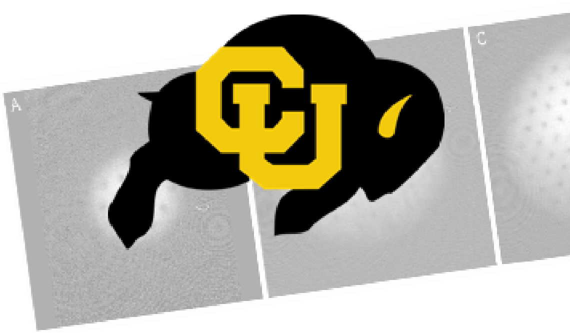
“Blackboard Seminar”

ε and 1/N Expansion of Fermi Gases Near the Unitarity Limit

Martin Veillette



~~"Blackboard Seminar"~~

Lightning Review of BCS-BEC Crossover

Martin Veillette

Fermi Atoms:

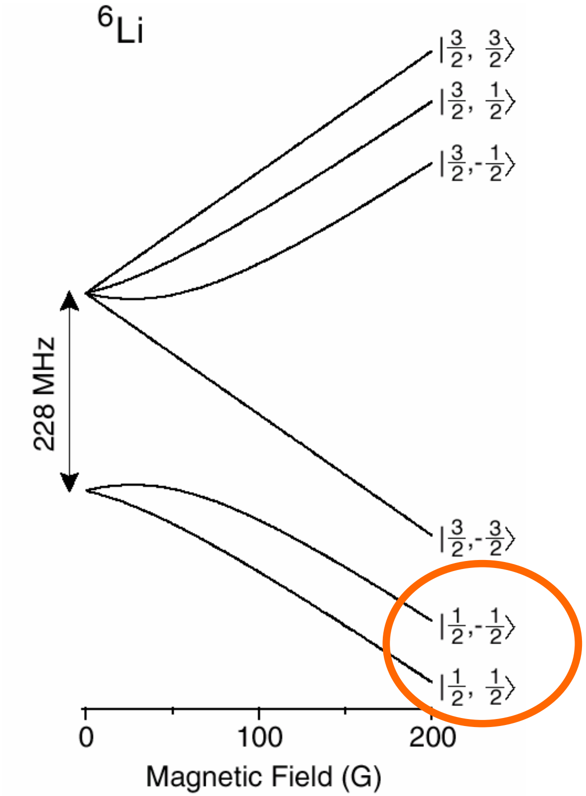
^{6}Li $I = 1, S = 1/2, L = 0$

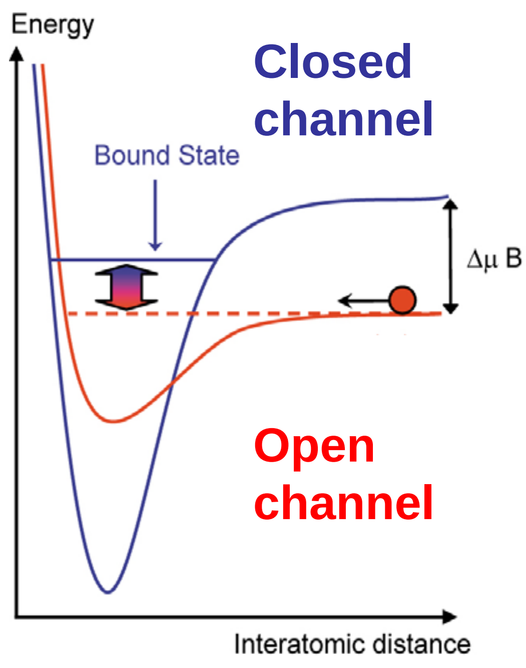
^{40}K $I = 4, S = 1/2, L = 0$

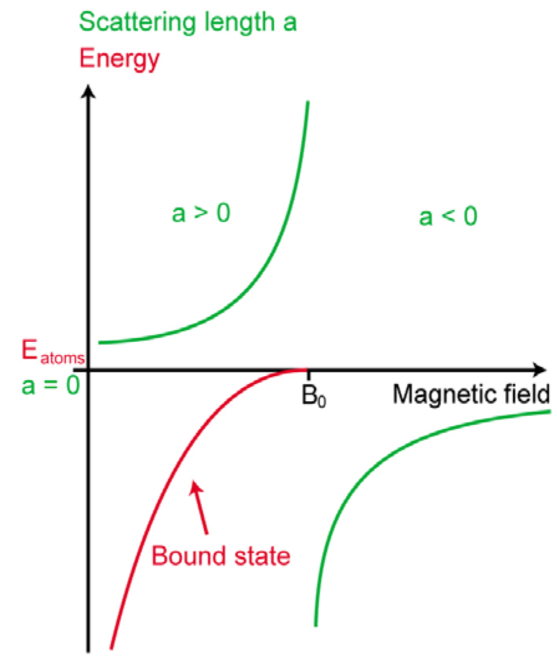
$$\mathbf{F} = \mathbf{I} + \mathbf{S}$$

"up" & "down" species: two different <u>hyperfine</u> states

e.g. ^{6}Li $|F = 1/2, m_F = \pm 1/2\rangle$

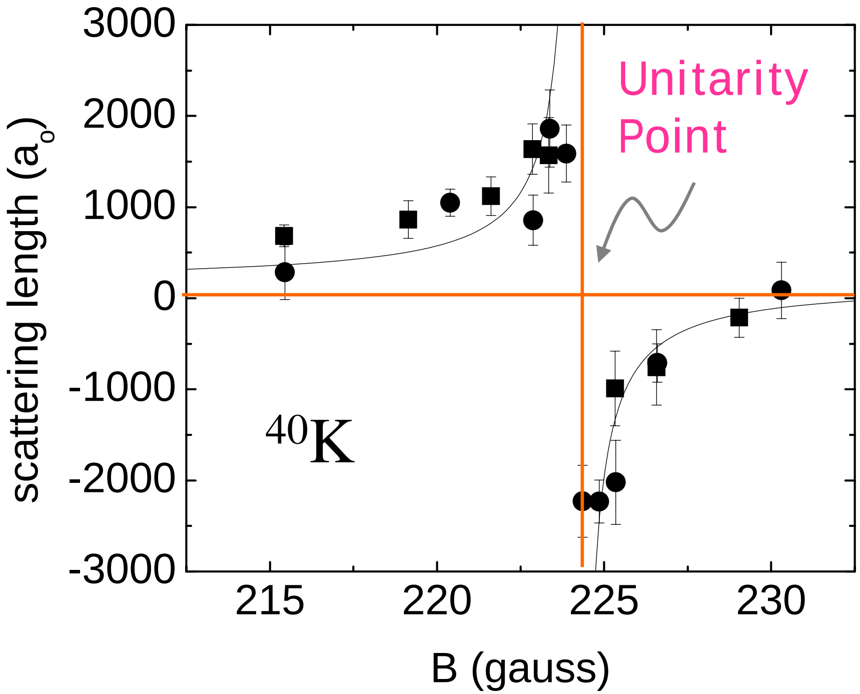






Feshbach Resonance (Broad Resonance)

$$\mathcal{H} = \sum_{\sigma=\uparrow,\downarrow} \int d^3\mathbf{r}\ \psi_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2 \nabla_\mathbf{r}^2}{2m} - \mu \right) \psi_\sigma(\mathbf{r}) + \lambda \int d^3\mathbf{r}\ \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}),$$



C.A. Regal and D.S. Jin, PRL, (2003)

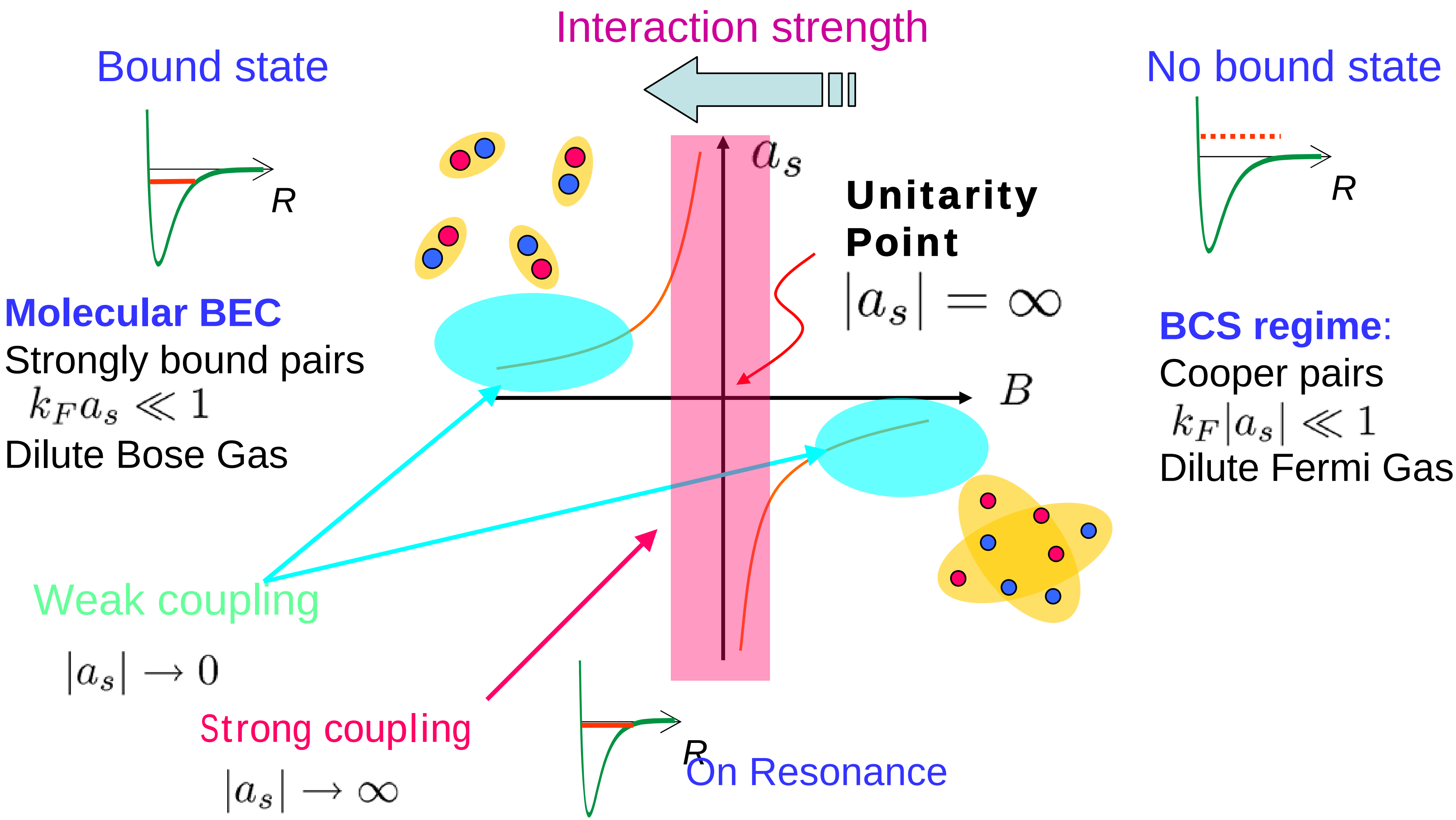
$$\frac{m}{4\pi\hbar^2 a_s} = \frac{1}{\lambda} + \frac{1}{V} \sum_\mathbf{k} \frac{1}{2\epsilon_\mathbf{k}},$$

$$a_s(B) = a_{\text{bg}} \left[1 - \frac{w}{B - B_{\text{res}}} \right]$$

$$F = \epsilon_F \times g \left(k_\mathrm{F} a_s, \frac{k_B T}{\epsilon_F} \right)$$

$$k_F = (3\pi^2 n)^{1/3}$$

BCS-BEC Crossover



At unitarity: No small parameter to expand around: hard problem

Bertsch: Challenge problem in many-body physics (1998): ground state of resonant gas

D.M. Eagles, Phys. Rev. **186**, *456 (1969);*
A.J. Leggett, J. Phys. (Paris), Colloq. **41**, *7 (1980);*

Mean-Field BCS-BEC Crossover at T=0

Variational approach: Ground state remain pair condensate form as one crosses from the fermion to the molecular side.

$$|G\rangle = O^{\dagger N/2}|\mathrm{vac}\rangle \qquad O^\dagger = \int f(\mathbf{r}_1 - \mathbf{r}_2)\psi^\dagger_\uparrow(\mathbf{r}_1)\psi^\dagger_\downarrow(\mathbf{r}_2)$$

Gap equation

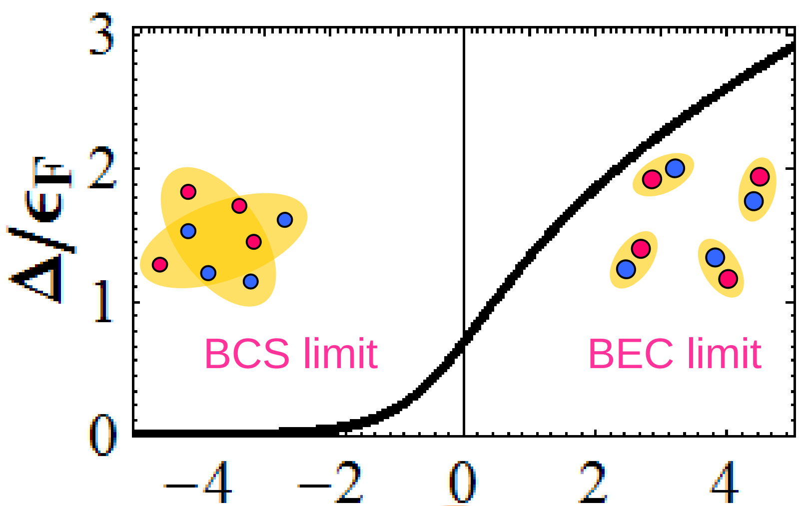
$$-\frac{m\Delta}{4\pi\hbar^2 a_s} = \frac{\Delta}{V}\sum_{\mathbf{k}}\left(\frac{1}{2E_{\mathbf{k}}} - \frac{1}{2\epsilon_{\mathbf{k}}}\right).$$

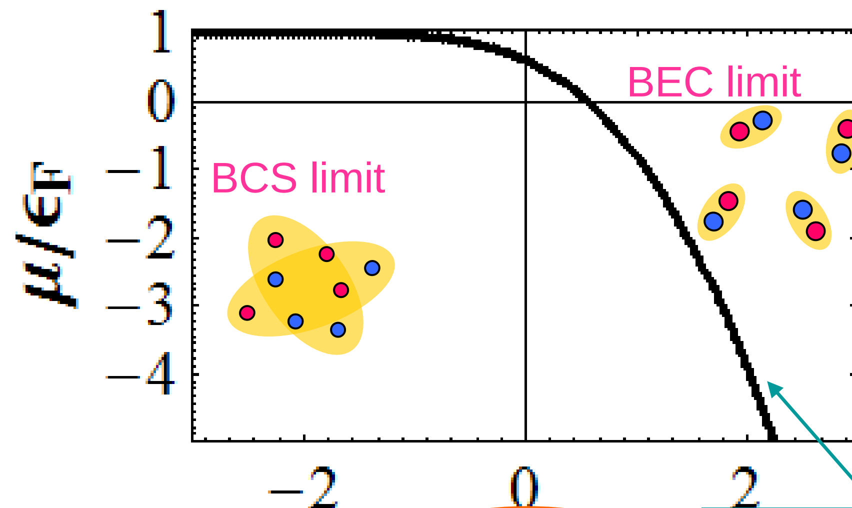
Particle Equation

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$$

$$\xi_{\mathbf{k}} = \epsilon_k - \mu$$

$$n = \frac{2}{V}\sum_{\mathbf{k}}\frac{1}{e^{-\beta E_{\mathbf{k}}} + 1}.$$





$1/k_F a$ ← Dimensionless Coupling constant → **$1/k_F a$**

$$\mu = -E_b/2 = -1/(2ma_s^2)$$

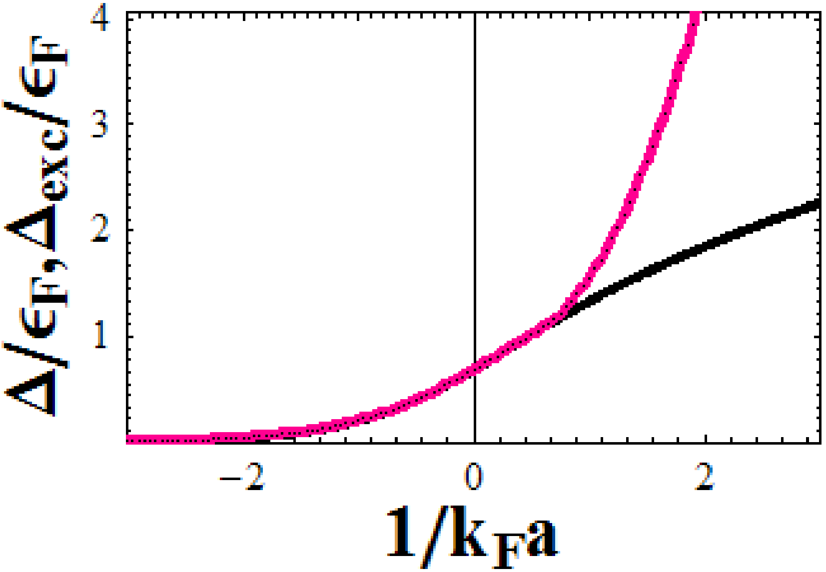
Absence of QPT from BCS to BEC

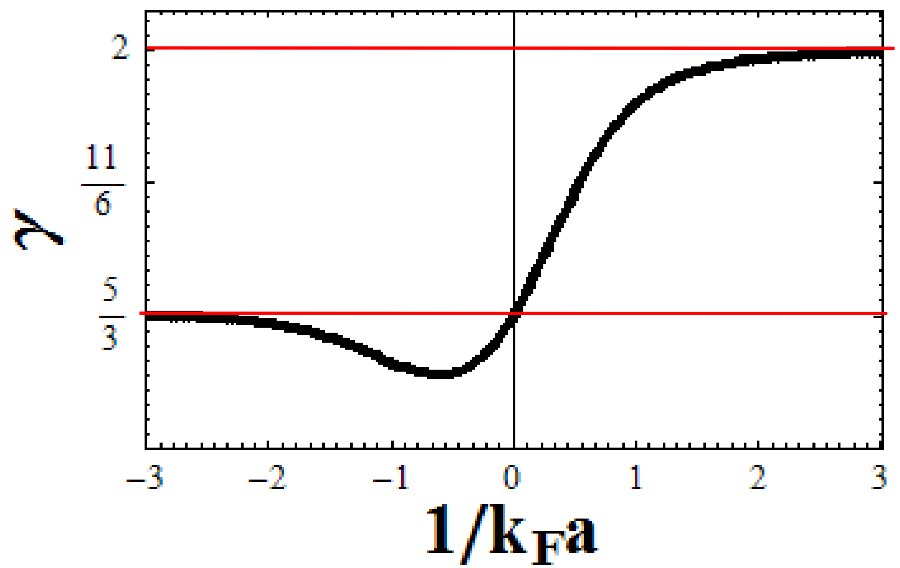
Smooth evolution of physical quantities from BCS to BEC limit.
Most are monotonic functions.

$$\Delta_{\rm exc} = \min_{k\geq 0}\sqrt{(\epsilon_k-\mu)^2+\Delta^2}$$

$$= \begin{cases} \Delta & (\mu>0) \\ \sqrt{|\mu|^2+\Delta^2} & (\mu<0) \end{cases}$$

$$\gamma = \frac{n}{P}\frac{\partial P}{\partial n}$$



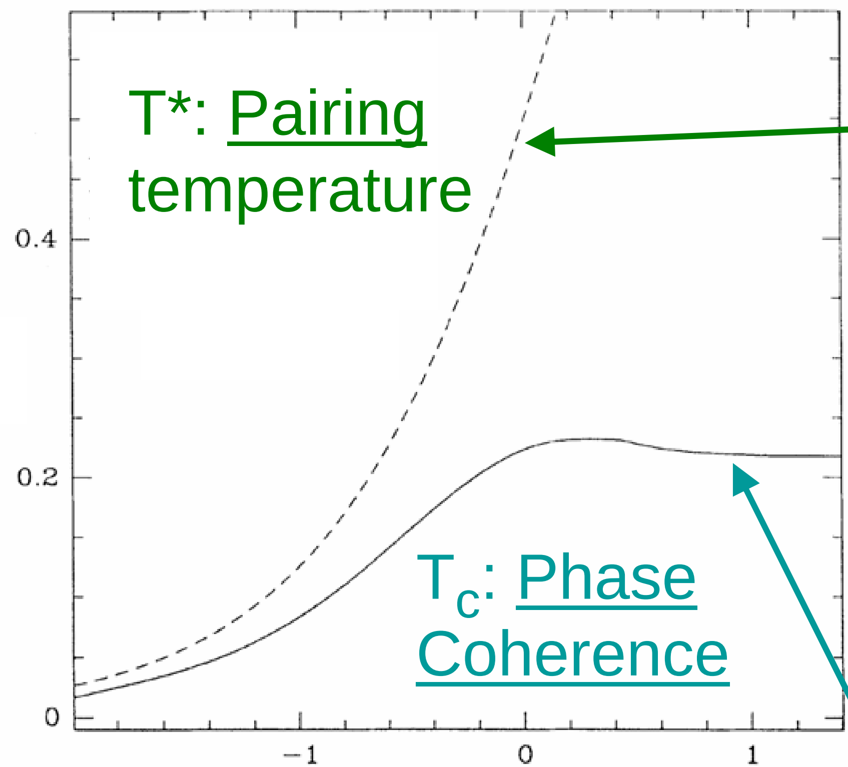


BCS-BEC Crossover at finite T

Mean-field approach (saddle point)

$$f = -\frac{m|\Delta|^2}{4\pi a_s} - \frac{1}{V}\sum_{\mathbf{k}}\left(E_{\mathbf{k}} - \xi_{\mathbf{k}} - \frac{|\Delta|^2}{2\epsilon_{\mathbf{k}}}\right) - \frac{2}{\beta V}\sum_{\mathbf{k}} \ln\left[1 + e^{-\beta E_{\mathbf{k}}}\right],$$

critical temperature: T_c/ϵ_F



scattering length: $1/(k_F a_s)$

$$-\frac{m}{4\pi\hbar^2 a_s} = \frac{1}{V}\sum_{\mathbf{k}}\left(\frac{\tanh(\beta\xi_{\mathbf{k}}/2)}{2\xi_{\mathbf{k}}} - \frac{1}{2\epsilon_{\mathbf{k}}}\right).$$

$$n = \frac{2}{V}\sum_{\mathbf{k}} \frac{1}{e^{-\beta\xi_{\mathbf{k}}} + 1}.$$

Gaussian fluctuations

$$S = S_{MF} + \sum \Gamma_q^{-1}|\Delta_q|^2.$$

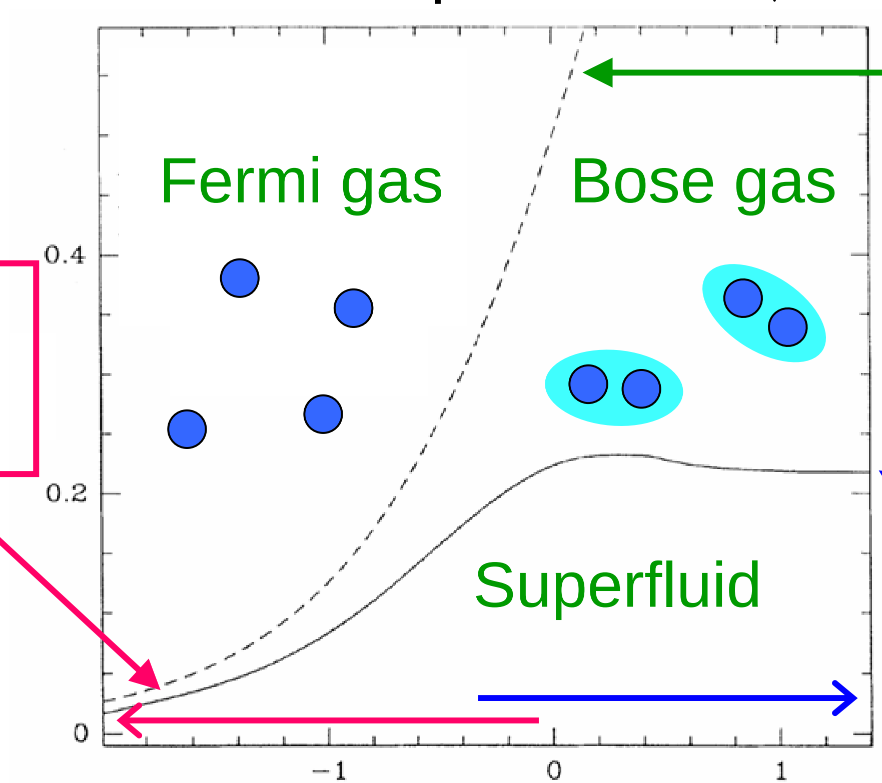
$$\Gamma_q^{-1} = \frac{-m}{4\pi a_s} + \frac{1}{V}\sum_{\mathbf{k}}^{q} \frac{1 - n_F(\xi_{\mathbf{k}+\mathbf{q}/2}) - n_F(\xi_{\mathbf{k}-\mathbf{q}/2})}{i\Omega - \xi_{\mathbf{k}+\mathbf{q}/2} - \xi_{\mathbf{k}-\mathbf{q}/2}} + \frac{1}{2\epsilon_{\mathbf{k}}}.$$

BEC $\Gamma_q^{-1} \sim i\Omega - (-E_b + \mathbf{q}^2/4m - 2\mu).$

$$n = n_o + \sum_{\mathbf{q}} n_B\left[-E_b + \mathbf{q}^2/4m - 2\mu\right].$$

BCS-BEC Crossover at finite T

critical temperature: T_c/ϵ_F



scattering length: $1/(k_F a_s)$

BCS behavior

$T_c/\epsilon_F \sim e^{-1/k_F|a_s|}$

Dissociation of bound boson

$E_b/[\log(E_b/\epsilon_F)]^{2/3}$

BEC behavior

$T_c/\epsilon_F \sim \text{const.}$

C. A. R. Sa de Melo, M. Randeria and J. R. Engelbrecht, PRL 71, 3202 (1993)

Dilute Bose Gas and Fermi Gas

Asymptotic results

BCS Limit $k_F|a_s| \ll 1$

$$\varepsilon = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\left[1 - \frac{10}{9\pi}k_F a_s - \frac{4}{21\pi^2}\left(11 - 2\ln 2\right)k_F^2 a_s^2 + \cdots\right]$$

$$k_B T_c = \frac{1}{4e^{1/3}}\frac{8e^{\gamma-2}}{\pi}\frac{\hbar^2 k_F^2}{2m}e^{\pi/(2k_F a_s)}$$

BEC limit $n_B a_B \ll 1$ $k_F a_s \ll 1$

$$\varepsilon = 2\pi\frac{\hbar^2}{m_B}n_B a_B\left[1 + \frac{128}{15\sqrt{\pi}}\sqrt{n_B a_B^3}\left(1 - \frac{5b}{a_B}\right) + \cdots\right]$$

$$k_B T_c = \frac{4\pi}{(\zeta[3/2])^{3/2}}\frac{\hbar^2}{m_B}n_B^{2/3}\left[1 + 1.32 n_B^{1/3}a_B + \left(19.75\ln(n_B^{1/3}a_B) + 75.7\right)n_B^{4/3}a_B^2 + \cdots\right]$$

$$a_B = 0.6 a_s$$

TABLE I. The critical temperature for a dilute Bose gas obtained by various analytic and numerical methods.

$\frac{\Delta T_c}{T_0} = 2.33 n^{1/3} a,$	leading order $1/N$ (Baym *et al.*, 2000)
$\frac{\Delta T_c}{T_0} = 1.71 n^{1/3} a,$	next-to-leading order $1/N$ (Arnold and Tomasik, 2000)
$\frac{\Delta T_c}{T_0} = (1.32 \pm 0.02) n^{1/3} a,$	lattice (Arnold and Moore, 2001a, 2001b)
$\frac{\Delta T_c}{T_0} = (1.29 \pm 0.05) n^{1/3} a,$	lattice (Kashurnikov *et al.*, 2001)
$\frac{\Delta T_c}{T_0} = 0.7 n^{1/3} a,$	one-bubble approximation (Holzmann *et al.*, 1999)
$\frac{\Delta T_c}{T_0} = 3.8 n^{1/3} a,$	one-bubble self-consistent approximation (Baym *et al.*, 1999, 2001)
$\frac{\Delta T_c}{T_0} = 2.5 n^{1/3} a,$	ladder-summation approximation (Baym *et al.*, 1999, 2001)
$\frac{\Delta T_c}{T_0} = 1.6 n^{1/3} a,$	bubble-summation approximation (Baym *et al.*, 1999, 2001)
$\frac{\Delta T_c}{T_0} = (1.27 \pm 0.11) n^{1/3} a,$	seven-loop variational perturbation theory (Kastening, 2004)
$\frac{\Delta T_c}{T_0} = (1.23 \pm\ ) n^{1/3} a,$	renormalization group in three dimensions (Ledowski *et al.*, 2003)
$\frac{\Delta T_c}{T_0} = (1.3 \pm 0.4) n^{1/3} a,$	simulations of classical field theory (Davis and Morgan, 2003)

Rev. Mod. Phys., Vol. 76, No. 2, April 2004

Unitarity Limit ($|a_s| = \infty$,Strong Coupling)

k_F is the only scale in the problem !

Simple system, with simple scaling → Energy per particle $\varepsilon = \xi \times \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}$

ξ is a universal parameter, related to many quantities

Chemical Potential $\mu = \xi \times \frac{\hbar^2 k_F^2}{2m}$

Bulk Modulus $B = \xi \times \frac{2}{3}\frac{k_F^3}{3\pi^2}\frac{\hbar^2 k_F^2}{2m}$

Pressure density $p = \xi \times \frac{2}{5}\frac{\hbar^2 k_F^2}{2m}$

First Sound $v = \sqrt{\xi} \times \sqrt{2/3}\frac{\hbar k_F}{m}$

Hard problem: No expansion parameter

References

❑ ε expansion (4- ε and 2+ ε)

Z. Nussinov and S. Nussinov, cond-mat/0410597, PRA (2006)

Y. Nishida and D.T. Son, cond-mat/0604500, cond-mat/0607835

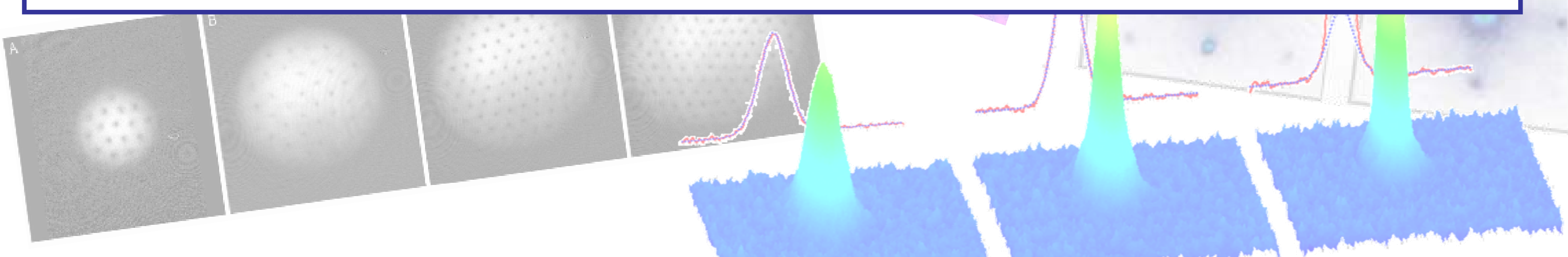
Y. Nishida, cond-mat/0608321

P. Arnold, J.E. Drut and D.T. Son, cond-mat/0608477

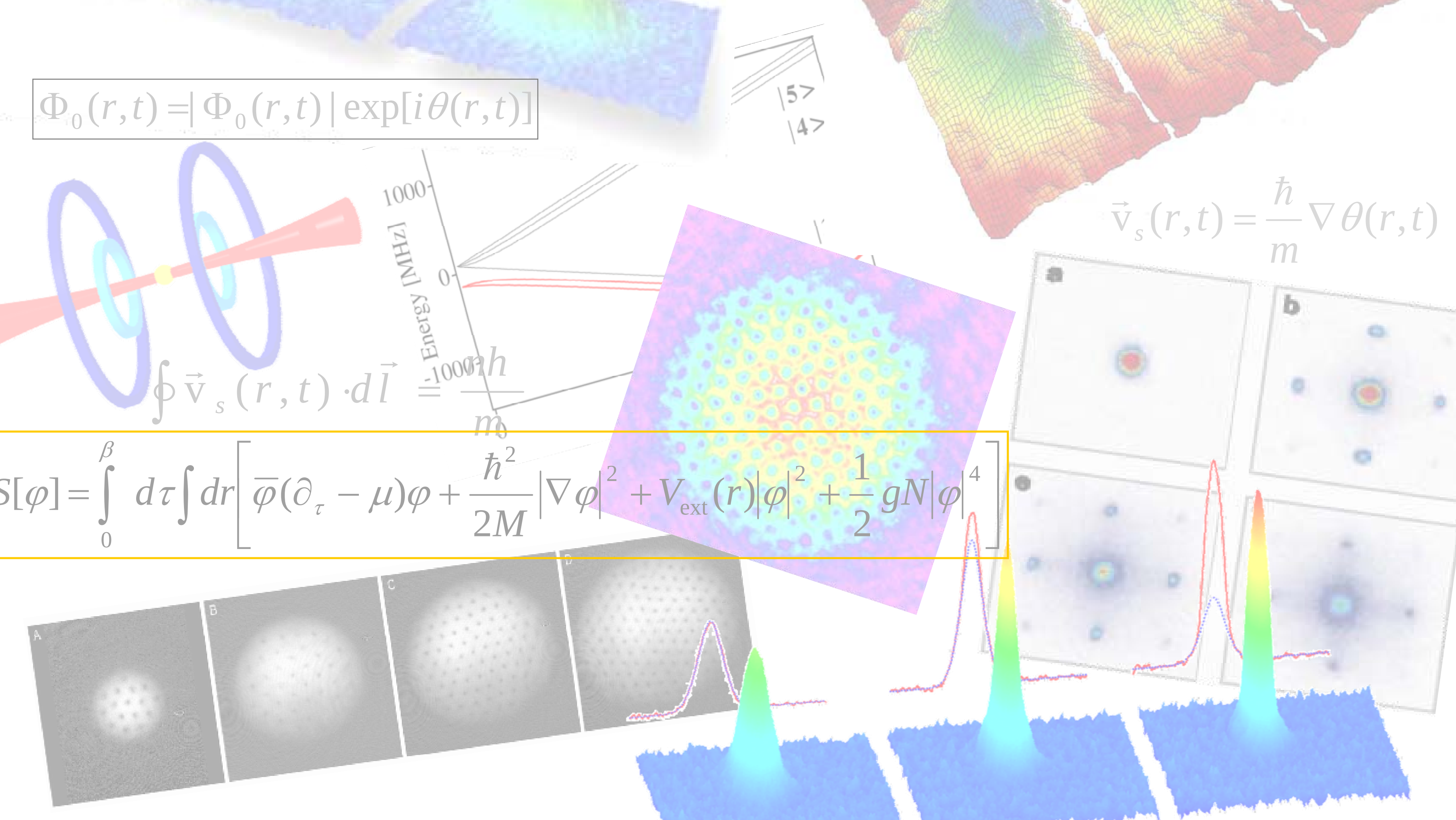
❑ Sp(2N)

M. Veillette, D. Sheehy and L. Radzihovsky, cond-mat/0610798

P. Nikolic and S. Sachdev, cond-mat/0609106



Supplementary Material

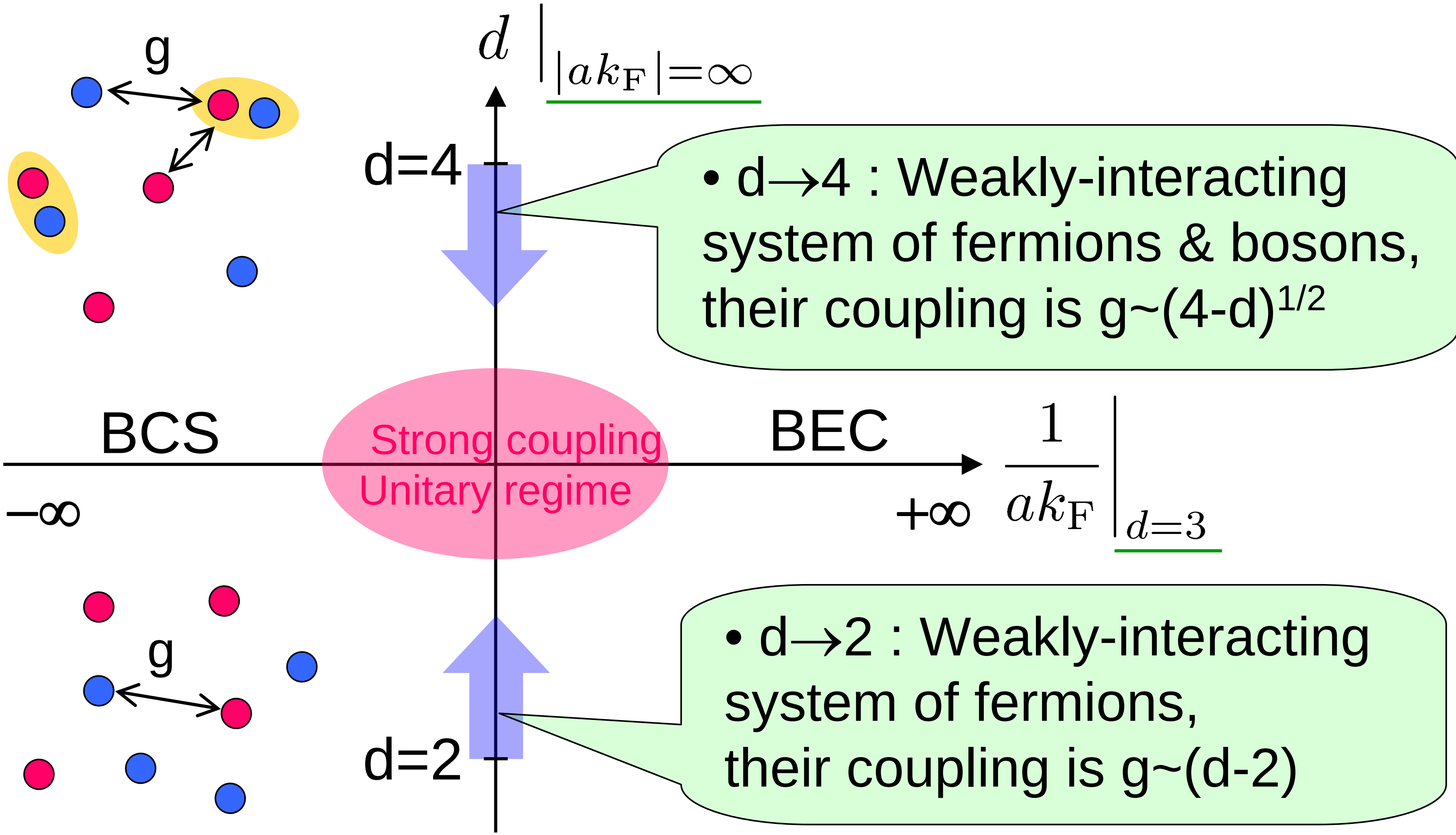


Cold Atoms
Basic Relations

TABLE I. – *Energy and length scales for trapped gaseous Bose-Einstein condensates. The hierarchy of energy and length scales simplifies the description of these quantum fluids. For each energy E, we define a length l by the relation $E = \hbar^2/2ml^2$ and indicate the relation between l and a common length scale. Numbers are typical for sodium BEC experiments. The values for the mean-field energy assume a density of* $\sim 10^{14}$ *cm*$^{-3}$.

Energy Scale E	$= h^2/2ml^2$	Length Scale		
limiting temperature for s-wave scattering	1 mK	scattering length	$a = l/2\pi$	$= 3$ nm
BEC transition temperature T_c	2 μK	separation between atoms	$n^{-1/3} = l/\sqrt{\pi}(2.612)^{1/3}$	$= 200$ nm
single-photon recoil energy	1.2 μK	optical wavelength	$\lambda = l$	$= 600$ nm
temperature T	1 μK	thermal de Broglie wavelength	$\lambda_{dB} = l/\sqrt{\pi}$	$= 300$ nm
mean field energy μ	300 nK	healing length	$\xi = l/2\pi$	$= 200$ nm
harmonic oscillator level spacing $\hbar\omega$	0.5 nK	oscillator length ($\omega \simeq 2\pi \cdot 10$Hz)	$a_{HO} = l/\sqrt{2}\pi$	$= 6.5$ μm

Epsilon expansion



Adapted from Nishida

Unitarity limit in d=4 and d=2

2-body wave function

$$R(r) \propto \frac{1}{r^{d-2}} - \frac{1}{a^{d-2}} + O(r)$$

Z.Nussinov and S.Nussinov,
cond-mat/0410597

Normalization at unitarity $a \to \infty$

$$\int \mathrm{d}^d \boldsymbol{x}\, R(r)^2 \propto \int_{r_0 \to 0}^{\infty} \mathrm{d}r\, \frac{1}{r^{d-3}}$$

diverges at $r \to 0$ for $d \geq 4$

Pair wave function is concentrated near its origin

→ Unitary Fermi gas for d≥4 is free “Bose” gas

At $d \leq 2$, any attractive potential leads to bound states

“$a \to \infty$” corresponds to zero interaction

→ Unitary Fermi gas for d≤2 is free Fermi gas

Adapted from Nishida

Saddle Point Approximation

$$f^{(0)} = N(-\frac{m|\Delta|^2}{4\pi a_s} - \frac{1}{V}\sum_{\mathbf{k}}\left(E_{\mathbf{k}} - \xi_{\mathbf{k}} - \frac{|\Delta|^2}{2\epsilon_{\mathbf{k}}}\right) - \frac{2}{\beta V}\sum_{\mathbf{k}} \ln\left[1 + e^{-\beta E_{\mathbf{k}}}\right]),$$

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$$

Gap equation

$$-\frac{m\Delta}{4\pi a_s} = \frac{\Delta}{V}\sum_{\mathbf{k}}\left(\frac{\tanh(\beta E_{\mathbf{k}}/2)}{2E_{\mathbf{k}}} - \frac{1}{2\epsilon_{\mathbf{k}}}\right).$$

$$\Delta_o^{(0)}/\epsilon_F = 0.6864,$$
$$\mu_o^{(0)}/\epsilon_F = 0.5906,$$
$$k_B T_c^{(0)}/\epsilon_F = 0.4965,$$
$$\mu_c^{(0)}/\epsilon_F = 0.7469.$$

⇒ $N = \infty$

Recover Standard Mean-Field Results

The $N = \infty$ phase is in the same universality class as $N = 1$

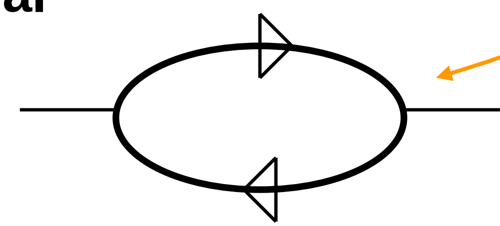
1/N expansion (Loop expansion)

Gaussian Fluctuations around saddle point

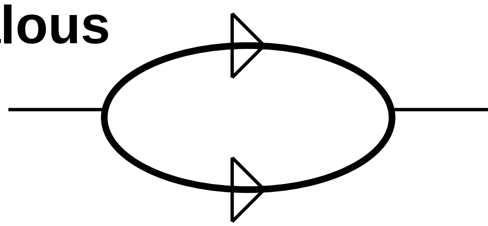
$$f = f_{(0)} + \frac{1}{N} f_{1/N}$$

$$S^{(1/N)} = \frac{1}{2} \sum_q \begin{pmatrix} \hat{b}^*(q) & \hat{b}(-q) \end{pmatrix} \begin{pmatrix} A(q) & B(q) \\ B^*(q) & A(-q) \end{pmatrix} \begin{pmatrix} \hat{b}(q) \\ \hat{b}^*(-q) \end{pmatrix},$$

Normal



Anomalous



$$\Delta = \Delta_o^{(0)} + \frac{1}{N}\delta\Delta_o + \dots,$$

$$\mu = \mu_o^{(0)} + \frac{1}{N}\delta\mu_o + \dots$$

$$T_c = T_c^{(0)} + \frac{1}{N}\delta T_c + \dots$$

$$\mu = \mu_c^{(0)} + \frac{1}{N}\delta\mu_c + \dots$$

Result

$$\Delta_o/\epsilon_F = 0.6864 - \frac{0.163}{N} + \ldots,$$

$$\mu_o/\epsilon_F = 0.5906 - \frac{0.312}{N} + \ldots$$

Zero Temperature

Result from 1/N $\xi = 0.5906 - 0.312/N + \cdots$

Exp with ^{40}K $\xi = 0.46^{+0.05}_{-0.12}$

$$T_c/\epsilon_F = 0.4964 - \frac{1.31}{N} + \ldots$$

$$\mu_c/\epsilon_F = 0.7469 - \frac{0.58}{N} + \ldots$$

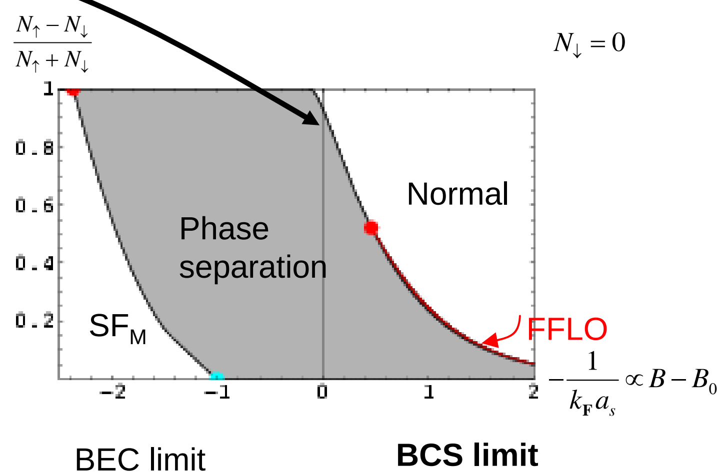
Critical Temperature

$$h_{c2}/\epsilon_F = 0.6929 + \frac{0.087}{N} + \ldots,$$

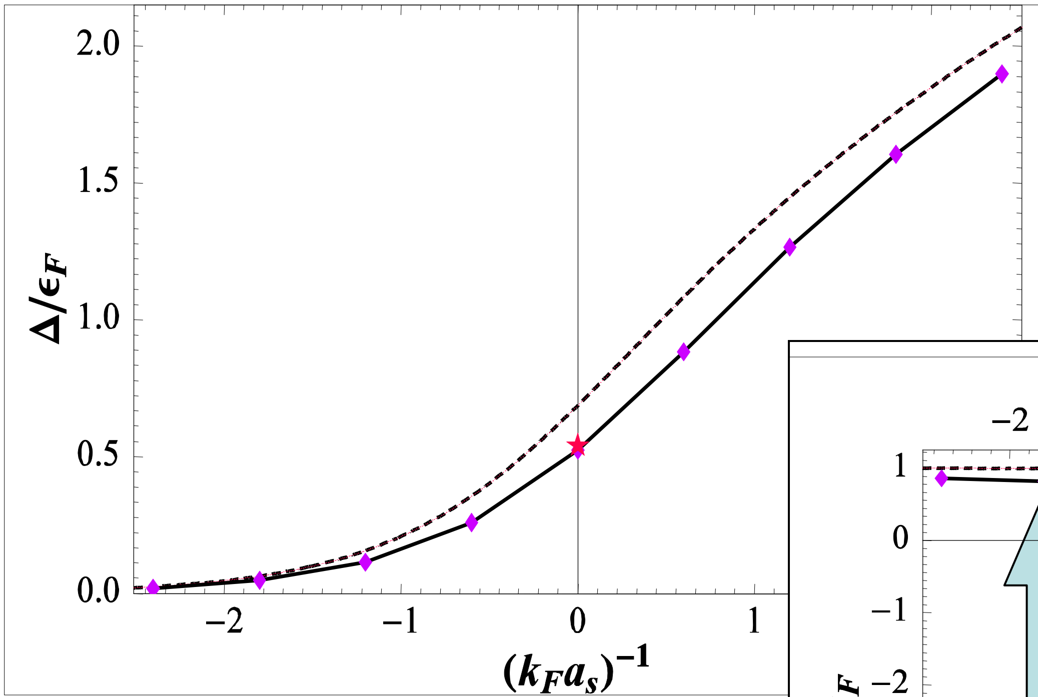
$$P_{c2} = 0.9326 - \frac{0.631}{N} + \ldots$$

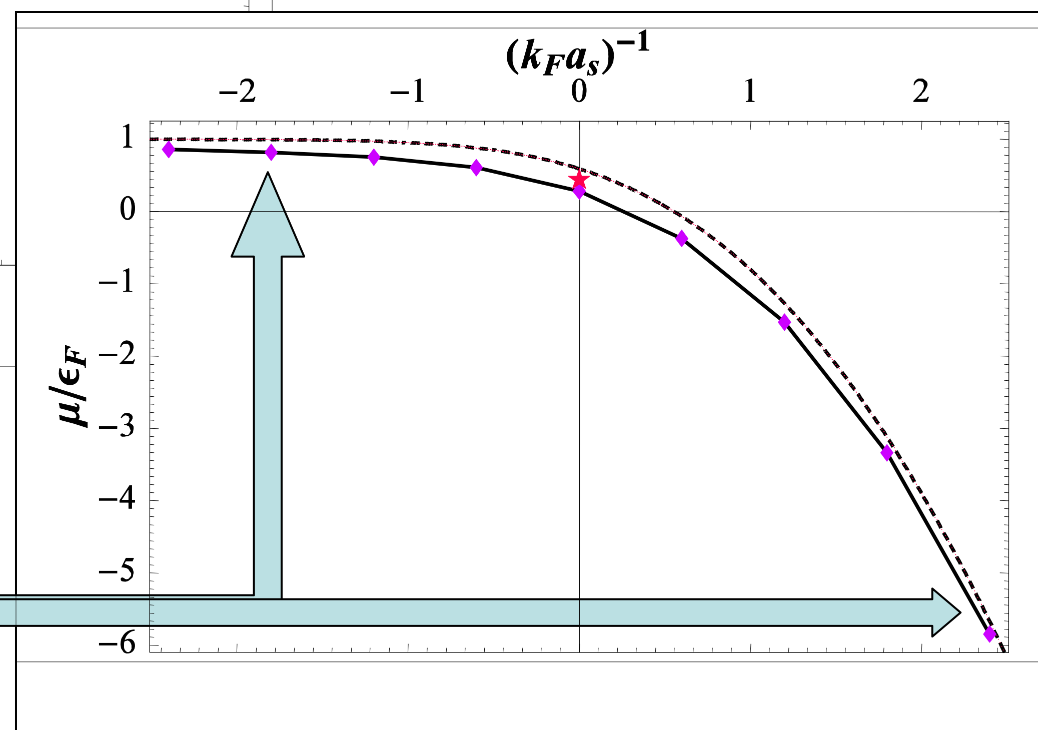
$$\mu/\epsilon_F = 0.8585 - \frac{0.458}{N} + \ldots$$

Population Imbalance



Large-N calculation away from Unitarity



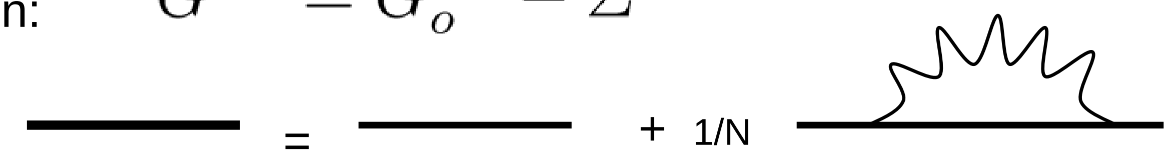


Recover Standard Result
Away from Unitarity

Excitation Gap Vs Order Parameter

- Excitation gap: Δ_{exc} Minimum energy to create excitation
- Order Parameter: $\Delta = \lambda\langle\psi^\dagger\psi^\dagger\rangle$
- In general $\Delta_{exc} \neq \Delta$ (but equal in Mean-field theory)

Δ_{exc} Calculation: $$G^{-1} = G_o^{-1} - \Sigma$$



$$\Delta_{exc} = \Delta + \frac{1}{2}\left(\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}\right)\Big|_{\substack{|\mathbf{k}|=\sqrt{2m\mu_o^{(0)}},\\ i\omega=\Delta_o^{(0)}}}$$

$$\Delta_{exc}/\epsilon_F = 0.6864 - 0.196/N + \mathcal{O}(1/N^2),$$

$$\Delta_o/\epsilon_F = 0.6864 - 0.163/N + \mathcal{O}(1/N^2),$$

Monte Carlo $\Delta_{mc}/\epsilon_F = 0.54$ (Carlson et al, 2003)