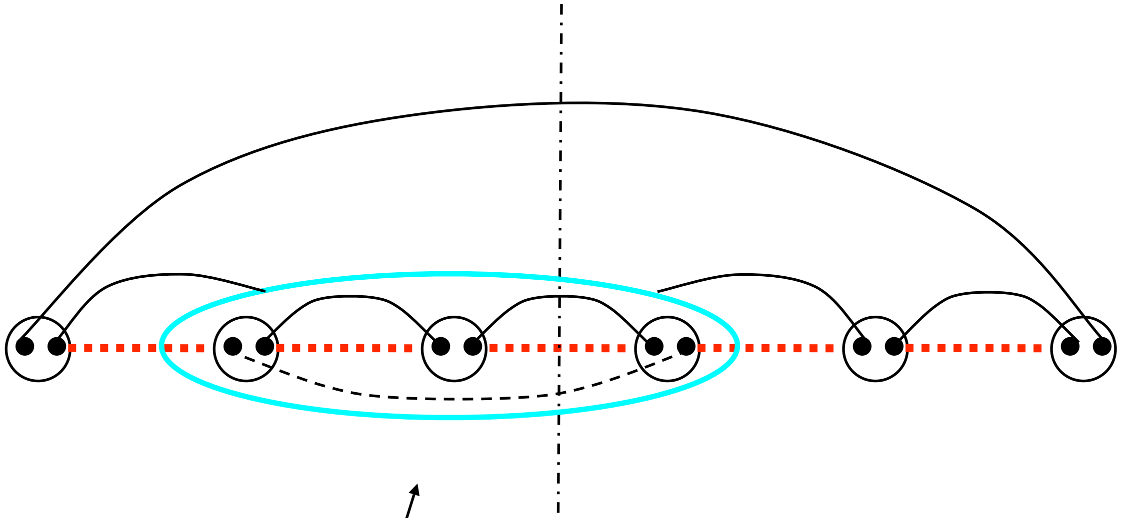

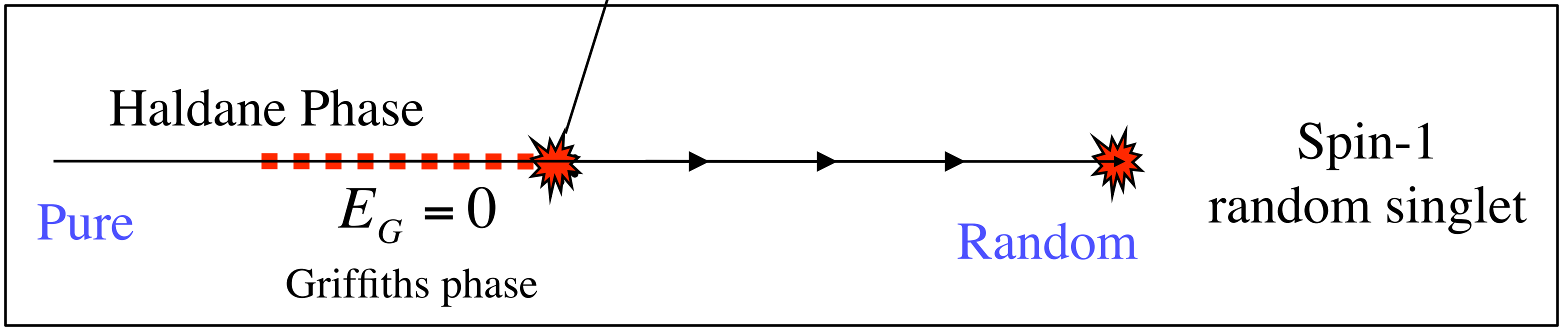

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Damle, Huse (2003).

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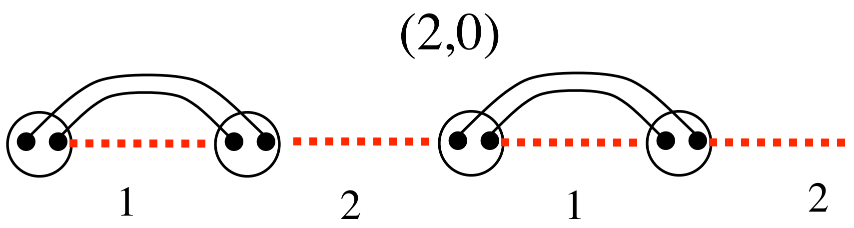

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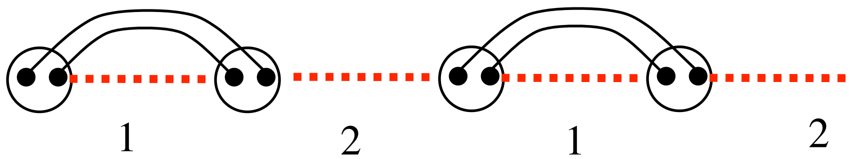

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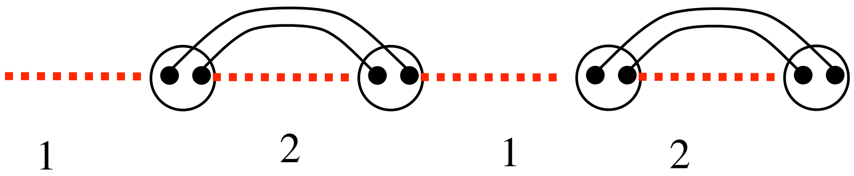

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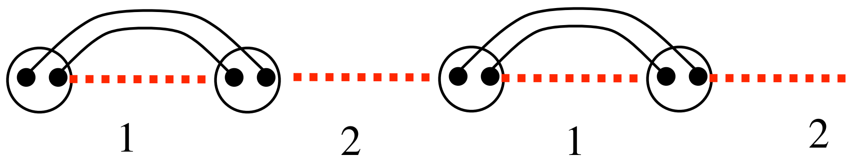

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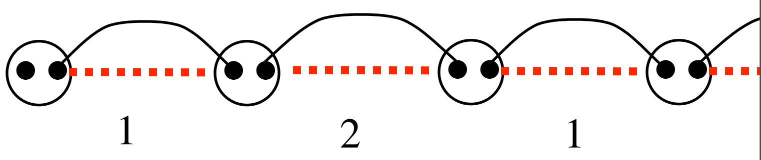

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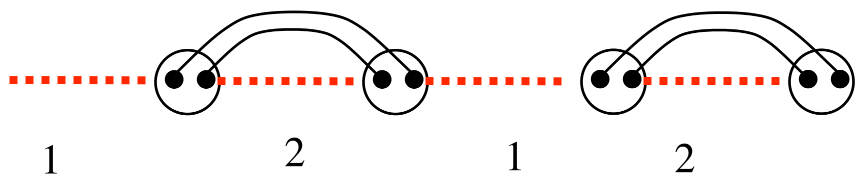

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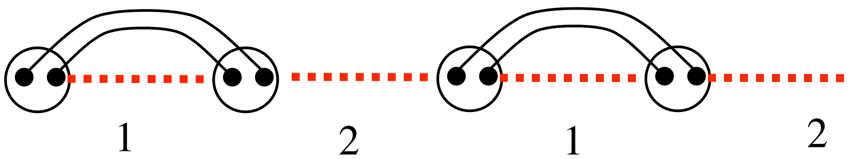

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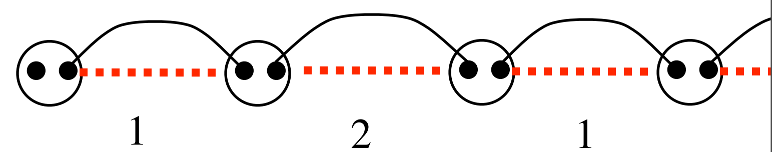

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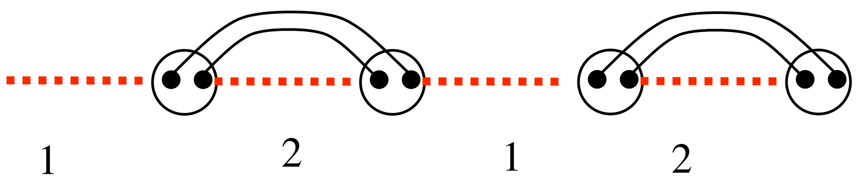

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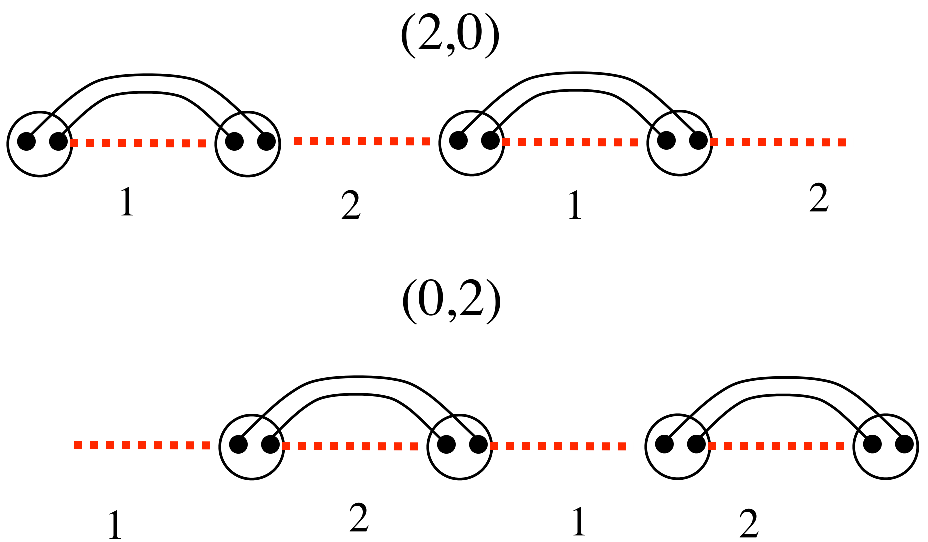

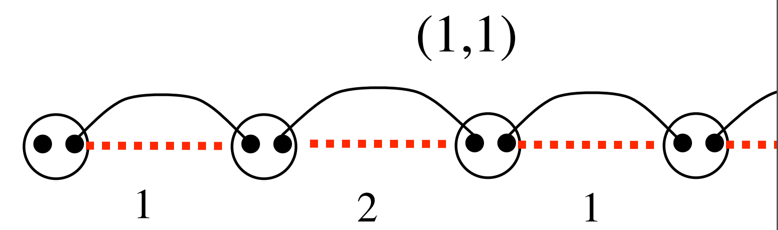

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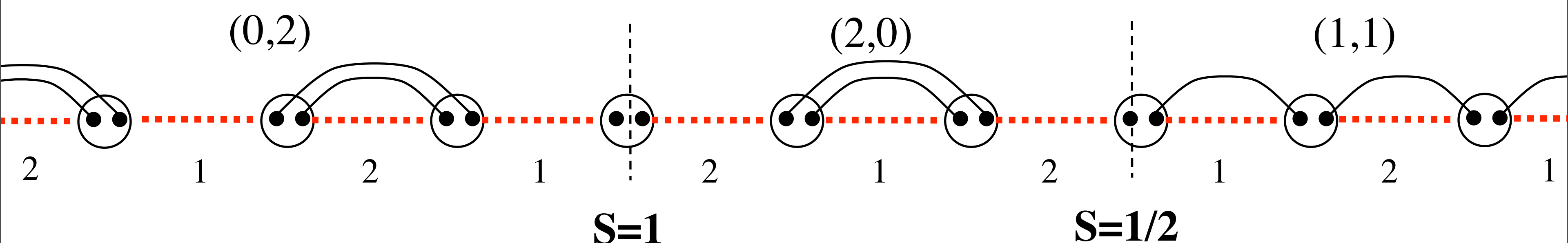


Spin-1 "$SU_2(2)$" critical point – Results from domain theory

- At criticality, all domain appear with equal probability.

- Energy-length scaling:

$$\Gamma = \ln \frac{1}{E} \sim L^{\psi}$$

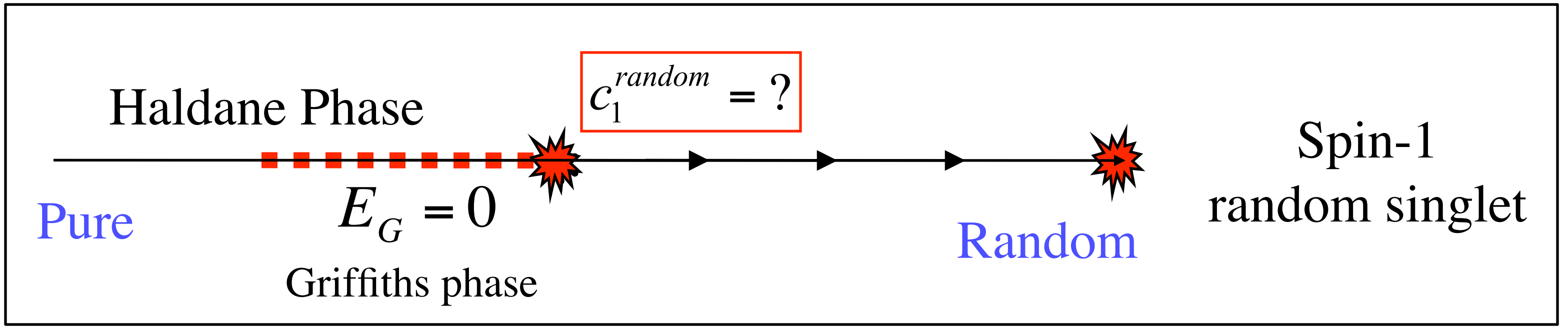
$$\psi = 1/3$$

- Fixed-point distribution:

$$\rho(J) = 1 / J^{1-\chi/\Gamma}$$

$$\chi = 2$$

Compare to $\chi = 1$ for random-singlet phase



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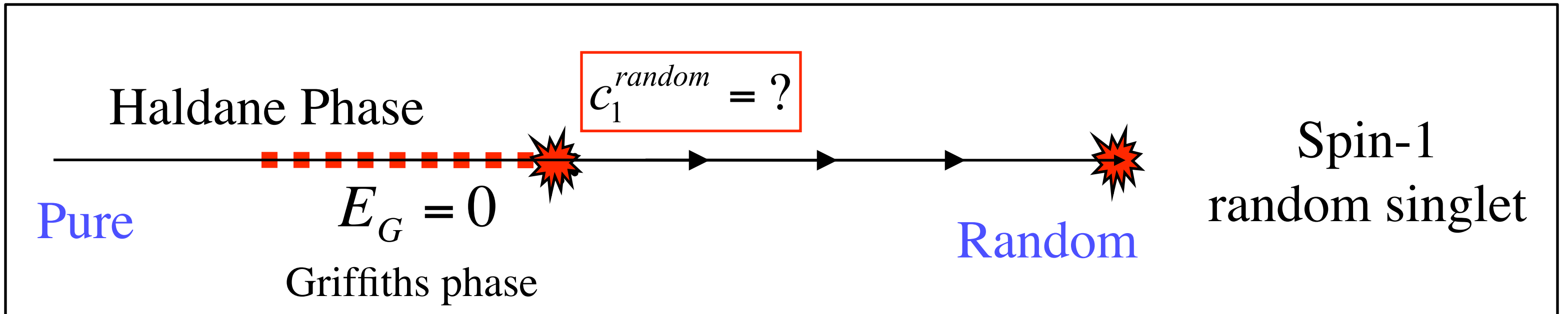
In general:

$$\psi = 1/(2S+1)$$

$$\chi = 2S$$



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Spin-1 Calculation

$$E_L = 2\frac{E_{singlet}}{\Delta l_{singlet}} \times \Delta l_L$$

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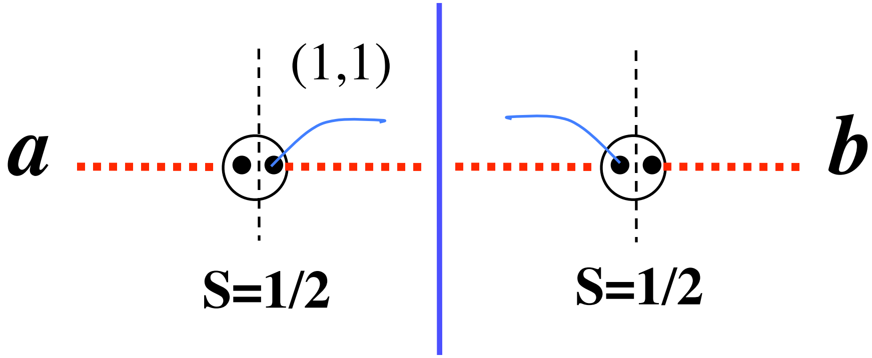

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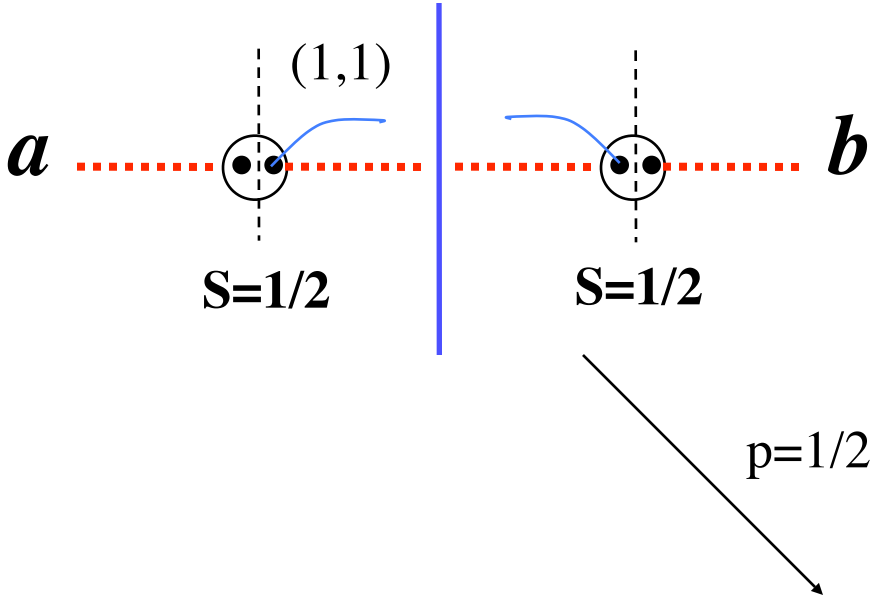

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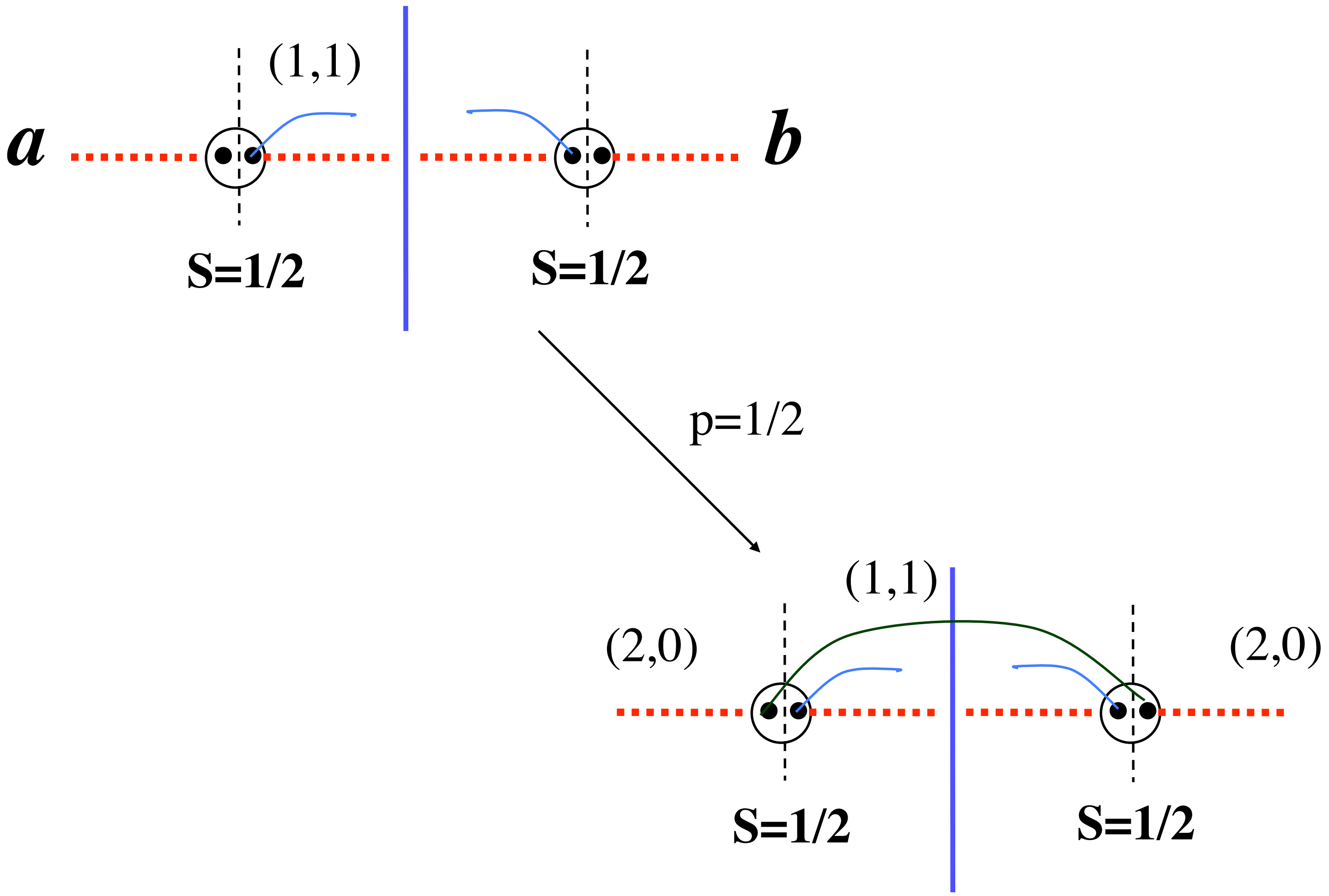

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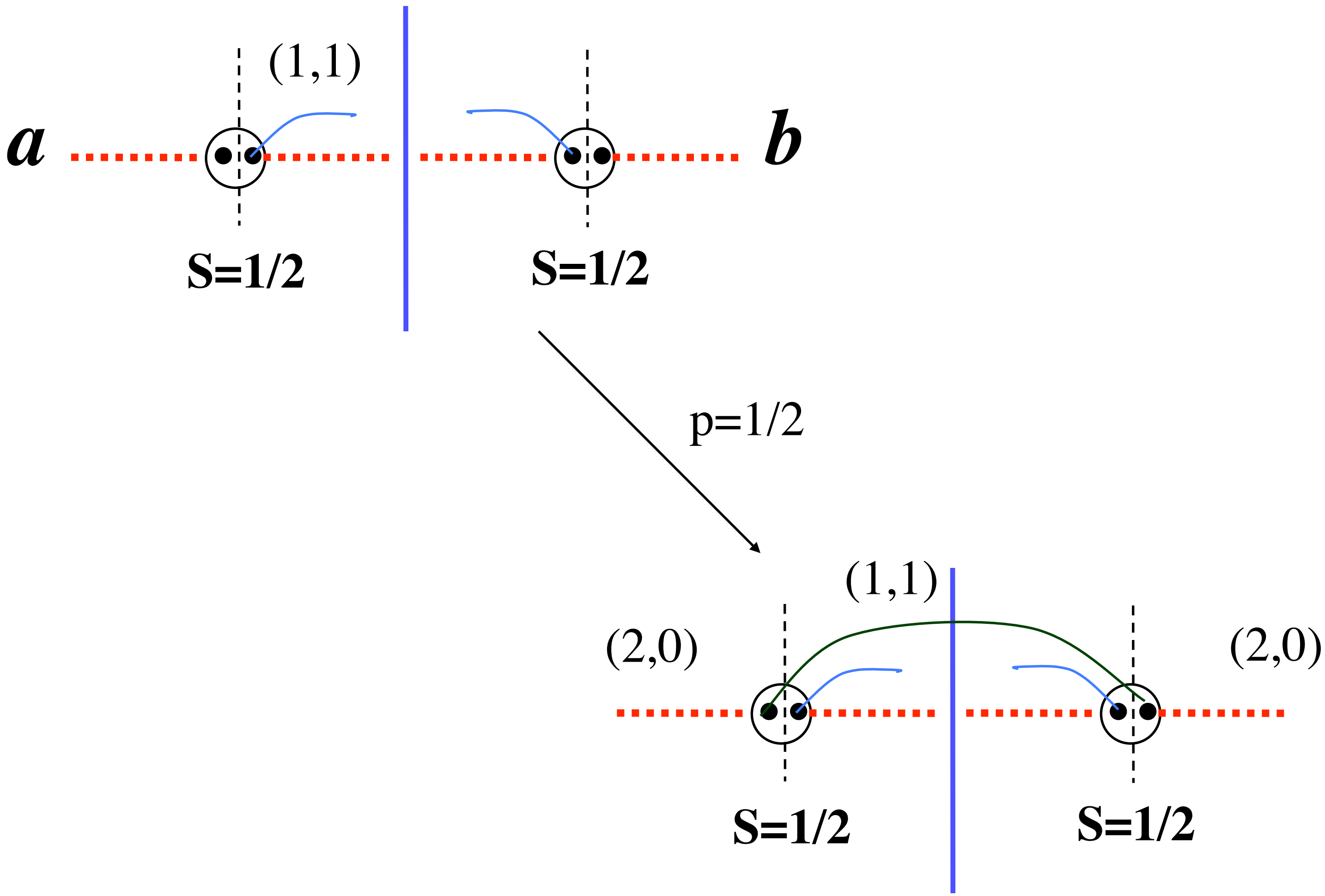

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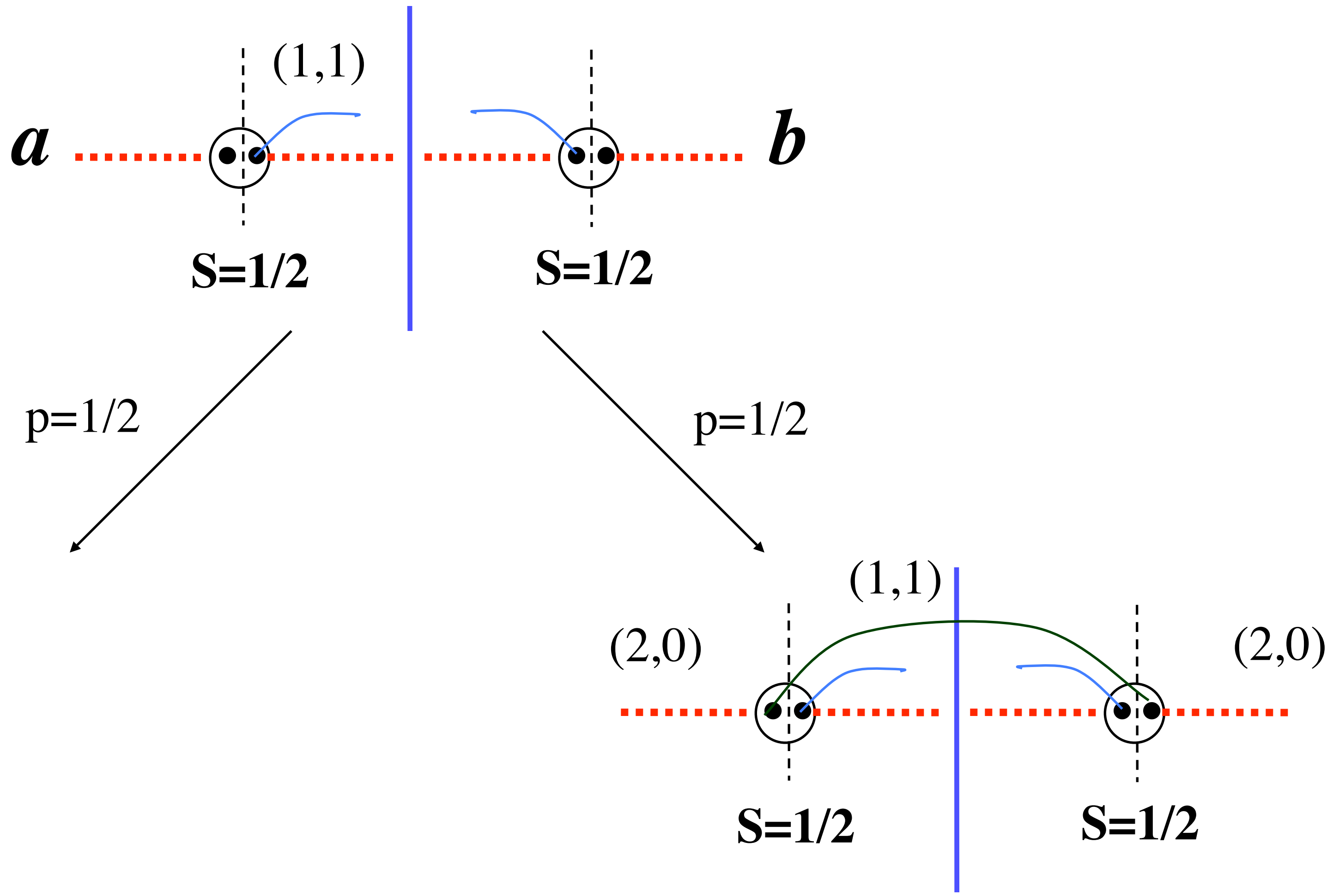

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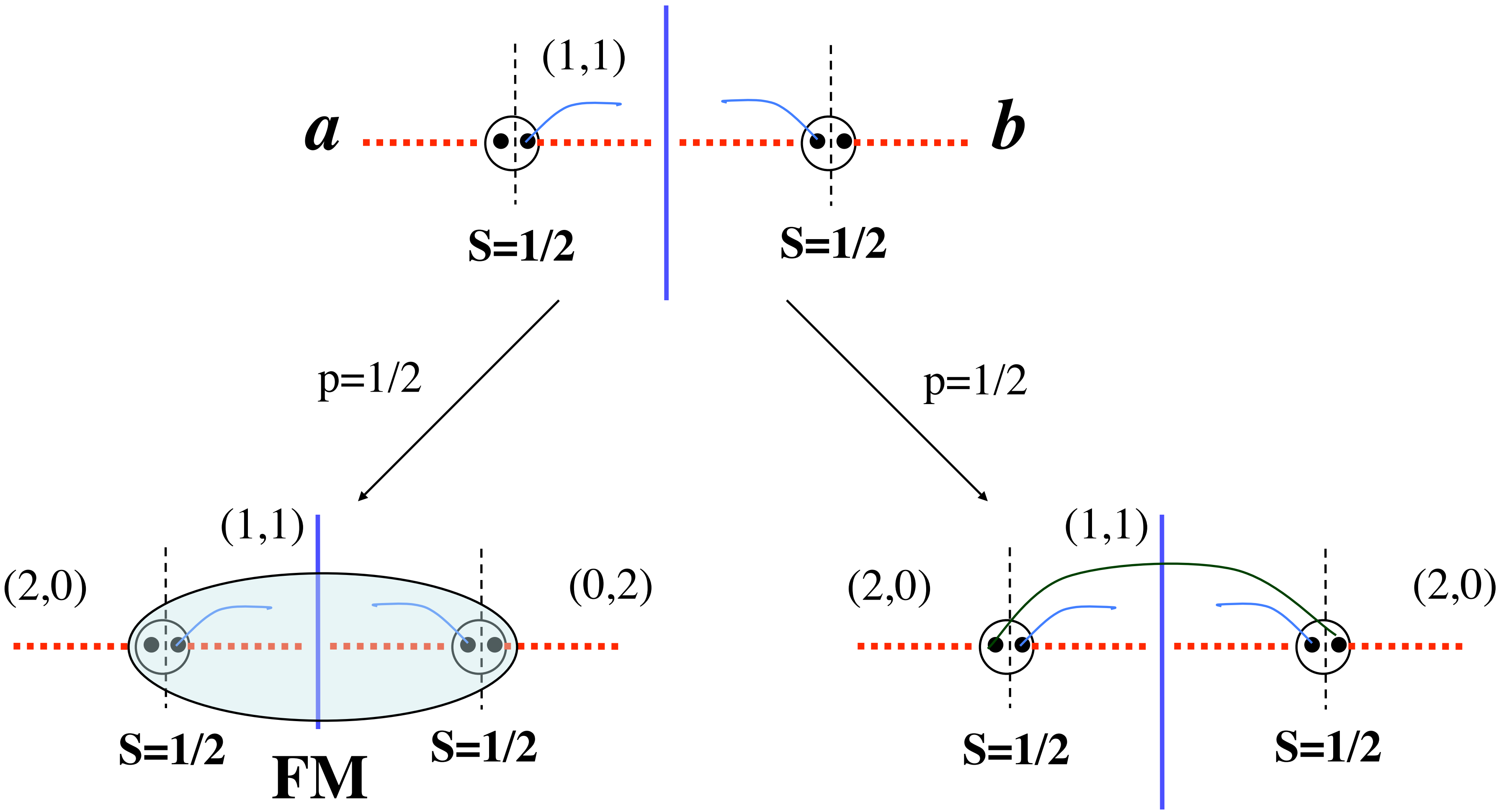

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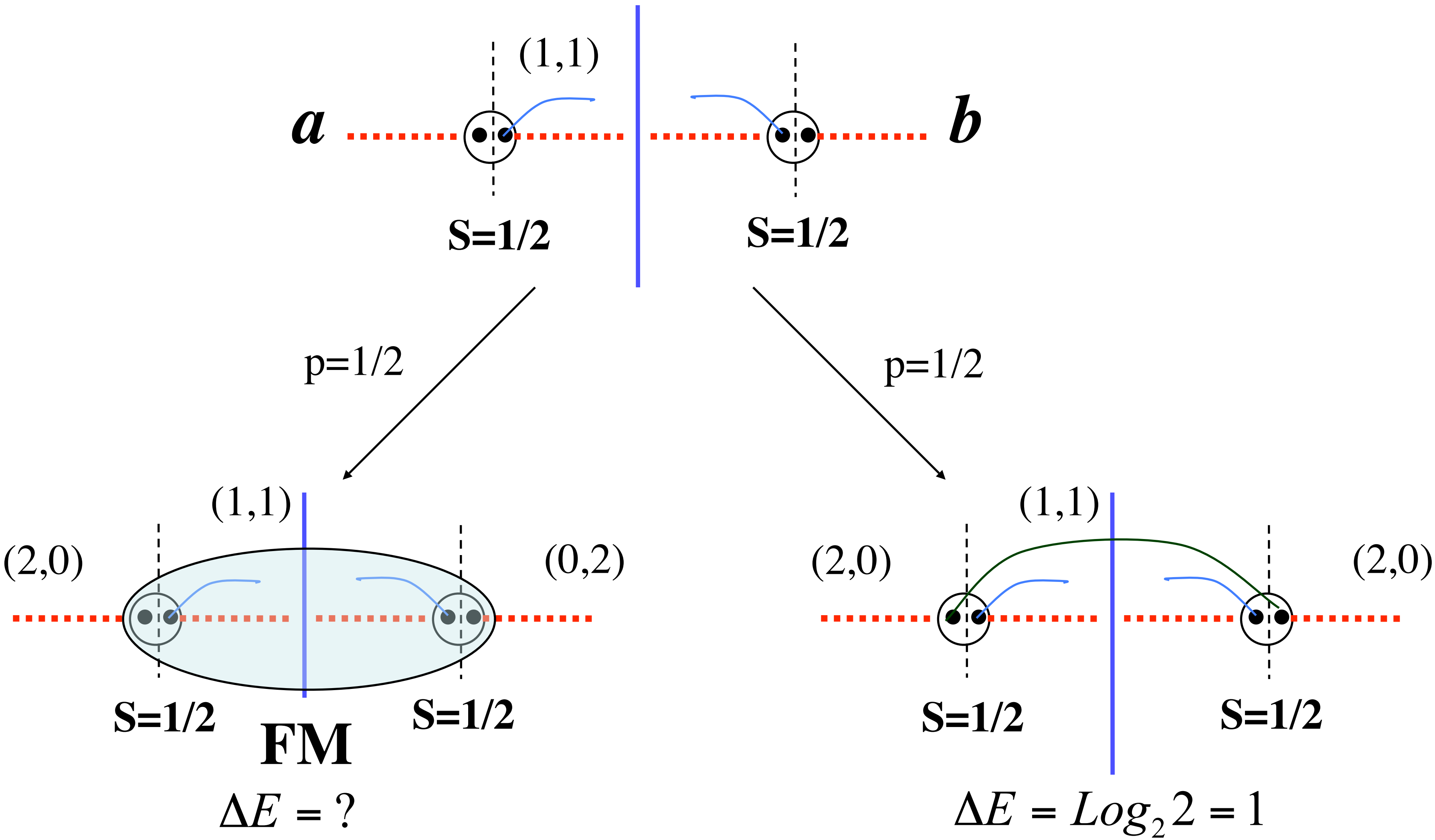

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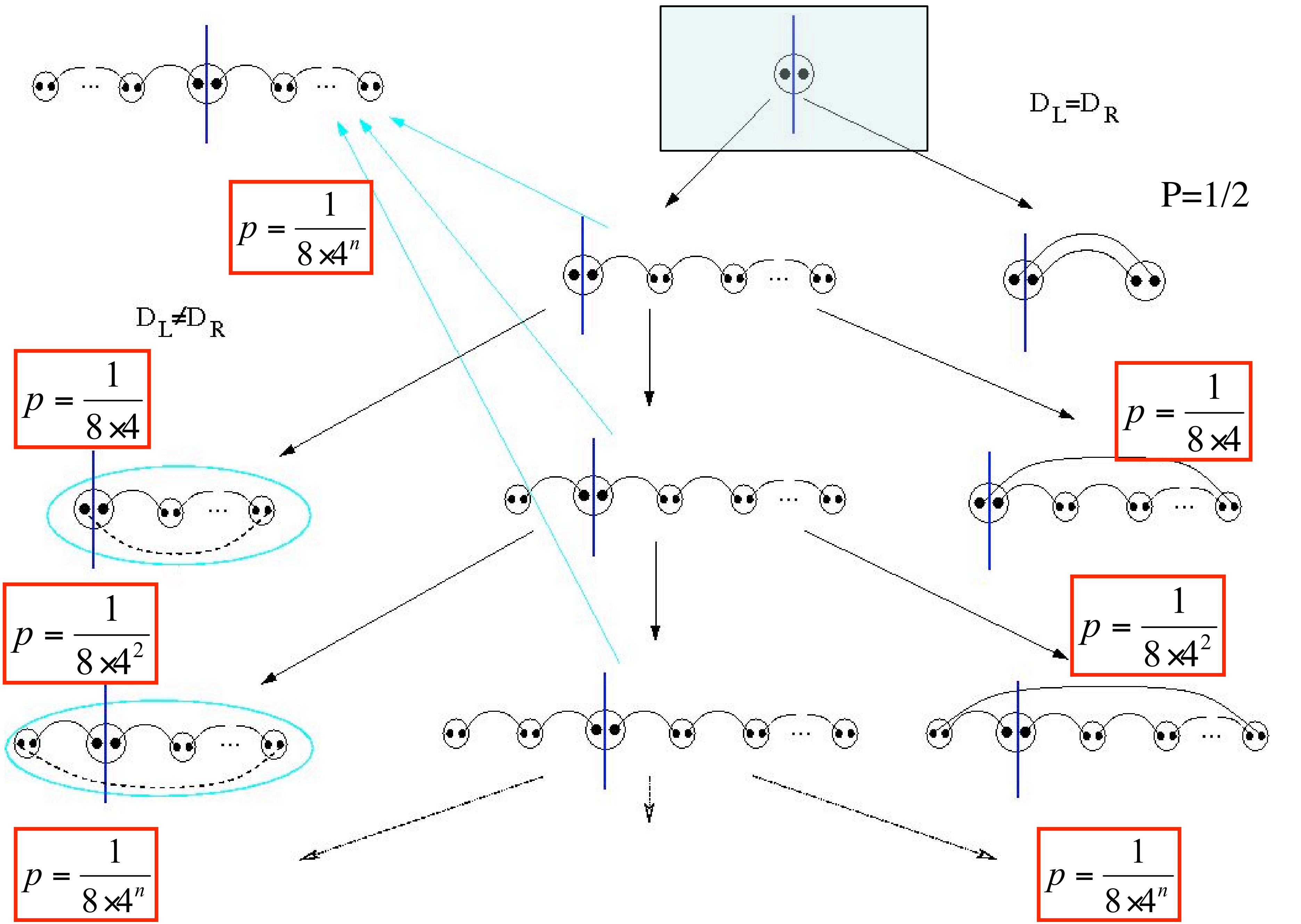

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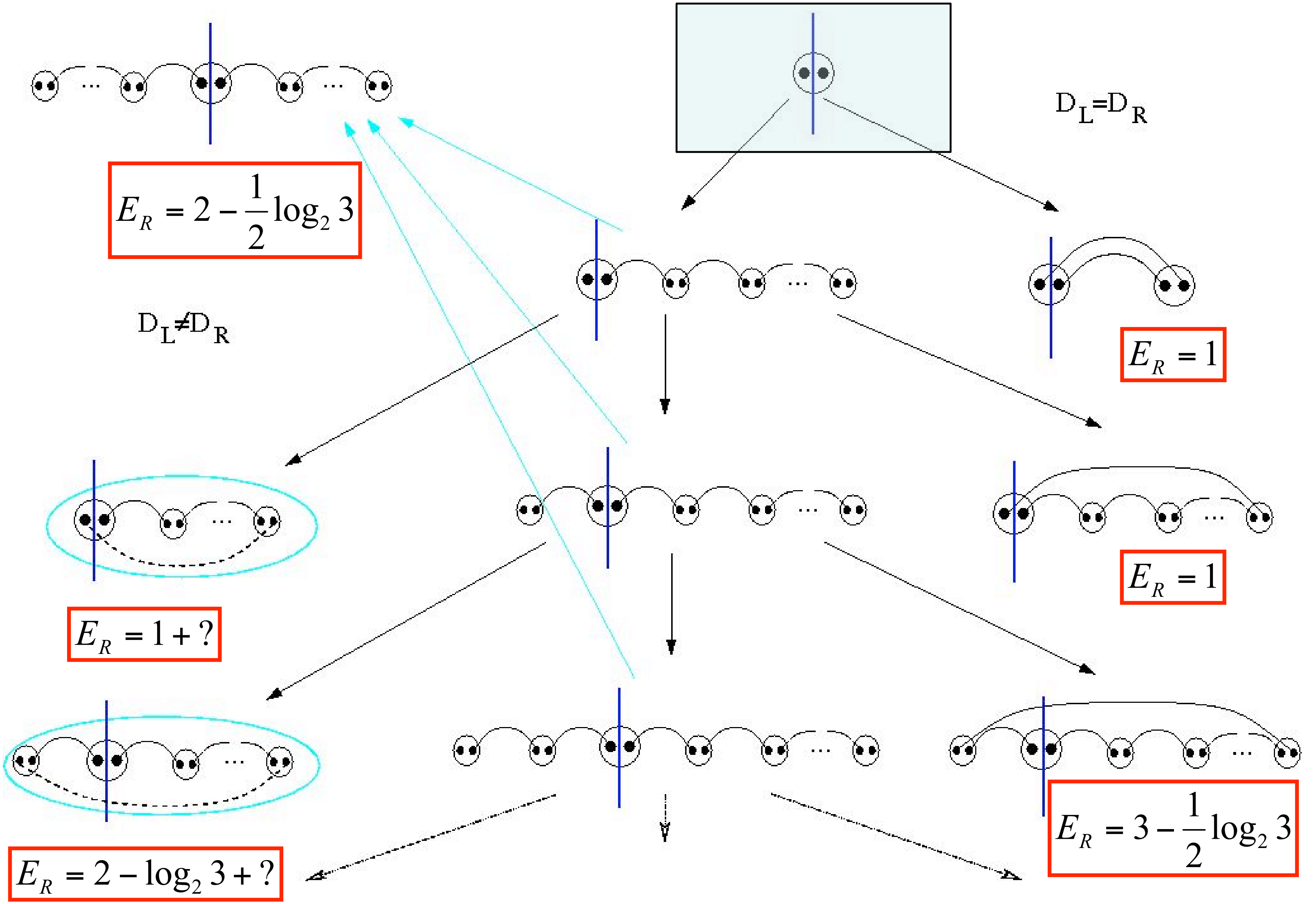

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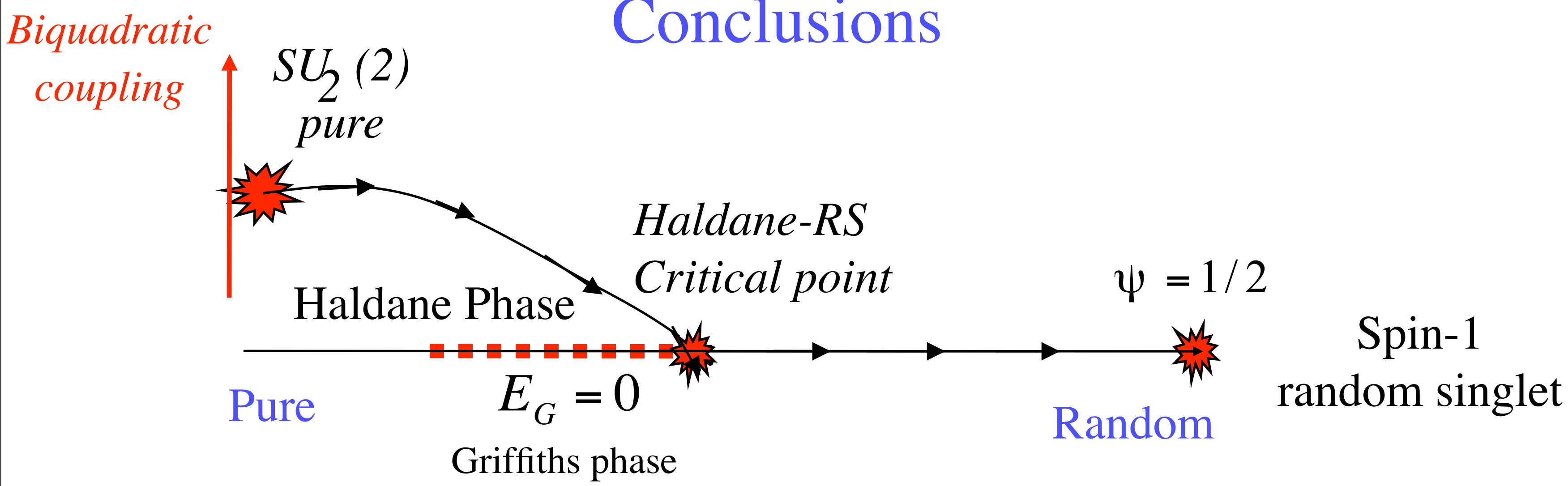
Conclusions



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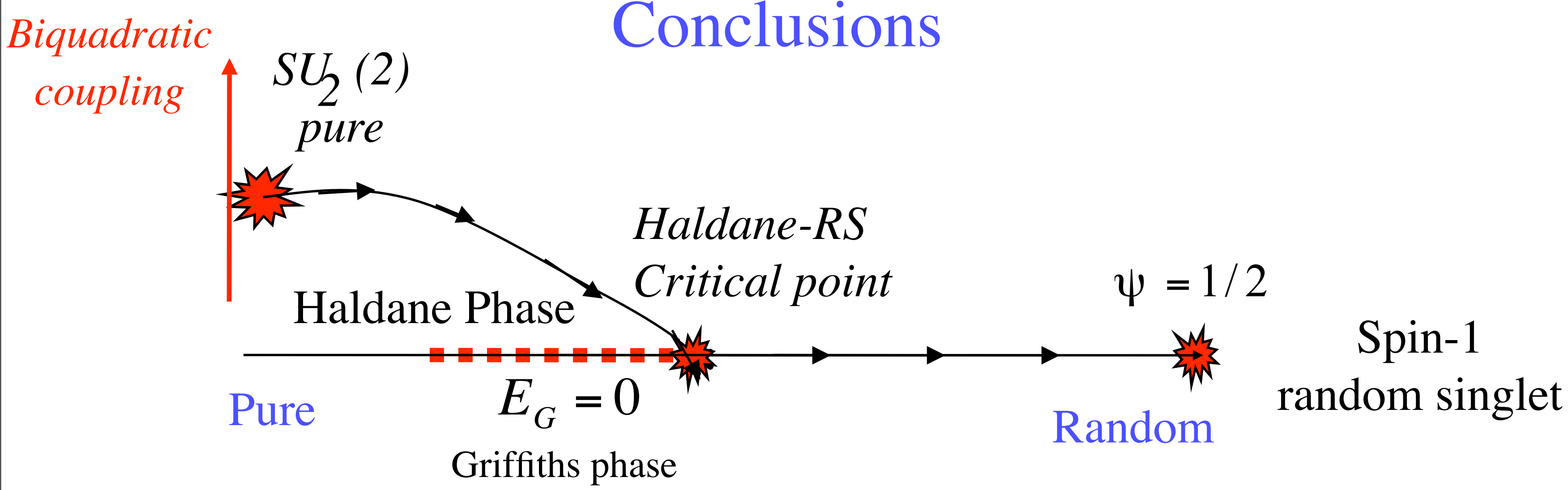

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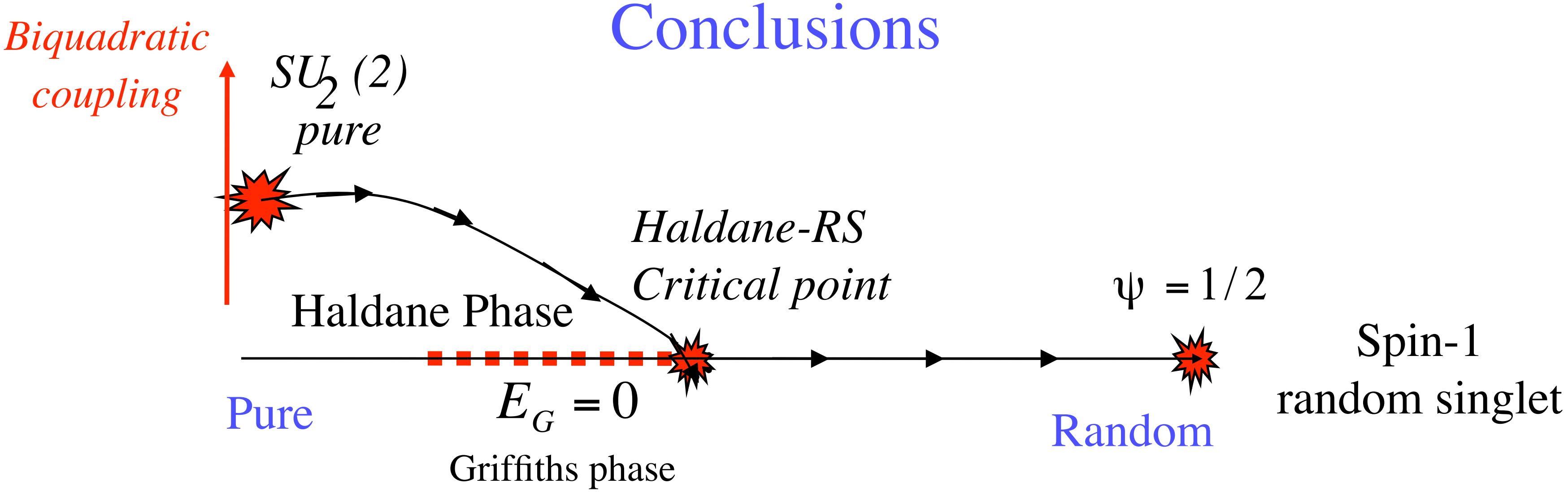

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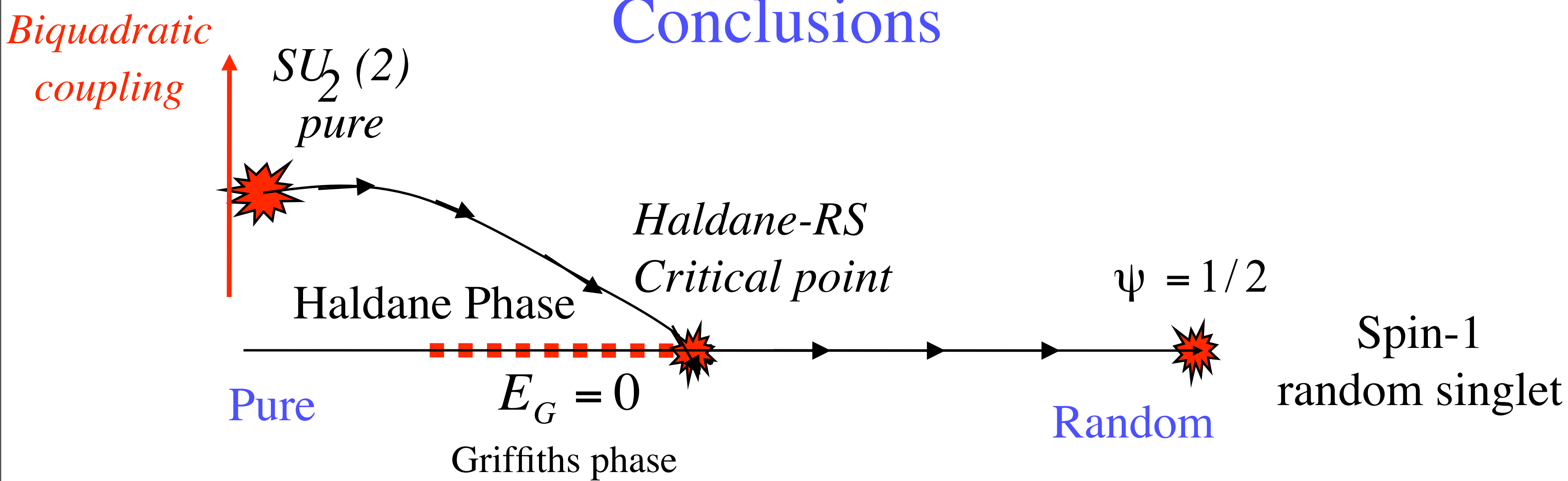

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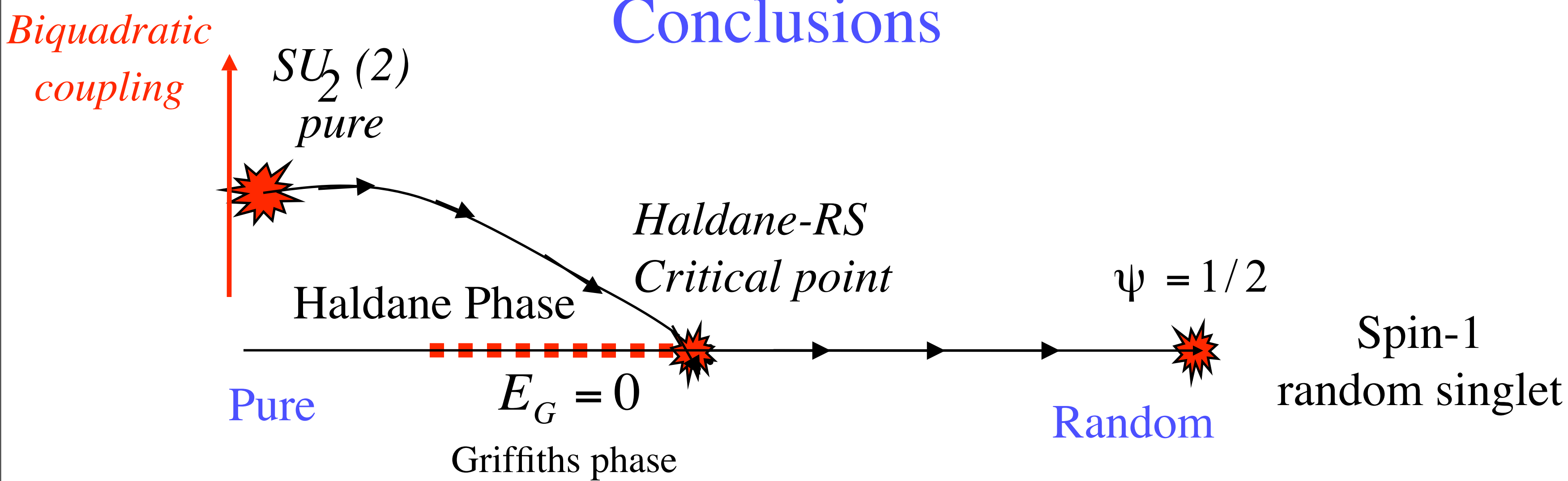

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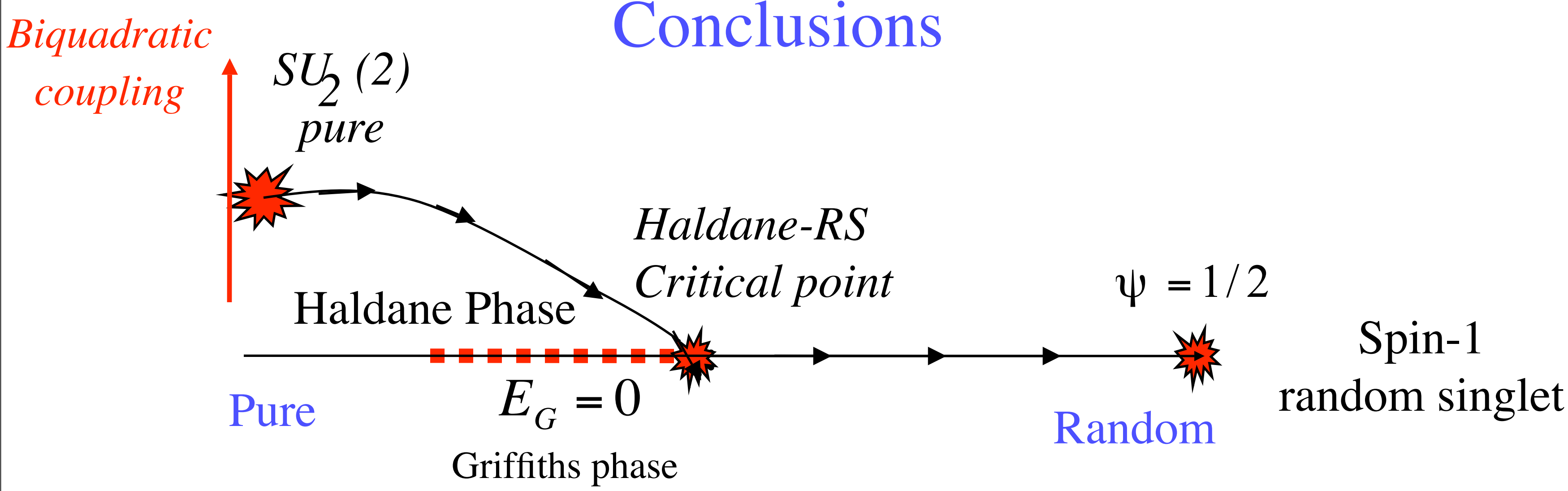

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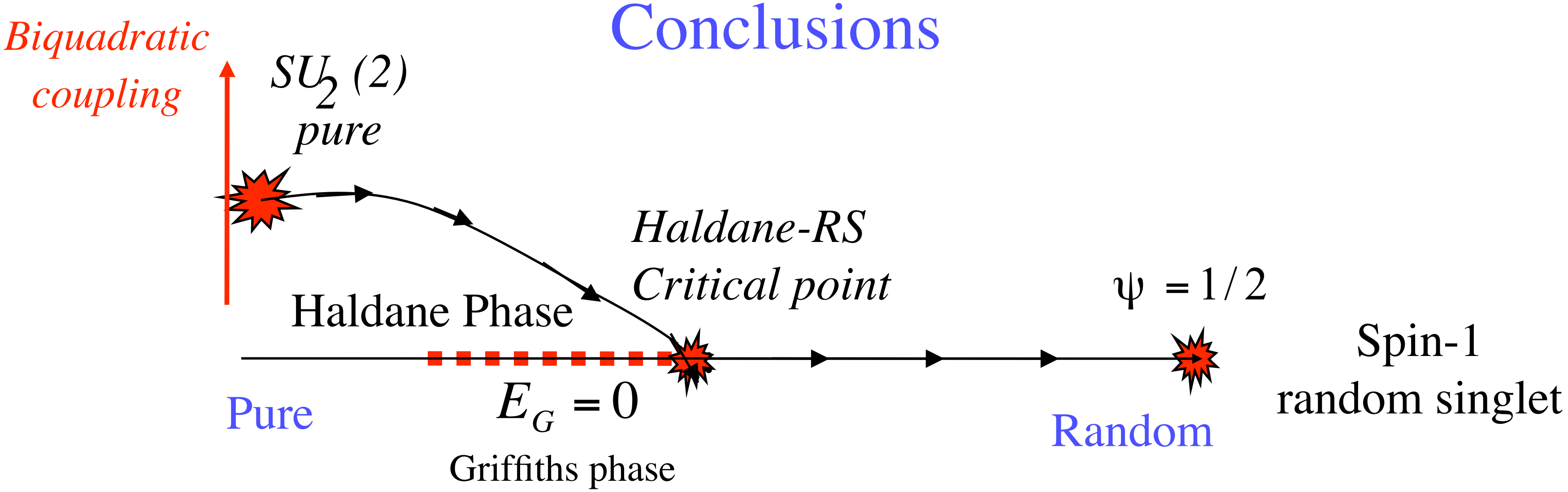

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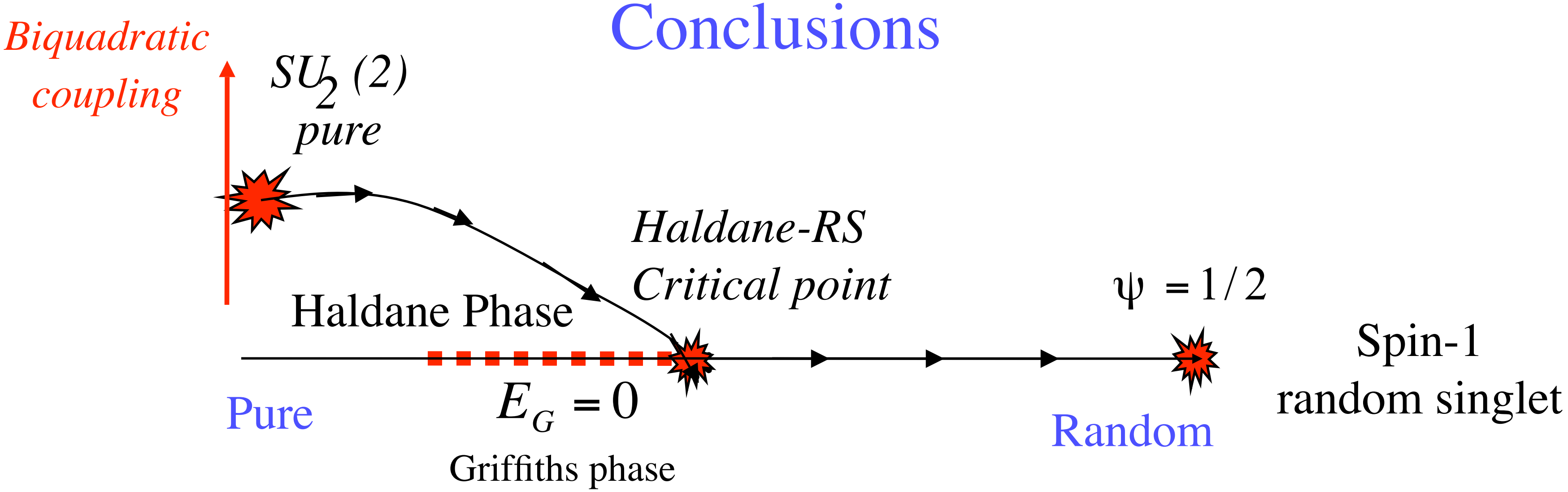

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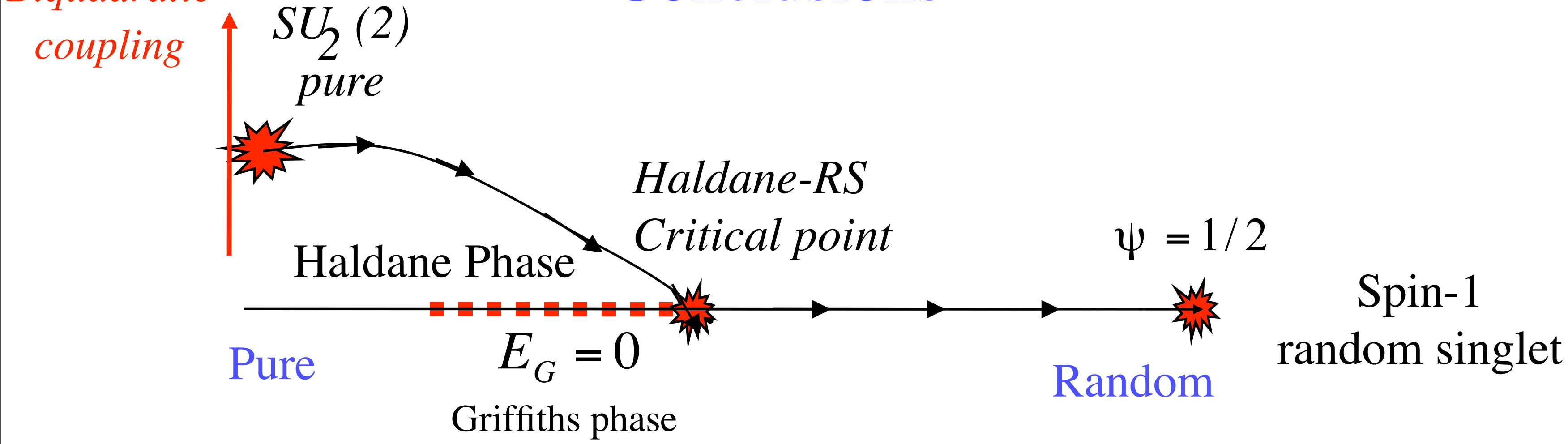

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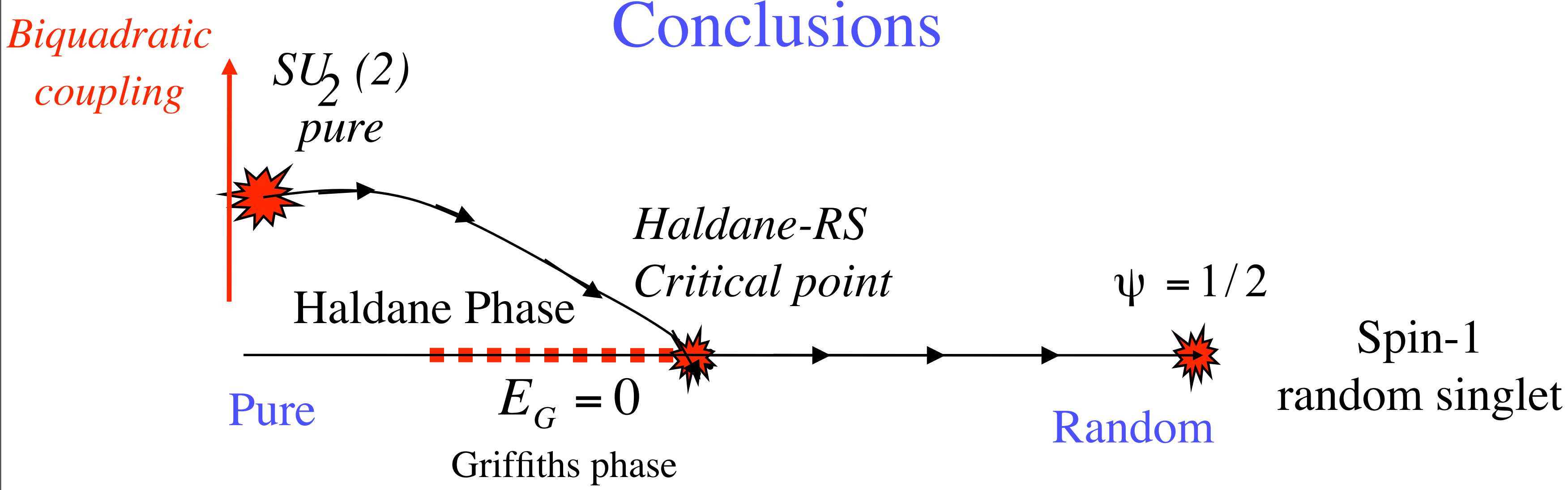

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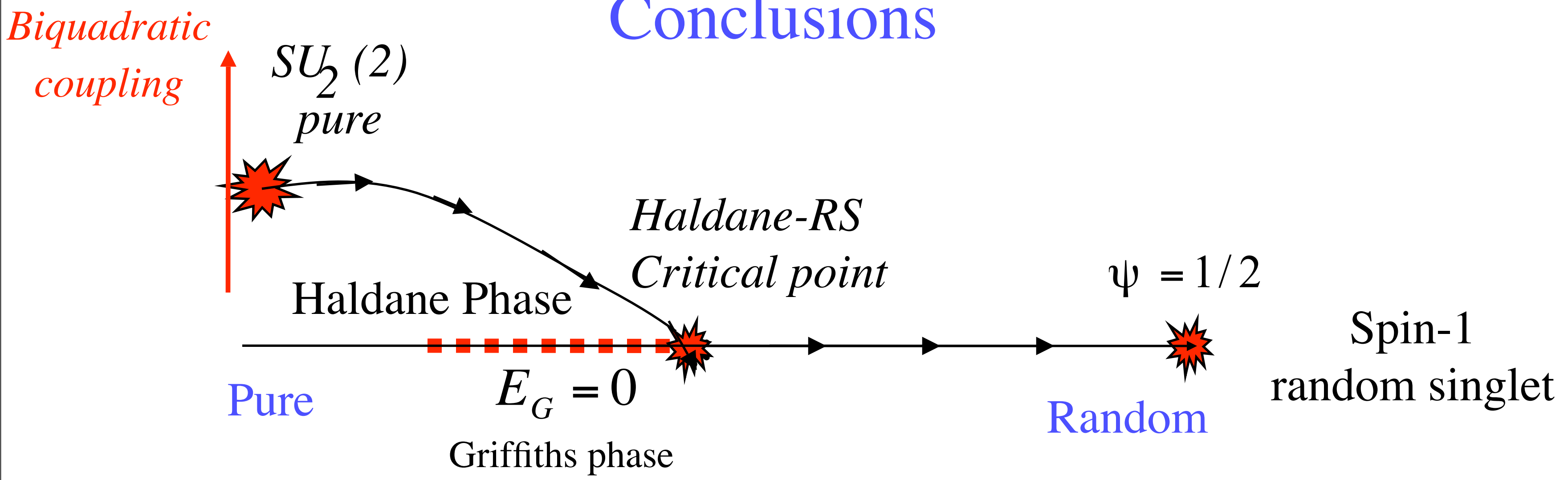

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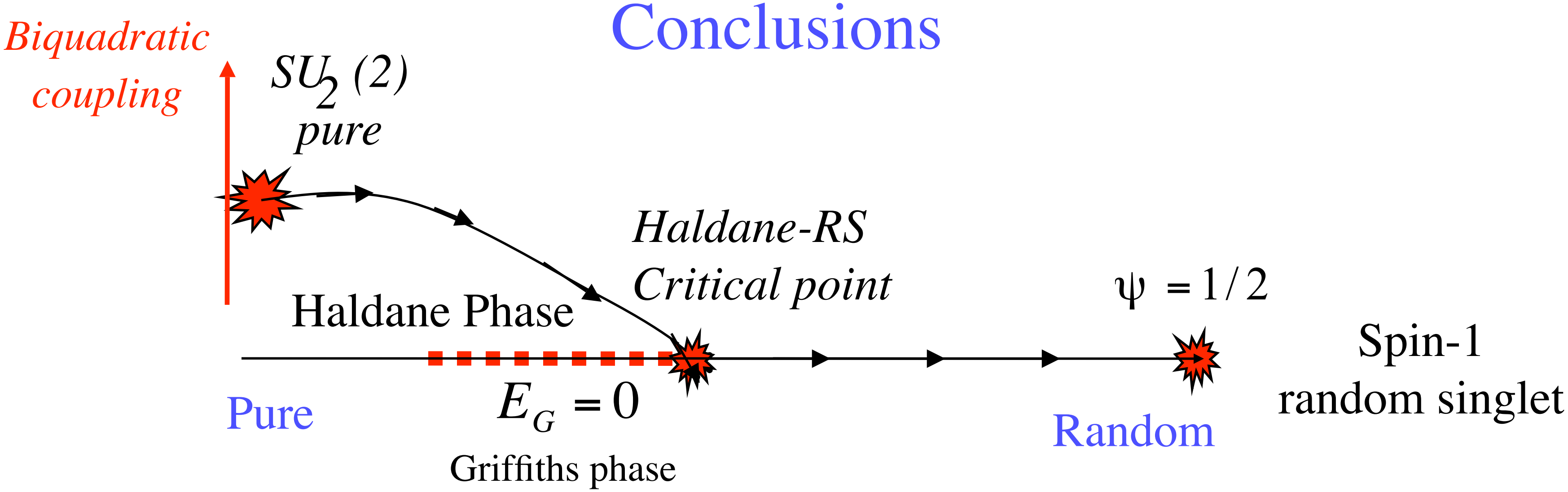

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May fail at S=42 ??? ☺