A New Look at the Two-Higgs-Doublet Model

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This talk is based on work that appears in:

1. J.F. Gunion and H.E. Haber, "The CP-conserving two-Higgs-doublet model: the approach to the decoupling limit," hep-ph/0207010, Phys. Rev. **D67**, 075019 (2003).

2. S. Davidson, J.F. Gunion and H.E. Haber, "Basis-independent methods for analyzing the two-Higgs-doublet model," in preparation. [See also G. Branco, L. Lavoura and J.P. Silva, *CP Violation* (Oxford University Press, Oxford, England, 1999), chapters 22 and 23.]

3. J.F. Gunion and H.E. Haber, "Conditions for explicit CP-Violation in the general two-Higgs-doublet model," in preparation.

Outline

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Motivation

The Higgs sector of the minimal supersymmetric extension of the Standard Model (MSSM) is a constrained 2HDM. However, whereas the tree-level model is constrained by SUSY, at one-loop all possible 2HDM interactions allowed by gauge invariance are generated (due to SUSY-breaking interactions).

Thus, the Higgs sector of the MSSM is in reality the most general 2HDM model (albeit with certain relations among the Higgs sector parameters determined by the fundamental parameters of the broken supersymmetric model).

The general 2HDM consists of two identical (hypercharge-one) scalar doublets Φ_1 and Φ_2. One can always redefine the basis, so the parameter $\tan\beta \equiv v_2/v_1$ is not meaningful!

To determine the physical quantities, one must develop basis-independent techniques.

The General Two-Higgs-Doublet Model

Consider the 2HDM potential in a *generic* basis:

$$\mathcal{V} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}] + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2$$

$$+ \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$$

$$+ \left\{ \tfrac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \left[\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2) \right] \Phi_1^\dagger \Phi_2 + \text{h.c.} \right\}$$

The vacuum expectation values (vev's) are $\langle \Phi_i^0 \rangle = v_i/\sqrt{2}$, with $\tan\beta \equiv v_2/v_1$ and $v^2 \equiv v_1^2 + v_2^2 = (246 \text{ GeV})^2$. The parameters m_{11}^2, m_{22}^2, m_{12}^2, $\lambda_1, \ldots, \lambda_7$ and v_1, v_2 would change under a U(2) transformation $\Phi_a \longrightarrow U_{a\bar{b}} \Phi_b$ (and $\Phi_{\bar{a}}^\dagger = \Phi_{\bar{b}}^\dagger U_{b\bar{a}}^\dagger$). To identify invariants, write :

$$\mathcal{V} = Y_{a\bar{b}} \Phi_{\bar{a}}^\dagger \Phi_b + \tfrac{1}{2} Z_{a\bar{b}c\bar{d}} (\Phi_{\bar{a}}^\dagger \Phi_b)(\Phi_{\bar{c}}^\dagger \Phi_d) \,,$$

where $Z_{a\bar{b}c\bar{d}} = Z_{c\bar{d}a\bar{b}}$ and hermiticity implies

$$Y_{a\bar{b}} = (Y_{b\bar{a}})^* \,, \qquad Z_{a\bar{b}c\bar{d}} = (Z_{b\bar{a}d\bar{c}})^* \,.$$

The barred indicies help keep track of which indices transform with U and which transform with $U^\dagger$. For example, $Y_{a\bar{b}} \longrightarrow U_{a\bar{c}} Y_{c\bar{d}} U_{d\bar{b}}^\dagger$ and $Z_{a\bar{b}c\bar{d}} \longrightarrow U_{a\bar{e}} U_{f\bar{b}}^\dagger U_{c\bar{g}} U_{h\bar{d}}^\dagger Z_{e\bar{f}g\bar{h}}$.

The scalar potential minimum condition is given by:

$$\widehat{v}_{\bar{a}}^{\,*}\left[Y_{a\bar{b}}+\tfrac{1}{2}v^2 Z_{a\bar{b}c\bar{d}}\widehat{v}_{\bar{c}}^{\,*}\widehat{v}_d\right]=0\,.$$

The most general U(1)$_{\rm EM}$-conserving vev is:

$$\langle\Phi_a\rangle=\frac{v}{\sqrt{2}}\left(\begin{array}{c}0\\ \widehat{v}_a\end{array}\right),\qquad \text{with}\qquad \widehat{v}_a\equiv\left(\begin{array}{c}c_\beta\,e^{i\eta}\\ s_\beta\,e^{i\xi}\end{array}\right).$$

In addition, if we define: $\widehat{w}_a\equiv -e^{i(\eta+\xi)}\,\epsilon_{ab}\,\widehat{v}_{\bar{b}}^{\,*}$ (with $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$), then $\widehat{w}_a\longrightarrow U_{a\bar{b}}\widehat{w}_b$ under a U(2)-transformation. Without loss of generality, we can always set $\eta=0$ with a U(1)$_Y$ rotation, and restrict the U(2) basis transformations to be of the form

$$U=\left(\begin{array}{cc}\cos\theta & e^{-i\xi}\,\sin\theta\\ -e^{i\chi}\,\sin\theta & e^{i(\chi-\xi)}\,\cos\theta\end{array}\right).$$

Here is one possible invariant: $Z_1\equiv Z_{a\bar{b}c\bar{d}}\,\widehat{v}_{\bar{a}}^{\,*}\,\widehat{v}_b\,\widehat{v}_{\bar{c}}^{\,*}\,\widehat{v}_d$. Explicitly,

$$Z_1=\lambda_1 c_\beta^4+\lambda_2 s_\beta^4+\tfrac{1}{2}\lambda_{345}s_{2\beta}^2+2s_{2\beta}\left[c_\beta^2{\rm Re}(\lambda_6 e^{i\xi})+s_\beta^2{\rm Re}(\lambda_7 e^{i\xi})\right],$$

where $\lambda_{345}\equiv\lambda_3+\lambda_4+{\rm Re}(\lambda_5 e^{i\xi})$, $c_\beta\equiv\cos\beta$, $s_\beta\equiv\sin\beta$, *etc.*

We now define the Higgs basis as the basis in which $\langle\Phi_a^0\rangle=v/\sqrt{2}$ and $\langle\Phi_b^0\rangle=0$. In this basis, $\widehat{v}=(1,0)$ and $\widehat{w}=(0,1)$,* and so Z_1 is the coefficient of $\frac{1}{2}(\Phi_a^\dagger\Phi_a)^2$ in the scalar potential.

*One is free to rotate $\Phi_b\longrightarrow e^{i\chi}\Phi_b$, in which case $\widehat{w}=(0,e^{i\chi})$.

The complete list:

$$Y_1 \equiv Y_{a\bar{b}}\,\widehat{v}^*_{\bar{a}}\,\widehat{v}_b\;,$$

$$Y_2 \equiv Y_{a\bar{b}}\,\widehat{w}^*_{\bar{a}}\,\widehat{w}_b\;,$$

$$Y_3 \equiv Y_{a\bar{b}}\,\widehat{v}^*_{\bar{a}}\,\widehat{w}_b\;,$$

and

$$Z_1 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{v}^*_{\bar{a}}\,\widehat{v}_b\,\widehat{v}^*_{\bar{c}}\,\widehat{v}_d\;,$$

$$Z_2 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{w}^*_{\bar{a}}\,\widehat{w}_b\,\widehat{w}^*_{\bar{c}}\,\widehat{w}_d\;,$$

$$Z_3 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{v}^*_{\bar{a}}\,\widehat{v}_b\,\widehat{w}^*_{\bar{c}}\,\widehat{w}_d\;,$$

$$Z_4 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{w}^*_{\bar{a}}\,\widehat{v}_b\,\widehat{v}^*_{\bar{c}}\,\widehat{w}_d\;,$$

$$Z_5 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{v}^*_{\bar{a}}\,\widehat{w}_b\,\widehat{v}^*_{\bar{c}}\,\widehat{w}_d\;,$$

$$Z_6 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{v}^*_{\bar{a}}\,\widehat{v}_b\,\widehat{v}^*_{\bar{c}}\,\widehat{w}_d\;,$$

$$Z_7 \equiv Z_{a\bar{b}c\bar{d}}\,\widehat{v}^*_{\bar{a}}\,\widehat{w}_b\,\widehat{w}^*_{\bar{c}}\,\widehat{w}_d\;.$$

When the above invariants are evaluated in the generic basis, one finds:

$$
\begin{aligned}
Y_1 &= m_{11}^2 c_\beta^2 + m_{22}^2 s_\beta^2 - \mathrm{Re}(m_{12}^2 e^{i\xi}) s_{2\beta}\,, \\
Y_2 &= m_{11}^2 s_\beta^2 + m_{22}^2 c_\beta^2 + \mathrm{Re}(m_{12}^2 e^{i\xi}) s_{2\beta}\,, \\
Y_3 &= \tfrac{1}{2}(m_{22}^2 - m_{11}^2) s_{2\beta} - \mathrm{Re}(m_{12}^2 e^{i\xi}) c_{2\beta} - i\,\mathrm{Im}(m_{12}^2 e^{i\xi})\,,
\end{aligned}
$$

and

$$
\begin{aligned}
Z_1 =& \lambda_1 c_\beta^4 + \lambda_2 s_\beta^4 + \tfrac{1}{2}\lambda_{345} s_{2\beta}^2 + 2 s_{2\beta}\left[c_\beta^2 \mathrm{Re}(\lambda_6 e^{i\xi}) + s_\beta^2 \mathrm{Re}(\lambda_7 e^{i\xi})\right]\,, \\
Z_2 =& \lambda_1 s_\beta^4 + \lambda_2 c_\beta^4 + \tfrac{1}{2}\lambda_{345} s_{2\beta}^2 - 2 s_{2\beta}\left[s_\beta^2 \mathrm{Re}(\lambda_6 e^{i\xi}) + c_\beta^2 \mathrm{Re}(\lambda_7 e^{i\xi})\right]\,, \\
Z_3 =& \tfrac{1}{4} s_{2\beta}^2 \left[\lambda_1 + \lambda_2 - 2\lambda_{345}\right] + \lambda_3 - s_{2\beta} c_{2\beta} \mathrm{Re}[(\lambda_6 - \lambda_7) e^{i\xi}]\,, \\
Z_4 =& \tfrac{1}{4} s_{2\beta}^2 \left[\lambda_1 + \lambda_2 - 2\lambda_{345}\right] + \lambda_4 - s_{2\beta} c_{2\beta} \mathrm{Re}[(\lambda_6 - \lambda_7) e^{i\xi}]\,, \\
Z_5 =& \tfrac{1}{4} s_{2\beta}^2 \left[\lambda_1 + \lambda_2 - 2\lambda_{345}\right] + \mathrm{Re}(\lambda_5 e^{2i\xi}) + i c_{2\beta} \mathrm{Im}(\lambda_5 e^{2i\xi})\,, \\
& - s_{2\beta} c_{2\beta} \mathrm{Re}[(\lambda_6 - \lambda_7) e^{i\xi}] - i s_{2\beta} \mathrm{Im}[(\lambda_6 - \lambda_7) e^{i\xi})]\,, \\
Z_6 =& -\tfrac{1}{2} s_{2\beta}\left[\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_{345} c_{2\beta} - i \mathrm{Im}(\lambda_5 e^{2i\xi})\right] + c_\beta c_{3\beta} \mathrm{Re}(\lambda_6 e^{i\xi})\,, \\
& + s_\beta s_{3\beta} \mathrm{Re}(\lambda_7 e^{i\xi}) + i c_\beta^2 \mathrm{Im}(\lambda_6 e^{i\xi}) + i s_\beta^2 \mathrm{Im}(\lambda_7 e^{i\xi})\,, \\
Z_7 =& -\tfrac{1}{2} s_{2\beta}\left[\lambda_1 s_\beta^2 - \lambda_2 c_\beta^2 + \lambda_{345} c_{2\beta} + i \mathrm{Im}(\lambda_5 e^{2i\xi})\right] + s_\beta s_{3\beta} \mathrm{Re}(\lambda_6 e^{i\xi}) \\
& + c_\beta c_{3\beta} \mathrm{Re}(\lambda_7 e^{i\xi}) + i s_\beta^2 \mathrm{Im}(\lambda_6 e^{i\xi}) + i c_\beta^2 \mathrm{Im}(\lambda_7 e^{i\xi})\,.
\end{aligned}
$$

The Higgs basis is obtained from the generic basis via:

$$\Phi_a = \quad \Phi_1 c_\beta + e^{-i\xi}\,\Phi_2 s_\beta\,, \qquad \Phi_b = -\Phi_1 s_\beta + e^{-i\xi}\,\Phi_2 c_\beta\,,$$

where

$$\Phi_a = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \varphi_a^0 + iG^0\right) \end{pmatrix}, \qquad \Phi_b = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(\varphi_b^0 + iA\right) \end{pmatrix},$$

and $G^\pm$ and G^0 are the Goldstone fields. In the Higgs basis,

$$\begin{aligned}\mathcal{V} =& Y_1 \Phi_a^\dagger \Phi_a + Y_2 \Phi_b^\dagger \Phi_b + [Y_3 \Phi_a^\dagger \Phi_b + \mathrm{h.c.}] + \tfrac{1}{2} Z_1 (\Phi_a^\dagger \Phi_a)^2 \\ &+ \tfrac{1}{2} Z_2 (\Phi_b^\dagger \Phi_b)^2 + Z_3 (\Phi_a^\dagger \Phi_a)(\Phi_b^\dagger \Phi_b) + Z_4 (\Phi_a^\dagger \Phi_b)(\Phi_b^\dagger \Phi_a) \\ &+ \left\{ \tfrac{1}{2} Z_5 (\Phi_a^\dagger \Phi_b)^2 + \left[Z_6\,(\Phi_a^\dagger \Phi_a) + Z_7 (\Phi_b^\dagger \Phi_b) \right] \Phi_a^\dagger \Phi_b + \mathrm{h.c.} \right\}\,.\end{aligned}$$

The corresponding scalar potential minimum conditions are:

$$Y_1 = -\tfrac{1}{2} Z_1 v^2\,, \qquad Y_3 = -\tfrac{1}{2} Z_6 v^2\,,$$

and the charged Higgs squared-mass is given by: $m_{H^\pm}^2 = Y_2 + \frac{1}{2} Z_3 v^2$. The decoupling limit is achieved in the limit of $Y_2 \gg v^2$, subject to the conditions $|Z_i| \lesssim \mathcal{O}(1)$.

The CP-conserving 2HDM

The Higgs spectrum in the CP-conserving model contains a CP-odd state, A, a charged scalar pair, $H^\pm$, and two CP-even states obtained by diagonalizing a 2×2 squared-mass matrix:

$$H = (\sqrt{2}\mathrm{Re}\,\Phi_1^0 - v_1)\cos\alpha + (\sqrt{2}\mathrm{Re}\,\Phi_2^0 - v_2)\sin\alpha\,,$$

$$h = -(\sqrt{2}\mathrm{Re}\,\Phi_1^0 - v_1)\sin\alpha + (\sqrt{2}\mathrm{Re}\,\Phi_2^0 - v_2)\cos\alpha\,.$$

The angle α (like β) is basis-dependent. Define $\widehat{n} = (c_\alpha, s_\alpha)$ in the generic basis. Then the basis independent quantities:

$$\widehat{n}_a \widehat{v}_a = \cos(\beta-\alpha) \equiv c_{\beta-\alpha}\,, \qquad \epsilon_{ab}\widehat{n}_a\widehat{v}_b = \sin(\beta-\alpha) \equiv s_{\beta-\alpha}\,.$$

are meaningful. The end result of the diagonalization is:

$$m^2_{H,h} = \tfrac{1}{2}\left[m_A^2 + v^2(Z_5+Z_1) \pm \sqrt{[m_A^2 + (Z_5 - Z_1)v^2]^2 + 4Z_6^2 v^4}\right]\,,$$

$$\sin[2(\beta-\alpha)] = 2s_{\beta-\alpha}c_{\beta-\alpha} = \frac{-2Z_6 v^2}{m_H^2 - m_h^2},$$

$$\cos[2(\beta-\alpha)] = c^2_{\beta-\alpha} - s^2_{\beta-\alpha} = \frac{(Z_1 - Z_5)v^2 - m_A^2}{m_H^2 - m_h^2}\,.$$

In addition, $m^2_{H^\pm} = m_A^2 + \frac{1}{2}v^2(Z_5 - Z_4)$.

The decoupling limit of the CP-conserving 2HDM

In the CP-conserving model, the decoupling limit corresponds to the limit where $m_A \gg v$, assuming $\lambda_i \lesssim \mathcal{O}(1)$. From the expressions for $\beta - \alpha$ just given, this limit corresponds to taking $\beta - \alpha \to \pi/2$; *i.e.*, $\cos(\beta - \alpha) \to 0$. In the approach to the decoupling limit, one finds:

$$m_A^2 \simeq v^2 \left[\frac{-Z_6}{c_{\beta-\alpha}} + Z_1 - Z_5 + \tfrac{3}{2} Z_6 c_{\beta-\alpha}\right] ,$$

$$m_h^2 \simeq v^2 (Z_1 + Z_6 c_{\beta-\alpha}) ,$$

$$m_H^2 \simeq m_A^2 + v^2 \left(Z_5 - Z_6 c_{\beta-\alpha}\right) ,$$

where $m_H > m_h$ implies $m_A^2 \gtrsim v^2(Z_1 - Z_5 + 2Z_6 c_{\beta-\alpha})$. Thus,

$$\boxed{\cos(\beta - \alpha) \simeq \frac{-Z_6 v^2}{m_A^2 - (Z_1 - Z_5)v^2} \simeq \frac{-Z_6 v^2}{m_H^2 - m_h^2}}$$

Implications

- $m_h \lesssim \mathcal{O}(v)$,
- $m_H \simeq m_A \simeq m_{H^\pm}$, [up to corrections of $\mathcal{O}(v^2/m_A)$] ,
- $\cos(\beta - \alpha) \lesssim \mathcal{O}\left(\frac{v^2}{m_A^2}\right)$.

2HDM Tree-Level Higgs Couplings

Higgs couplings to gauge bosons: suppression factors

$\cos(\beta-\alpha)$	$\sin(\beta-\alpha)$	no angle factor
HW^+W^-	hW^+W^-	
HZZ	hZZ	
ZAh	ZAH	$ZH^+H^-, \gamma H^+H^-$
$W^\pm H^\mp h$	$W^\pm H^\mp H$	$W^\pm H^\mp A$
$ZW^\pm H^\mp h$	$ZW^\pm H^\mp H$	$ZW^\pm H^\mp A$
$\gamma W^\pm H^\mp h$	$\gamma W^\pm H^\mp H$	$\gamma W^\pm H^\mp A$

Higgs self-couplings

Here are two examples:

$$g_{hhh} = -3v\Big[Z_1 s^3_{\beta-\alpha} + Z_{345} s_{\beta-\alpha} c^2_{\beta-\alpha} + 3Z_6 c_{\beta-\alpha} s^2_{\beta-\alpha} + Z_7 c^3_{\beta-\alpha}\Big]$$

$$g_{hhhh} = -3\Big[Z_1 s^4_{\beta-\alpha} + Z_2 c^4_{\beta-\alpha} + 2Z_{345} c^2_{\beta-\alpha} s^2_{\beta-\alpha} + 4Z_6 c_{\beta-\alpha} s^3_{\beta-\alpha} + 4Z_7 c^3_{\beta-\alpha} s_{\beta-\alpha}\Big]$$

Indeed, all couplings above depend only on basis-independent invariants.

Higgs couplings to fermion pairs

$$-\mathcal{L}_Y = \overline{Q}^0_L \widetilde{\Phi}_1 \eta_1^{U,0} U^0_R + \overline{Q}^0_L \Phi_1 \eta_1^{D,0} D^0_R + \overline{Q}^0_L \widetilde{\Phi}_2 \eta_2^{U,0} U^0_R + \overline{Q}^0_L \Phi_2 \eta_2^{D,0} D^0_R + \text{h.c.}\,,$$

where $\widetilde{\Phi}_i \equiv i\sigma_2 \Phi_i^*$, Q^0_L is the weak isospin quark doublet, and U^0_R, D^0_R are weak isospin quark singlets in an interaction eigenstate basis, and $\eta_1^{U,0}, \eta_2^{U,0}, \eta_1^{D,0}, \eta_2^{D,0}$ are matrices in flavor space. Identify the fermion mass eigenstates by employing the appropriate bi-unitary transformation of the quark mass matrices involving unitary matrices V_L^U, V_L^D, V_R^U, V_R^D, where $K \equiv V_L^U V_L^{D\,\dagger}$ is the CKM matrix. Then, define $\eta^Q \equiv (\eta_1^Q, \eta_2^Q)$, where $\eta_i^Q \equiv V_L^Q \eta_i^{Q,0} V_R^{Q\,\dagger}$, for $Q = U,\ D$. One can then introduce the invariant quantities:

$$\kappa^Q \equiv \widehat{v} \cdot \eta^Q = c_\beta \eta_1^Q + s_\beta \eta_2^Q\,,$$

$$\rho^Q \equiv \widehat{w} \cdot \eta^Q = -s_\beta \eta_1^Q + c_\beta \eta_2^Q\,.$$

The quark mass terms are identified by replacing the scalar fields with their vev's. V_L^U, V_L^D, V_R^U and V_R^D are then chosen so that κ^D and κ^U are diagonal with real non-negative entries. The resulting quark mass matrices are then diagonal:

$$M_D = \frac{v}{\sqrt{2}} \kappa^D\,, \qquad M_U = \frac{v}{\sqrt{2}} \kappa^U\,.$$

The Higgs-fermion Yukawa couplings are then given by:

$$
\begin{aligned}
-\mathcal{L}_Y =& \frac{1}{v}\,\overline{D}\left[M_D s_{\beta-\alpha} + \frac{v}{\sqrt{2}}(\rho^D P_R + {\rho^D}^\dagger P_L) c_{\beta-\alpha}\right] Dh \\
&+ \frac{1}{v}\,\overline{D}\left[M_D c_{\beta-\alpha} - \frac{v}{\sqrt{2}}(\rho^D P_R + {\rho^D}^\dagger P_L) s_{\beta-\alpha}\right] DH \\
&+ \frac{i}{\sqrt{2}}\overline{D}(\rho^D P_R - {\rho^D}^\dagger P_L) DA \\
&+ \frac{1}{v}\,\overline{U}\left[M_U s_{\beta-\alpha} + \frac{v}{\sqrt{2}}(\rho^U P_R + {\rho^U}^\dagger P_L) c_{\beta-\alpha}\right] Uh \\
&+ \frac{1}{v}\,\overline{U}\left[M_U c_{\beta-\alpha} - \frac{v}{\sqrt{2}}(\rho^U P_R + {\rho^U}^\dagger P_L) s_{\beta-\alpha}\right] UH \\
&- \frac{i}{\sqrt{2}}\overline{U}(\rho^U P_R - {\rho^U}^\dagger P_L) UA \\
&+ \left\{\overline{U}\left[K\rho^D P_R - {\rho^U}^\dagger K P_L\right] DH^+ + \text{h.c.}\right\}\,.
\end{aligned}
$$

where $P_{R,L} \equiv \frac{1}{2}(1 \pm \gamma_5)$. The Higgs-fermion Yukawa couplings depend only on invariant quantities: the quark masses, the ρ^Q and $\beta - \alpha$. In general, ρ^U and ρ^D are complex non-diagonal matrices. Hence, the Yukawa couplings exhibit tree-level Higgs-mediated flavor-changing neutral currents and CP-violating Higgs-fermion couplings. But, near the decoupling limit, the couplings of h approach their SM values.

In a one generation model, the Higgs-quark interaction produces the following Feynman rules of the form $-ig_{\phi f_1 \bar{f}_2}$:

$$g_{hq\bar{q}} = \frac{m_q}{v} s_{\beta-\alpha} + \tfrac{1}{\sqrt{2}}(S_q + i\gamma_5 P_q) c_{\beta-\alpha} \,,$$

$$g_{Hq\bar{q}} = \frac{m_q}{v} c_{\beta-\alpha} - \tfrac{1}{\sqrt{2}}(S_q + i\gamma_5 P_q) s_{\beta-\alpha} \,,$$

$$g_{Au\bar{u}} = -\tfrac{1}{\sqrt{2}}(iS_u\gamma_5 - P_u) \,,$$

$$g_{Ad\bar{d}} = \tfrac{1}{\sqrt{2}}(iS_d\gamma_5 - P_d) \,,$$

$$g_{H^+ d\bar{u}} = \tfrac{1}{2}[\rho^D(1+\gamma_5) - \rho^{U*}(1-\gamma_5)] \,,$$

where $S_q \equiv \mathrm{Re}\ \rho^Q$ and $P_q \equiv \mathrm{Im}\ \rho^Q$.

The MSSM Higgs sector is a type-II 2HDM, *i.e.*, $\eta_1^U = \eta_2^D = 0$. A basis-independent condition for type-II is: $\eta^U \cdot \eta^D = 0$. But, this means that $\tan\beta$ has been promoted to a physical parameter since $\eta_1^U = \eta_2^D = 0$ occurs only for one choice of β.† It follows that:

$$\tan\beta = -\frac{\rho^D}{\kappa^D} = \frac{\kappa^U}{\rho^U} \,.$$

These two definitions are consistent if $\kappa^U\kappa^D + \rho^U\rho^D = 0$ is satisfied.

† Actually two choices: β and $\beta - \pi/2$, corresponding to whether the Higgs doublet that couples to, say, the up-type quark is Φ_1 or Φ_2.

But this is equivalent to the type-II condition, $\eta^U \cdot \eta^D = 0$. Moreover, using $\kappa^Q = \sqrt{2} M_Q/v = \sqrt{2} v^{-1} \mathrm{diag}(m_u, m_d)$,

$$\rho^D = \frac{-\sqrt{2} m_d \tan\beta}{v}\,, \qquad \rho^U = \frac{\sqrt{2} m_u \cot\beta}{v}\,.$$

This then yields the well-known type-II Higgs-quark interactions:

$$h b\bar{b}: \; -\frac{\sin\alpha}{\cos\beta} = 1\sin(\beta-\alpha) - \tan\beta\cos(\beta-\alpha)\,,$$

$$h t\bar{t}: \quad \frac{\cos\alpha}{\sin\beta} = 1\sin(\beta-\alpha) + \cot\beta\cos(\beta-\alpha)\,,$$

$$H b\bar{b}: \quad \frac{\cos\alpha}{\cos\beta} = \cos(\beta-\alpha) + \tan\beta\,\sin(\beta-\alpha)\,,$$

$$H t\bar{t}: \quad \frac{\sin\alpha}{\sin\beta} = \cos(\beta-\alpha) - \cot\beta\,\sin(\beta-\alpha)\,.$$

In the more general (type-III) 2HDM, $\tan\beta$ is not a meaningful parameter. Nevertheless, one can introduce three $\tan\beta$-like parameters:‡

$$\tan\beta_d \equiv \frac{-\rho^D}{\kappa^D}\,, \qquad \tan\beta_u \equiv \frac{\kappa^U}{\rho^U}\,, \qquad \tan\beta_e \equiv \frac{-\rho^E}{\kappa^E}\,,$$

the last one corresponding to the Higgs-lepton interaction. In a type-III 2HDM, there is no reason for the three parameters above to coincide.

‡ Interpretation: In the Higgs basis, up and down-type quarks interact with both Higgs doublets. But, clearly there exists some basis (*i.e.*, a rotation by angle β_u from the Higgs basis) for which only one of the two up-type quark Yukawa couplings is non-vanishing. This defines the physical angle β_u.

Conditions for explicit CP-violation

Here, we consider the conditions for Higgs-mediated CP-violation due to an explicitly CP-violating Higgs potential.[§]

Theorem: The Higgs potential is CP-conserving if and only if there exists a basis in which all Higgs potential parameters are real.

Potentially complex Higgs potential parameters are: m_{12}^2, λ_5, λ_6 and λ_7. Of course, these are basis-dependent. Nevertheless, the following result should be noted:

Theorem: In a generic basis, the following is a sufficient (but not a necessary) condition for an explicitly CP-conserving 2HDM scalar potential:

$$\begin{aligned}\mathrm{Im}\,([m_{12}^2]^2\lambda_5^*) &= \mathrm{Im}\,(m_{12}^2\lambda_6^*) = \mathrm{Im}\,(m_{12}^2\lambda_7^*) \\ &= \mathrm{Im}\,(\lambda_5^*\lambda_6^2) = \mathrm{Im}\,(\lambda_5^*\lambda_7^2) = \mathrm{Im}\,(\lambda_6^*\lambda_7) = 0\,.\end{aligned}$$

[§]We defer the question of whether CP is spontaneously broken if the Higgs potential is manifestly CP-conserving.

Clearly, the latter is not good enough. We shall instead provide a set of basis-independent conditions. The complete set of conditions is summarized by the following result:

Theorem: The following is a necessary and sufficient condition for an explicitly CP-conserving 2HDM scalar potential:

$$I_{YZZZ} = I_{YYZZ} = I_{6Z} = I_{3Y3Z} = 0\,,$$

where

$$\begin{aligned}
I_{YZZZ} &\equiv \mathrm{Im}(Z^{(1)}_{a\bar{c}} Z^{(1)}_{e\bar{b}} Z_{b\bar{e}c\bar{d}} Y_{d\bar{a}})\,, \\
I_{YYZZ} &\equiv \mathrm{Im}(Y_{a\bar{b}} Y_{c\bar{d}} Z_{b\bar{a}d\bar{f}} Z^{(1)}_{f\bar{c}})\,, \\
I_{6Z} &\equiv \mathrm{Im}(Z_{a\bar{b}c\bar{d}} Z^{(1)}_{b\bar{f}} Z^{(1)}_{d\bar{h}} Z_{f\bar{a}j\bar{k}} Z_{k\bar{j}m\bar{n}} Z_{n\bar{m}h\bar{c}})\,, \\
I_{3Y3Z} &\equiv \mathrm{Im}(Y_{q\bar{f}} Y_{h\bar{b}} Y_{g\bar{a}} Z_{e\bar{h}f\bar{q}} Z_{c\bar{e}d\bar{g}} Z_{a\bar{c}b\bar{d}})\,.
\end{aligned}$$

Above, we have introduced:

$$Z^{(1)}_{a\bar{d}} \equiv \delta_{b\bar{c}} Z_{a\bar{b}c\bar{d}} = Z_{a\bar{b}b\bar{d}}\,.$$

Explicit results

$$
\begin{aligned}
I_{6Z} &= 2|\lambda_5|^2 \mathrm{Im}[(\lambda_7^*\lambda_6)^2] - \mathrm{Im}[\lambda_5^{*\,2}(\lambda_6-\lambda_7)(\lambda_6+\lambda_7)^3] \\
&\quad +2\mathrm{Im}(\lambda_7^*\lambda_6)\left[|\lambda_5|^2[|\lambda_6|^2+|\lambda_7|^2-(\lambda_1-\lambda_2)^2] - 2(|\lambda_6|^2-|\lambda_7|^2)^2\right] \\
&\quad +(\lambda_1-\lambda_2)\mathrm{Im}\Big[[\Lambda^* - 2\lambda_5^*(\lambda_6+\lambda_7)](\lambda_7-\lambda_6)(\lambda_7^*\lambda_6-\lambda_6^*\lambda_7)\Big] \\
&\quad -(\lambda_1-\lambda_2)\mathrm{Im}(\lambda_5^*\Lambda^2) - 2(|\lambda_6|^2-|\lambda_7|^2)\mathrm{Im}[\lambda_5^*\Lambda(\lambda_6+\lambda_7)] \\
&\quad +(\lambda_1-\lambda_2)|\lambda_5|^2\mathrm{Im}[\lambda_5^*(\lambda_6+\lambda_7)^2]\,,
\end{aligned}
$$

$$
\begin{aligned}
I_{YZZZ} &= 2(|\lambda_6|^2-|\lambda_7|^2)\mathrm{Im}[Y_{12}(\lambda_6^*+\lambda_7^*)] \\
&\quad +(Y_{11}-Y_{22})\Big[\mathrm{Im}[\lambda_5^*(\lambda_6+\lambda_7)^2] - (\lambda_1-\lambda_2)\mathrm{Im}(\lambda_7^*\lambda_6)\Big] \\
&\quad +(\lambda_1-\lambda_2)\Big[\mathrm{Im}(Y_{12}\Lambda^*) - \mathrm{Im}[Y_{12}\lambda_5^*(\lambda_6+\lambda_7)]\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
I_{YYZZ} &= (\lambda_1-\lambda_2)\mathrm{Im}(Y_{12}^2\lambda_5^*) - 2|Y_{12}|^2\mathrm{Im}(\lambda_7^*\lambda_6) \\
&\quad -\mathrm{Im}[(Y_{12}\lambda_6^*)^2] + \mathrm{Im}[(Y_{12}\lambda_7^*)^2] + (Y_{11}-Y_{22})^2\mathrm{Im}(\lambda_7^*\lambda_6) \\
&\quad -(Y_{11}-Y_{22})\left[\mathrm{Im}(Y_{12}\Lambda^*) + \mathrm{Im}(Y_{12}\lambda_5^*(\lambda_6+\lambda_7))\right]\,,
\end{aligned}
$$

where

$$
\Lambda \equiv (\lambda_2-\lambda_3-\lambda_4)\lambda_6 + (\lambda_1-\lambda_3-\lambda_4)\lambda_7\,.
$$

The expression for I_{3Y3Z} is very long and will not be given here.

Remarks

- All invariants of cubic order or less are manifestly real.
- The imaginary part of any potentially complex quartic or quintic invariant is a real linear combintation of I_{YZZZ} and I_{YYZZ}.
- The imaginary part of any potentially complex sixth order invariant constructed solely out of the Higgs self-couplings is proportional to I_{6Z}.
- The imaginary part of any potentially complex sixth order invariant that is both cubic in Y and Z respectively is a real linear combination of I_{YZZZ}, I_{YYZZ} and I_{3Y3Z}.
- The imaginary part of any other potentially complex sixth order invariant is a real linear combintations of I_{YZZZ} and I_{YYZZ}.
- The imaginary part of any potentially complex invariant of order seven or above is a real linear combinations of I_{YZZZ}, I_{YYZZ}, I_{6Z} and I_{3Y3Z}.

To see that all four invariants introduced above are required, we first note that there always exists a basis in which $\lambda_7 = -\lambda_6$. [Proof: noting that

$$Z^{(1)} = \begin{pmatrix} \lambda_1 + \lambda_4 & \lambda_6 + \lambda_7 \\ \lambda_6^* + \lambda_7^* & \lambda_2 + \lambda_4 \end{pmatrix}$$

is an hermitian matrix, we can always diagonalize it.] In the $\lambda_7 = -\lambda_6$ basis (this basis is not unique),

$$I_{6Z} = -(\lambda_1 - \lambda_2)^3 \mathrm{Im}(\lambda_5^* \lambda_6^2)\,,$$

$$I_{YZZZ} = -(\lambda_1 - \lambda_2)^2 \mathrm{Im}(Y_{12} \lambda_6^*)\,,$$

$$I_{YYZZ} = (\lambda_1 - \lambda_2) \left[\mathrm{Im}(Y_{12}^2 \lambda_5^*) + (Y_{11} - Y_{22}) \mathrm{Im}(Y_{12} \lambda_6^*) \right]\,.$$

First, suppose that $\lambda_1 \neq \lambda_2$. Then consider three cases:

1. $Y = 0$ $[\Longrightarrow I_{YZZZ} = I_{YYZZ} = I_{3Y3Z} = 0]$
2. $\lambda_6 = 0$ and $Y_{11} = Y_{22}$ $[\Longrightarrow I_{6Z} = I_{YZZZ} = I_{3Y3Z} = 0]$
3. $\lambda_5 = 0$ and $Y_{11} = Y_{22}$ $[\Longrightarrow I_{6Z} = I_{YYZZ} = 0]$.

In case 3, $I_{3Y3Z} = 0$ if in addition $\mathrm{Re}(Y_{12} \lambda_6^*) = 0$. Then, in each case there is only one potentially complex invariant.

In a basis where $\lambda_6 = -\lambda_7$,

$$
\begin{aligned}
I_{3Y3Z} &= 2\mathrm{Im}(Y_{12}^3 \lambda_6 (\lambda_5^*)^2) - 4\mathrm{Im}(Y_{12}^3 (\lambda_6^*)^3) \\
&\quad + [(Y_{11} - Y_{22})^2 - 6|Y_{12}|^2](Y_{11} - Y_{22})\mathrm{Im}(\lambda_6^2 \lambda_5^*) \\
&\quad - (\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)(Y_{11} - Y_{22})\mathrm{Im}(Y_{12}^2 (\lambda_6^*)^2) \\
&\quad + (\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)\mathrm{Im}(Y_{12}^3 \lambda_5^* \lambda_6^*) \\
&\quad - (\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)\left[(Y_{11} - Y_{22})^2 - |Y_{12}|^2\right]\mathrm{Im}(Y_{12} \lambda_6 \lambda_5^*) \\
&\quad + 2(2|\lambda_6|^2 - |\lambda_5|^2)\left[(Y_{11} - Y_{22})^2 - |Y_{12}|^2\right]\mathrm{Im}(Y_{12} \lambda_6^*) \\
&\quad + \left[(\lambda_1 - \lambda_3 - \lambda_4)(\lambda_2 - \lambda_3 - \lambda_4) + 2|\lambda_6|^2 - |\lambda_5|^2\right] \\
&\qquad\qquad \times (Y_{11} - Y_{22})\mathrm{Im}(Y_{12}^2 \lambda_5^*)\,.
\end{aligned}
$$

If $\lambda_6 = 0$ and $Y_{11} = Y_{22}$, then $I_{3Y3Z} = 0$. In this case, only I_{YYZZ} is potentially complex.

If $\lambda_5 = 0$ and $Y_{11} = Y_{22}$, then

$$I_{3Y3Z} = -16\,[\mathrm{Re}(Y_{12}\lambda_6^*)]^2\,\mathrm{Im}(Y_{12}\lambda_6^*)\,.$$

Thus, if in addition $\mathrm{Re}(Y_{12}\lambda_6^*) = 0$, then only I_{YZZZ} is potentially complex.

If $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$ then $I_{6Z} = I_{YZZZ} = I_{YYZZ} = 0$. Nevertheless, CP can still be violated if $I_{3Y3Z} \neq 0$.

The case of $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$ is noteworthy, since if these relations are true in one basis then they are true in all bases. Moreover, since $I_{6Z} = 0$ in this case, one can conclude that if $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$ then there must exist a basis in which λ_5, λ_6 and λ_7 are all real. In this basis only $Y_{12} \equiv -m_{12}^2$ is potentially complex and we find:

$$
\begin{aligned}
I_{3Y3Z} = 2\left[\lambda_5^2 + \lambda_5(\lambda_1 - \lambda_3 - \lambda_4) - 2\lambda_6^2\right] \,\mathrm{Im}\, Y_{12} \\
\times \Big[4\lambda_6\,(\mathrm{Re}\, Y_{12})^2 - (Y_{11} - Y_{22})(\lambda_3 + \lambda_4 + \lambda_5 - \lambda_1)\,\mathrm{Re}\, Y_{12} \\
-(Y_{11} - Y_{22})^2\lambda_6\Big] \,.
\end{aligned}
$$

Note that, *e.g.*, if $\lambda_5^2 + \lambda_5(\lambda_1 - \lambda_3 - \lambda_4) - 2\lambda_6^2 = 0$, then it is possible to have $\mathrm{Im}\, m_{12}^2 = 0$ (and all the λ_i zero) and yet the Higgs potential is CP-conserving! This would then imply that it should be possible to transform to another basis in which *all* the Higgs potential parameters are simultaneously real. Numerical experiments suggest that this is always possible.

Conditions for a CP-conserving Higgs sector

If $I_{6Z} = I_{YZZZ} = I_{YYZZ} = I_{3Y3Z} = 0$, then the Higgs potential is CP-conserving. This means that there exists a basis in which all Higgs potential parameters are real. If the vev is complex in this basis, then the model exhibits spontaneous CP-violation. The corresponding basis-independent conditions have been given by Lavoura and Silva and by Botella and Silva. They define two invariants:

$$I_1 \equiv v_{\bar{a}}^* Y_{a\bar{b}} Z_{b\bar{c}c\bar{d}} v_d \,, \qquad I_2 \equiv v_{\bar{b}}^* v_{\bar{e}}^* Y_{b\bar{c}} Y_{e\bar{f}} Z_{c\bar{a}f\bar{d}} v_a v_d \,.$$

Then, the Higgs sector (excluding the couplings to fermions) conserves CP if and only if I_1 and I_2 are real. Evaluating the invariants I_1 and I_2 in the Higgs basis and applying the scalar potential minimum condition, one finds

$$\mathrm{Im}(Z_6 Z_7^*) \propto \mathrm{Im}\ I_1 \,, \qquad \mathrm{Im}(Z_6^2 Z_5^*) \propto \mathrm{Im}\ I_2 \,.$$

These authors also define additional invariants that include the Higgs-fermion Yukawa matrices. One can then write down the appropriate basis-independent conditions for a completely CP-conserving Higgs sector.

Unfinished business

• Express all Higgs couplings in terms of invariants in the most general CP-violating 2HDM (and explore the approach to the decoupling limit).

• Evaluate the one-loop radiative corrections to various Higgs processes in terms of the so-called physical $\tan\beta$-like parameters. (There are three such parameters in the one-generation model.)

• Which $\tan\beta$-like parameters will be measured in precision Higgs studies at a linear collider (LC)? How can one best treat the full three-generation model to one loop order? What simplications can be exploited in the MSSM?

• What is the physical significance of the four explicit CP-violating invariants? Which (if any) could be measured in precision Higgs studies at an LC?