Magic Tricks for Scattering Amplitudes

Z. Bern

KITP, January, 2004

Abstract

Feynman diagram calculations are generally extremely complicated. Yet there can be a hidden beauty. Here we discuss methods for uncovering this.

Some Review Articles:
M. Mangano and S.J. Parke, Phys. Rept. 200:301,1991
Z. Bern, hep-ph/9304249
L. Dixon, hep-ph/9601359
Z. Bern, L. Dixon and D. Kosower, hep-ph/9602280
Z. Bern, gr-qc/0206071

Outline

- Motivation

  (a) "Industrial" calculations – collider physics program

  (b) Uncovering hidden properties of gauge and gravity theories

- Surprising structures

  (a) Simple formulas for sums of large numbers of Feynman diagrams

  (b) Gravity $\sim$ (gauge theory) $\times$ (gauge theory)

  (c) Curves in twistor space – link to topological string theory

- Tricks of the trade

  (a) Helicity

  (b) Supersymmetry

  (c) String theory ideas

  (d) Unitarity and analytic properties

  (e) Guessing

  (f) Twistor space – latest magic.

- Examples of amplitudes with arbitrary numbers of legs

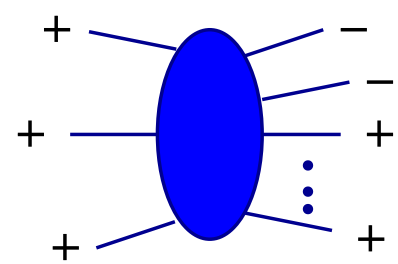
Motivation

Recall Edward Witten's talk

Gauge theory scattering amplitudes $\leftrightarrow$ topological string theory.

QCD Parke-Taylor helicity scattering amplitudes played a prominent role:

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}$$



Plenty of other known examples in gauge and gravity theories.

What magic tricks were used to obtain these?

Infinite numbers of Feynman diagrams summed.

Motivation

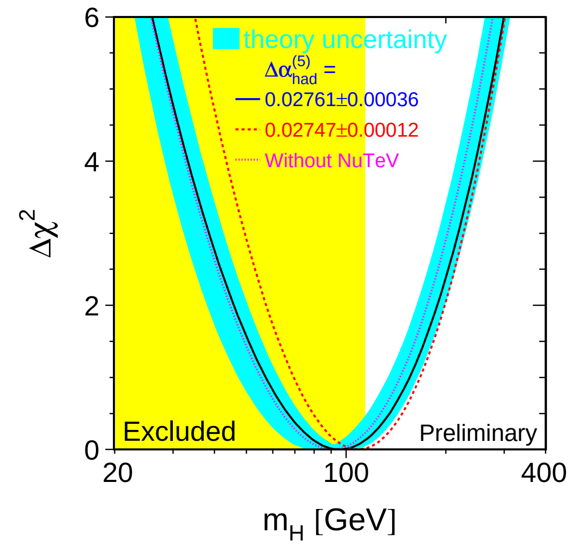
Applications of Feynman diagrams to collider physics

The quest for precision

- Uncover deviations from the Standard Model.
- Match experimental precision.

  From LEP: $\alpha_s = 0.121 \pm 0.001(\text{exp}) \pm 0.006(\text{theory})$ Bethke (2000)
- Need multi-leg scattering amplitudes because α_s is large (+ large logs).
- Constrain new physics: Higgs boson $M_H \leq 200$ GeV (95% CL)

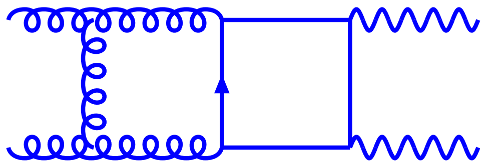
As long as there are colliders we need to push the theoretical precision.



This talk is more about investigating the structure of field theory

Major Advance of Past Few Years

Generic two-loop computations involving more than 1 kinematic variable is a new art only a few years old.



New loop integration technology!

Key to Progress

In the past few years the field of high loop computations has gotten a tremendous boost due to the influx of energetic bright young people.

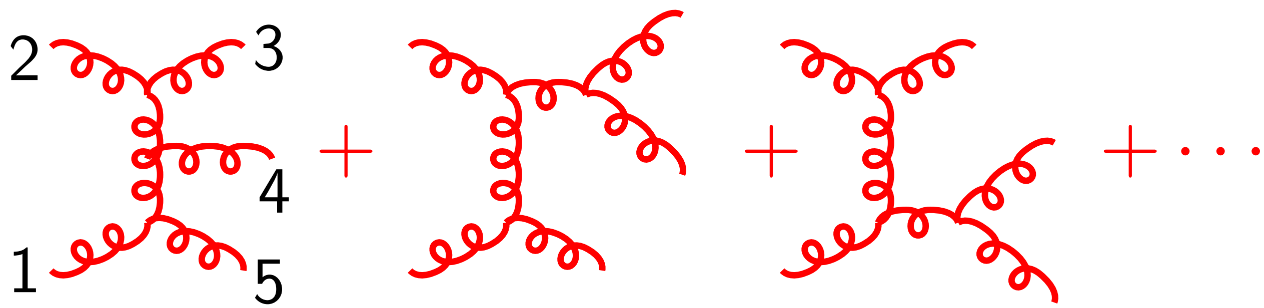
Babis Anastasiou, Andrzej Czarnecki, Daniel de Florian, Thomas Gehrmann, Massimiliano Grazzini, Robert Harlander, Gudrun Heinrich, Bill Kilgore, Pierpaolo Mastrolia, Kirill Melnikov, Sven Moch, Zoltan Nagy, Carlo Oleari, Matthias Steinhauser, Peter Uwer, Doreen Wackeroth, Stefan Weinzierl, and many others

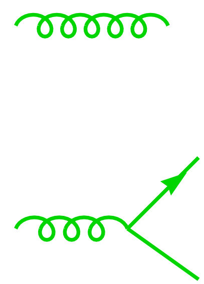
One major goal of the KITP collider program is to apply this breakthrough to improving theoretical precision at colliders.

Example of hidden structure

Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

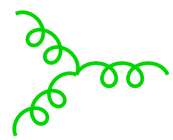
Described by following Feynman diagrams:





$\frac{\eta_{\mu\nu}}{p^2}$

$g\gamma_\mu$



$g[(p_1 - p_2)_\rho \eta_{\mu\nu} + (p_2 - p_3)_\mu \eta_{\nu\rho} + (p_3 - p_1)_\nu$

If you evaluate these using textbook methods you will only discover that this is a very disgusting mess.

Result of a brute force calculation (actually only a small part of it):

$$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$$

Chinese Magic

Vector polarizations

Xu, Zhang and Chang
F.A.Berends, R.Kleiss, P.De Causmaecker
R. Gastmans and T. T. Wu
J.F. Gunion and Z. Kunszt
& many others

$$\varepsilon_\mu^+(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\, k \rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\, q]}$$

More sophisticated version of circular polarization: $\varepsilon_\mu^\pm = (0, 1, \pm i, 0)$

All required properties of polarization vectors satisfied:

$$\varepsilon_i^2 = 0\,, \qquad k \cdot \varepsilon(k,q) = 0\,, \qquad \varepsilon^+ \cdot \varepsilon^- = -1$$

Notation

$$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\, l \rangle = \langle k_{j\,-} | k_{l\,+} \rangle = \sqrt{2k_j \cdot k_l}\; e^{i\phi}$$

$$\epsilon_{\dot a \dot b}\tilde\lambda_j^{\dot a}\tilde\lambda_l^{\dot b} \longleftrightarrow [j\, l] = \langle k_{j\,+} | k_{l\,-} \rangle = -\sqrt{2k_j \cdot k_l}\; e^{-i\phi}$$

Changes in reference momentum q are equivalent to gauge transformations.

Graviton polarization tensors are the squares of these!

$$\varepsilon_{\mu\nu}^{++} = \varepsilon_\mu^+ \varepsilon_\nu^+\,, \qquad 2 = 1 + 1$$

Five Gluon Results with Helicity

Following contains the complete physical content as the messy formula:.

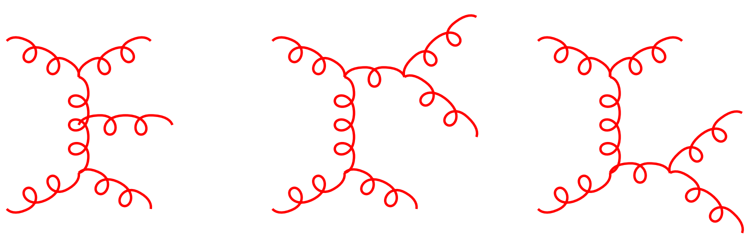
$$A_5(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5(1^-, 2^-, 3^+, 4^+, 5^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}$$

These are color stripped amplitudes.

$$\mathcal{A}_5(1,2,3,4,5) = \sum_{\rm perms} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})\, A_5(1^-, 2^-, 3^+, 4^+, 5^+)$$

Motivated by the Chan-Paton factors of open string theory. Mangano and Parke

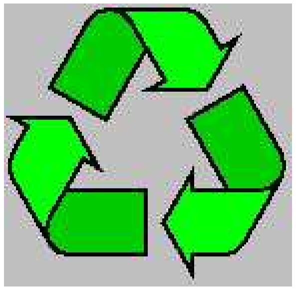


Feynman diagrams scramble together kinematics and color.

General Themes

Throughout this talk there will two general themes:

- Deeper theoretical understanding $\longrightarrow$ calculational improvements.
- Recycling is good!



Examples:

1. Helicity
2. Supersymmetry and applications to QCD
3. Recursive methods
4. Unitarity sewing method – quantum loops from trees
5. Gravity amplitudes from gauge theory ones
6. Twistor space – uncovers previously unknown important structure.

Factorization Properties

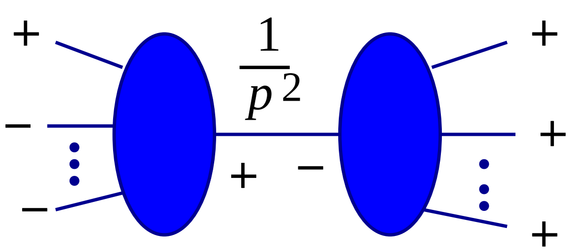
Parke and Taylor guessed the n-point maximally helicity violating (MHV) amplitudes:

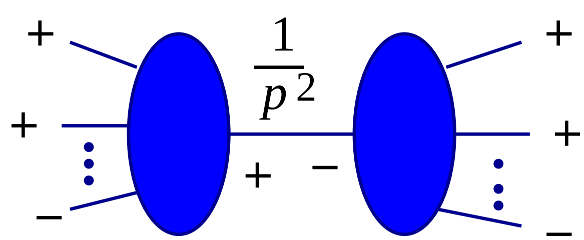
$$A_n(1^{\pm}, 2^+, 3^+, \ldots, n^+) = 0$$

$$A_n(1^-, 2^-, 3^+, 4^+, n^+) = i \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n\,1\rangle}$$

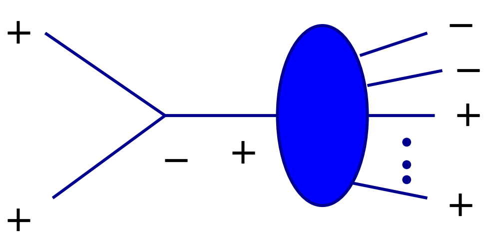
Tree amplitudes must satisfy very stringent properties.

Every pole corresponds to a propagating physical particle.





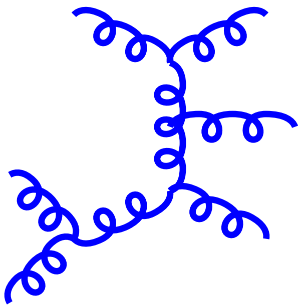
No multi-particle poles!

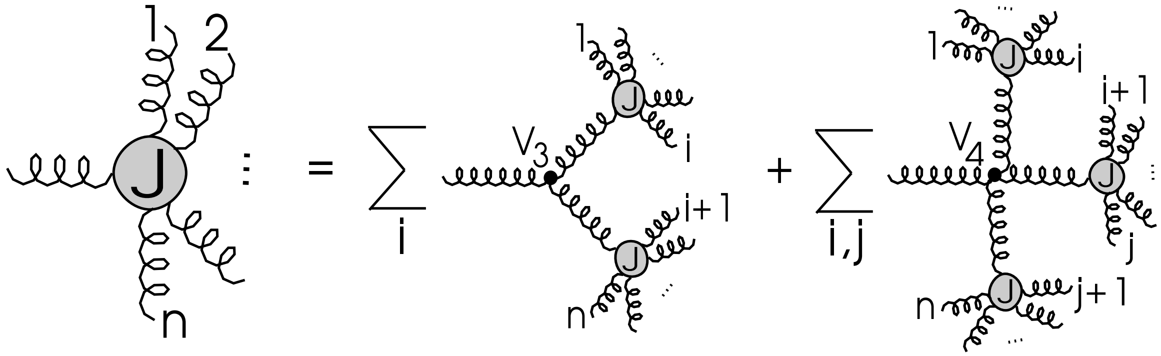


Collinear & soft singularities universal!

Berends-Giele Recursion Relations

Feynman diagram beg to be evaluated recursively





J^μ is the Berends-Giele current. For MHV can solve analytically!

$$J^\mu(1^-,2^+,\ldots,n^+) \;=\; \frac{\langle 1^-|\gamma^\mu \not{P}_{2,n}|1^+\rangle}{\sqrt{2}\,\langle 1\,2\rangle\cdots\langle n\,1\rangle} \sum_{m=3}^{n} \frac{\langle 1^-|\not{k}_m \not{P}_{1,m}|1^+\rangle}{P^2_{1,m-1}P^2_{1,m}}\,,$$

Dotting with ε^- on the free leg and cleaning up gives:

$$A_n^{\rm tree}(1^-,2^-,3^+,4^+,\ldots,n^+) \;=\; i\,\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}$$

Parke-Taylor amplitude is proven!

Infinite number of Feynman diagrams solved at once!

Some applications of recursive methods:

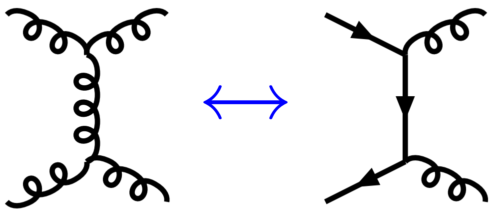
- Proof of Parke-Taylor formula — Berends and Giele
- Amplitudes with three negative helicities — Kosower
- Numerical evaluation of high point tree amplitudes — Berends, Giele, Kuijf; Mangano et al
- MHV gauge theory loop amplitudes — Mahlon

Supersymmetry

Grisaru, Pendleton and van Nieuwenhuizen
S.J. Parke and T. Taylor; Z. Kunszt

Supersymmetry relates bosons and fermions.

Does susy exist in nature? Not yet known.



Susy teaches us important properties about amplitudes.

Difference between $N = 1$ super-Yang-Mills theory and QCD?

QCD Quarks: Fundamental color representation.

Gluinos: Adjoint color representation.

But we already saw: color can be stripped away.

$$[Q(p), g^{\pm}(k)] = \mp\Gamma^{\pm}(k,p)\,\tilde{g}^{\pm}(k)\,, \qquad [Q(p), \tilde{g}^{\pm}(k)] = \mp\Gamma^{\mp}(k,p)\,g^{\pm}(p)$$

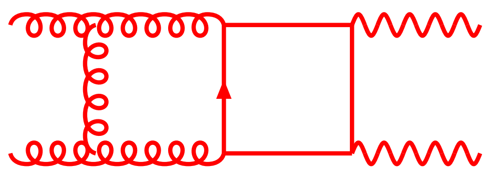
$$\langle 0|[Q, g^- g^- \tilde{g}^+ g^+ g^+]|0\rangle = 0$$

$$A_5(1_g^-, 2_q^-, 3_{\bar{q}}^+, 4_g^+, 5_g^+) = \frac{\langle 1\,3\rangle}{\langle 1\,2\rangle} A_5(1_g^-, 2_g^-, 3_g^+, 4_g^+, 5_g^+)\,.$$

Sample Susy Applications

- Relate gluon amplitudes to simpler scalar amplitudes Parke and Taylor (1985)
- Used to obtain 6 gluon non-MHV amplitudes from quark amplitudes back in the days when this was really tough* – 220 Feynman diagrams.

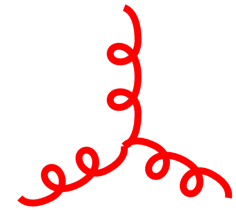
  Z. Kunszt (1985)
- Check on 4-loop QCD β-function computed by van Ritbergen, Vermaseren and Larin (1998). Jack, Jones, North (1997)
- Check on two-loop $gg \to gg$ QCD amplitudes ZB, De Freitas, Dixon (2002)



* Today, using the very latest "twistor space" wizardry you can do each helicity on the back of an envelope.

Gravity

Consider the gravity and Yang-Mills Lagrangians:



$$\mathcal{L}_{\rm gravity} = \sqrt{g}\, R\,, \qquad \qquad \mathcal{L}_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}$$

Yang-Mills three vertex: $V^{abc}_{3\mu\nu\rho}(k_1,k_2,k_3) = f^{abc}\,(k_1-k_2)_\rho \eta_{\mu\nu} +$ cyclic

Compare to gravity

$$\begin{aligned} &G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\ &\quad {\rm sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2 \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2 \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ &\qquad + P_6(k_1\cdot k_2 \eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ &\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ &\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2 \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$

Naive conclusions: (a) Gravity is an unholy mess and (b) perturbative expansions of two theories have little to do with each other.

But this can't be true! String theory unifies gravity and gauge theory.

String Theory Intuition

Basic string theory fact:

$$\text{closed string} \sim (\text{left-mover open string}) \times (\text{right-mover open string})$$

In the field theory or infinite string tension limit this should imply

$$\text{gravity} \sim (\text{gauge theory}) \times (\text{gauge theory})$$

1) How do we make this precise?

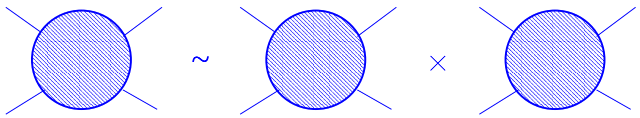
2) How can we exploit this?

3) How can this be understood from the Einstein-Hilbert Lagrangian?

W. Siegel hep-th/9308133; Z. Bern and A. Grant hep-th/9904026

Kawai-Lewellen-Tye Tree-Level Relations

At tree-level, KLT (1985) presented some remarkable relations between closed and open string amplitudes.



In the field theory limit ($\alpha' \to 0$)

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

$$s_{ij} = (k_i + k_j)^2$$

where we have stripped all coupling constants. M_n is gravity amplitude and A_n is color stripped gauge theory amplitude.

$$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

These relations hold for any external string states.

Explicit all n formula: hep-th/9811140 Appendix A

Also holds for classes of higher dimension operators.

Niels Emil Bjerrum-Bohr
hep-th/0302131, hep-th/0305062

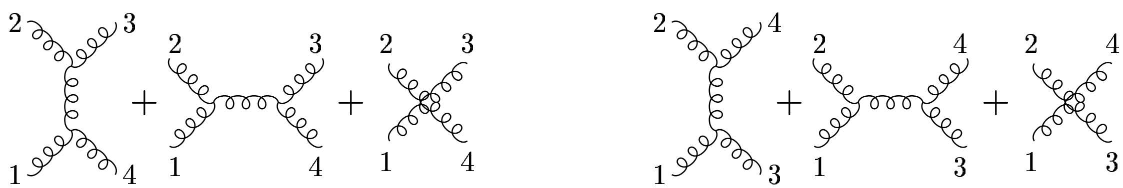
Magical Examples

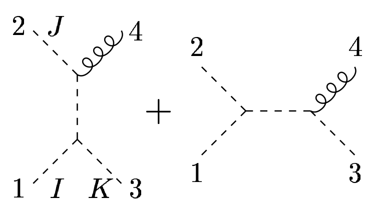
$$M_4^{\rm tree}(1_h^-, 2_h^-, 3_h^+, 4_h^+) = \left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\rm tree}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A_4^{\rm tree}(1_g^-, 2_g^-, 4_g^+, 3_g^+)$$

$$= \left(\frac{\kappa}{2}\right)^2 s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} \times \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,4\rangle \langle 4\,3\rangle \langle 3\,1\rangle}$$

$$M_4^{\rm tree}(1_g^-, 2_g^-, 3_g^+, 4_h^+) = g\frac{\kappa}{2} s_{12} A_4^{\rm tree}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A_4^{\rm tree}(1_s^I, 2_s^J, 4_g^+, 3_s^K)$$

$$= g\frac{\kappa}{2} s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} \times f^{IJK} \frac{[4\,3]\,\langle 3\,2\rangle}{\langle 2\,4\rangle}$$





All n generalizations

We already know the maximal helicity violation (MHV) pure gluon tree of QCD:

$$A_n(1^-,2^-,3^+,\ldots,n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

Gravity obtained by pushing above through KLT formulae.
After cleaning up: Berends, Giele and Kuijf

$$M_n^{\rm tree}(1^-,2^-,3^+,\ldots,n^+) = -i\,\langle 1\,2\rangle^8 \left[\frac{[1\,2]\,[n-2\;\;n-1]}{\langle 1\;\;n-1\rangle\;N(n)}\left(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle i\,j\rangle\right)\prod_{l=3}^{n-3}\left(-\langle n^-|\not{K}_{l+1,n-1}|l^-\rangle\right) + {\rm Perms}\right],$$

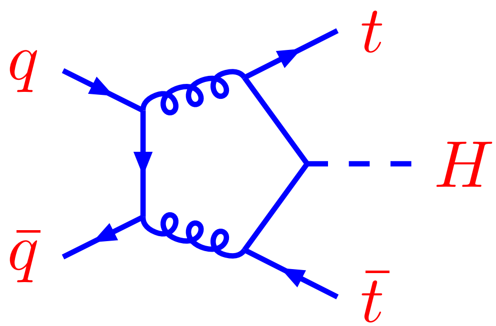
where $N(n) = \prod_{i<j}^{n}\langle i\,j\rangle$

Same idea works for gravity coupled to matter. Bern, De Freitas, Wong

Key idea: If you know a gauge theory tree amplitude, you immediately know corresponding gravity amplitudes!

State of the Art at One Loop

Five point is state of the art for generic calculations. $e.g., pp \to \bar{t}tH$.



Reina, Dawson and Wackeroth (2001)
Beenakker, Dittmaier, Kramer, Plumper, Spira (2001)

At 6 points complete answers only for very special theories: $N = 4$ supersymmetric Yang-Mills and the Yukawa Model.

Bern, Dixon, Dunbar and Kosower (1994)
Binoth, Guillet, Heinrich and Schubert (2001)

Arbitrary numbers of legs worked out in QCD, susy gauge theories and also in gravity theories, but limited to MHV helicity configurations.

Bern, Chalmers, Dixon, Kosower (1994); Mahlon (1994); Bern, Dixon, Dunbar and Kosower (1994)
Bern, Dixon, Perelstein, Rozowsky (1999)

Here I discuss mainly methods used to obtain arbitrary numbers of legs.

Application of String Theory

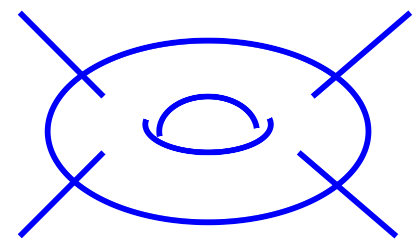
Bern and Kosower (1992)
Bern, Dixon and Kosower (1993)

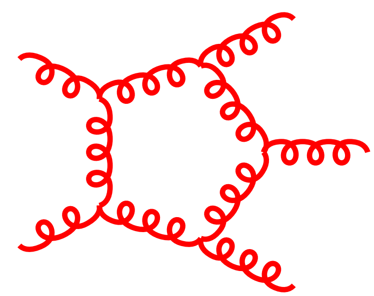
What can string theory teach us about loop level Feynman diagrams?

Every order of string perturbation theory has only one string diagram.

479 Feynman diagrams for $gg \to ggg$ with no apparent simple relation between diagrams.

Led to 'string-based' calculational rules which were used for first one-loop five point calculation: $gg \to ggg$.





- Application of helicity at loop level and 5 pt structure
- One-loop color decomposition
- Supersymmetry decompositions of loops.
  QCD = susy + non-susy.
- Non-linear gauges
- Gravity $\sim$ (gauge theory) $\times$ (gauge theory) at loops. Bern, Dunbar, Shimada
- Points to arbitrary numbers of legs.

Amplitudes via Unitarity

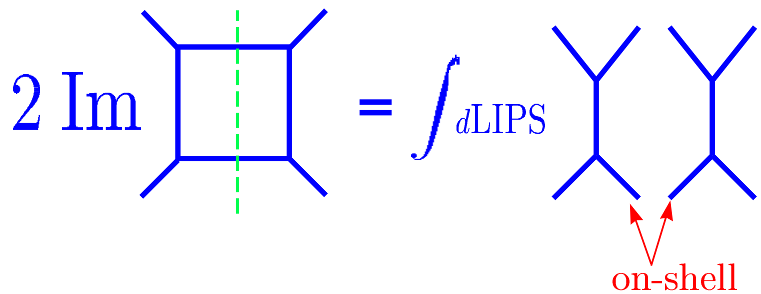
Bern, Dixon, Dunbar and Kosower
Bern and Morgan

Basic property: The scattering matrix is unitary: $S^\dagger S = 1$.

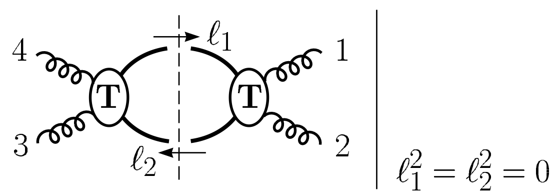
We will use this well known property of the S-matrix to obtain all quantum corrections.

Taking $S = 1 + i\,T$ gives

$2\,\text{Im}\,T = T^\dagger T$ or



To maintain gauge invariance, sum over all Feynman diagrams on either side of the cut.



From unitarity we can obtain the imaginary parts of loop amplitudes from tree amplitudes.

To obtain the complete quantum S-matrix we also need real parts, especially rational functions.

Generic form of a loop amplitude:

$$A \sim \ln(-s - i\epsilon) + \text{rational} + \text{other logs}$$
$$\sim \ln(s) - i\pi + \text{rational} + \text{other logs}$$

The $i\pi$ term is fixed by unitarity and the $\ln(s)$ can be reconstructed from this.

However rational terms seemingly can't be reconstructed.

Problem seems basic. Consider complex function

$$a(\ln(s) - i\pi) + b$$

You can get a from imaginary part but not b.

Trick: Use analytic properties as a functions of space-time dimension!

Analytic Properties for $D \neq 4$

Consider:

$$A_4^{\text{1-loop}}(1^+, 2^+, 3^+, 4^+) = \frac{1}{48\pi^2}$$

Has no imaginary part! How do we construct real rational parts from nothing?

Magic Trick: Continue the amplitude to $D = 4 - 2\epsilon$ dimensions.

From dimensional analysis in massless theories:

$$A^{D=4-2\epsilon} \sim \int d^{4-2\epsilon}p \, \cdots \sim \sum_i (s_i)^{-\epsilon} \times \text{rational}_i + \cdots$$

$$\sim \sum_i \text{rational}_i (1 - \epsilon \ln s_i) + \cdots$$

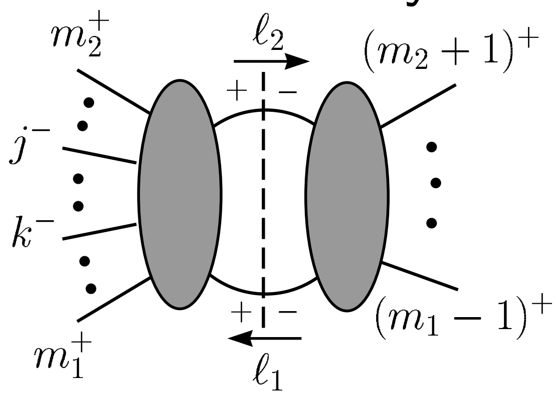
Thus:

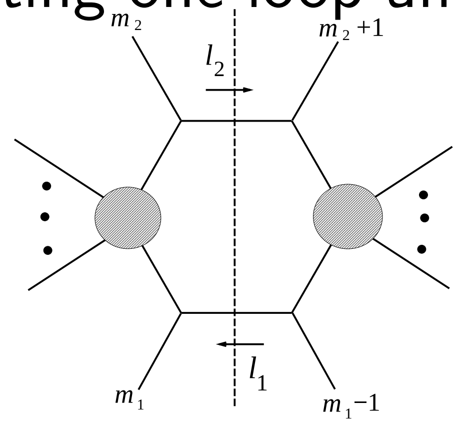
$$\text{rational} = \sum_i \text{rational}_i$$

From $\mathcal{O}(\epsilon)$ branch cuts can reconstruct $\mathcal{O}(\epsilon^0)$ rational terms.

Arbitrary Number of Legs at One Loop

Consider cuts of maximally helicity violating one-loop amplitudes.





Bern, Dixon

Dunbar and Kosower

The tree-level Parke-Taylor amplitudes for n gluons have a remarkable property:

$$A^{\rm tree}(\ell_1^+, m_1^+, \cdots, k^-, \cdots, j^-, \cdots, m_2^+, \ell_2^+)$$

$$= \frac{\langle k\, j\rangle^4}{\langle \ell_1\, m_1\rangle \langle m_1,\, m_1+1\rangle \cdots \langle m_2-1,\, m_2\rangle \langle m_2\, \ell_2\rangle \langle \ell_1\, \ell_1\rangle}$$

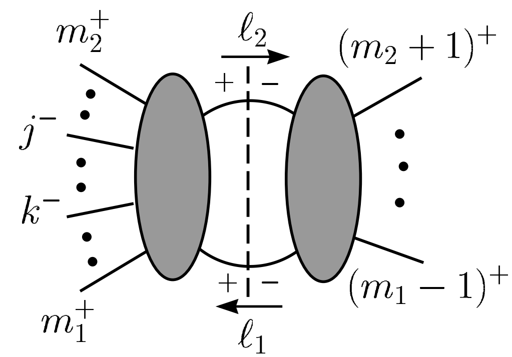
Only 2 denominators in each tree have non-trivial dependence on loop momentum.

Together with 2 cut propagators the 4 denominators from the trees give at worst a hexagon integral (which simplifies easily in susy cases).

At one loop calculated:

- All MHV amplitudes in maximal $N = 4$ super-Yang-Mills theory.
- All MHV amplitudes in $N = 1$ super-Yang-Mills
- All helicities for maximal $N = 4$ super-Yang-Mills at six=points.

Comments on QCD:



For QCD amplitudes, using Parke-Taylor amplitudes does not quite work.

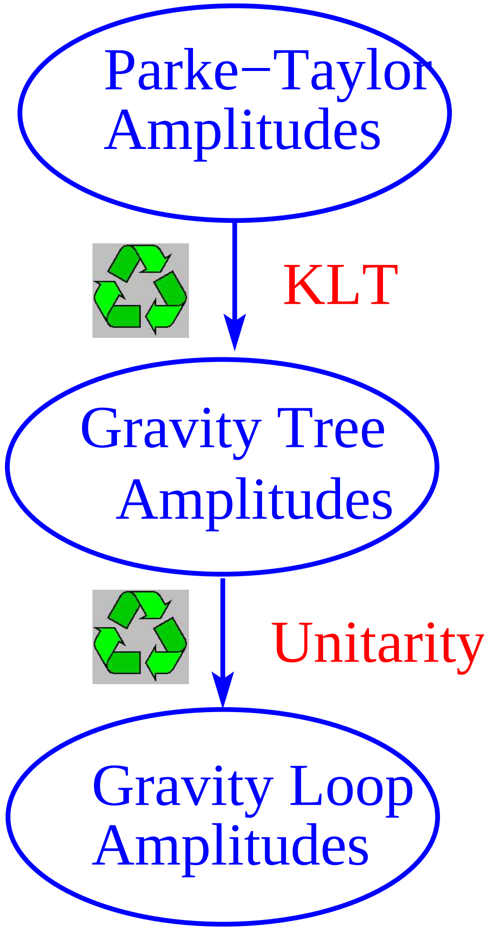
Must be careful about D-dimensional momenta

In susy theories it's OK because of better UV properties: $D = 4 - 2\epsilon$. If you make $\mathcal{O}(\epsilon)$ error it's OK as long as you don't hit $1/\epsilon$.

In QCD we calculated up to six-point allowing for an all-n guess, proven by Mahlon.

Gravity Loops

To obtain n-point gravity loops we combine the ideas.



At one-loop n-points:

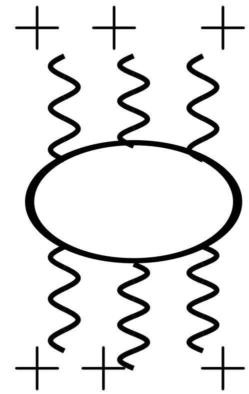
- $N = 8$ maximally susy gravity MHV amplitudes
  Bern, Dixon, Perelstein, Rozowsky
- Identical helicity Einstein gravity with any matter ($n \leq 6$). Rest were guessed.

At n-loops:

- Maximally susy gravity is less divergent in the UV than previously thought.

Bern, Dixon, Dunbar, Rozowsky and Perelstein
Howe and Stelle

Guessing answers

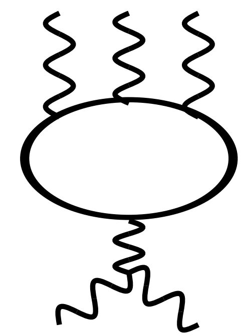


Consider the one-loop identical helicity n-photon amplitude of massless QED.

Can we guess the answer?

- Tree amplitude vanishes (photons don't couple to photons)
- The answer can have no logarithms. $D = 4$ unitarity cuts vanish.
- Dimension of amplitudes $\sim p^{4-n}$.

$$\sim \frac{1}{(\langle a_1\, a_2 \rangle)^{n-4}}$$



Does not exist

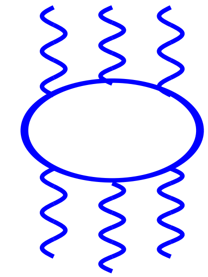
- There can be no kinematic poles in the answer.

What rational function with dimension p^{4-n} has no kinematic poles?

Bern, Chalmers, Dixon, Koswer

Mahlon

If you guessed $A_{n\gamma}(1^+, 2^+, \ldots, n^+) = 0 \qquad (n > 4)$



You are right!

For other helicities not so simple because easy to find combinations of logs and dilogs which vanish in factorization limits.

But it demonstrates the power of understanding the analytic properties of scattering amplitudes.

How about a more complicated example? Try 1 loop gravity.

A non-trivial guess

Bern, Dixon, Perelstein, Rozowsky

The one-loop all-plus helicity n-graviton amplitude of Einstein gravity was constructed by guessing.

Key: Universal soft graviton emission $\longrightarrow$ analytic properties.

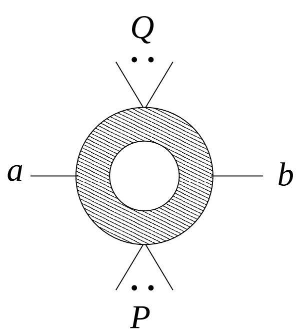
$$M_n^{\text{1-loop}}(1^+, 2^+, \cdots, n^+) = -i\frac{N_s}{(4\pi)^2}\frac{1}{2^{n+2}\cdot 240}\sum_{\substack{b>a\\P,Q}}^{n} h(a,P,b)h(b,Q,a)\operatorname{tr}^3[\not k_a \not P \not k_b \not Q]$$

where

$$h(a,\{1,2,\ldots,n\},b) \equiv (-1)^n \frac{[1\,2]}{\langle 1\,2\rangle}\frac{1}{\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,\, n\rangle}$$

$$\times\frac{\langle a^-|(1+2)|3^-\rangle\cdots\langle a^-|\not P_{1,n-1}|n^-\rangle}{\langle 1\,b\rangle\langle 1\,a\rangle\langle 2\,a\rangle\cdots\langle n-1,\, a\rangle\langle n\,a\rangle\langle n\,b\rangle}$$

$+$ Perms



The h function has simple properties as a graviton momentum becomes soft, $k_i \to 0$. N_s counts the number of bosonic minus fermionic states.

The ansatz has been proven for up to six external legs using unitarity methods.

Problem with guessing is that it only works in special cases.

Twistor Space

E. Witten hep-th/0312155

In a recent paper Witten showed that after Fourier transform to twistor space points lie on curves and a link made to a topological string theory.

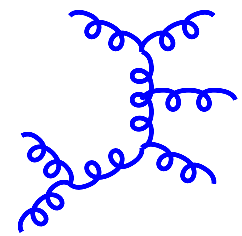
$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j \mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$$

Explicit link to the topological B model – explicit calculations

Roiban, Spradlin, Volovich

But can it help us calculate better? Decisively, Yes! Cachazo, Svrcek and Witten

Consider non-MHV amplitudes from 220 diagrams



$$A_6 = 8g^4 \left[\frac{\alpha^2}{t_{123} s_{12} s_{23} s_{45} s_{56}} + \frac{\beta^2}{t_{234} s_{23} s_{34} s_{56} s_{61}} + \frac{\gamma^2}{t_{345} s_{34} s_{45} s_{61} s_{12}} + \frac{t_{123}\beta\gamma + t_{234}\gamma\alpha + t_{345}\alpha\beta}{s_{12} s_{23} s_{34} s_{45} s_{56} s_{61}} \right]$$

e.g. for $A_6(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$

$$\alpha = 0, \qquad \beta = [23]\langle 56\rangle\langle 1|\not{k}_2 + \not{k}_3|4\rangle, \qquad \gamma = [12]\langle 45\rangle\langle 3|\not{k}_1 + \not{k}_2|6\rangle$$

It sure doesn't look simple! Hidden structure uncovered in twistor space.

Twistor Space Magic

Cachazo, Svrcek and Witten
forthcoming paper

The simple structure of the curves in twistor space for non-MHV amplitudes implies that there must be a way to express non-MHV in terms of MHV. (For details wait for the paper.)

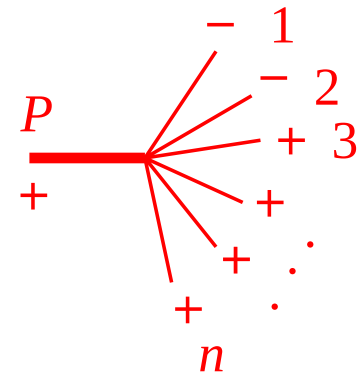
Here I want to show you practical consequences of this observation:

Continue spinor off-shell ($P^2 \neq 0$):
$$\langle j\, P\rangle = \eta \sum_{k=1}^{n} \langle j\, k\rangle\, [k\, q]$$

where $P = k_1 + k_2 + \cdots k_n$ and q auxiliary, satisfying $q^2 = 0$.

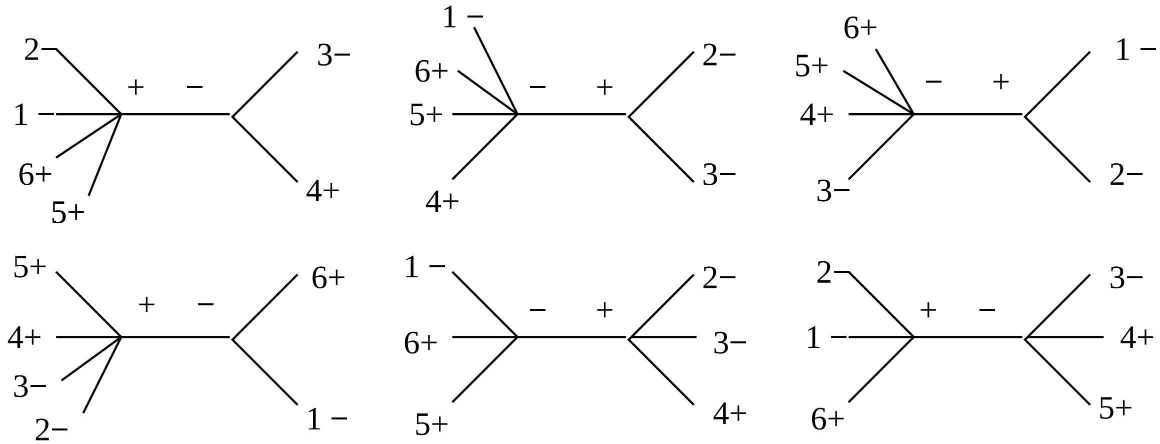
Use this to define an off-shell "MHV vertex"

$$V(1^-, 2^-, 3^+, \ldots, n^+, P^+) = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle \cdots \langle n-1,\, n\rangle\, \langle n\, P\rangle\, \langle P\, 1\rangle}$$



Build non-MHV amplitudes by sewing together MHV vertices.

Cachazo, Svrcek and Witten



$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle}$$

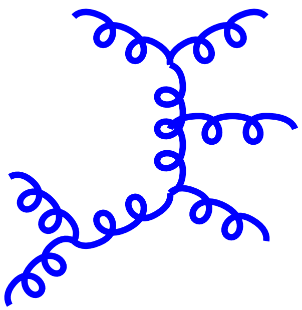
$$+ \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle}$$

$$+ \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle}$$

$$+ \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle}$$

$$+ \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle}$$

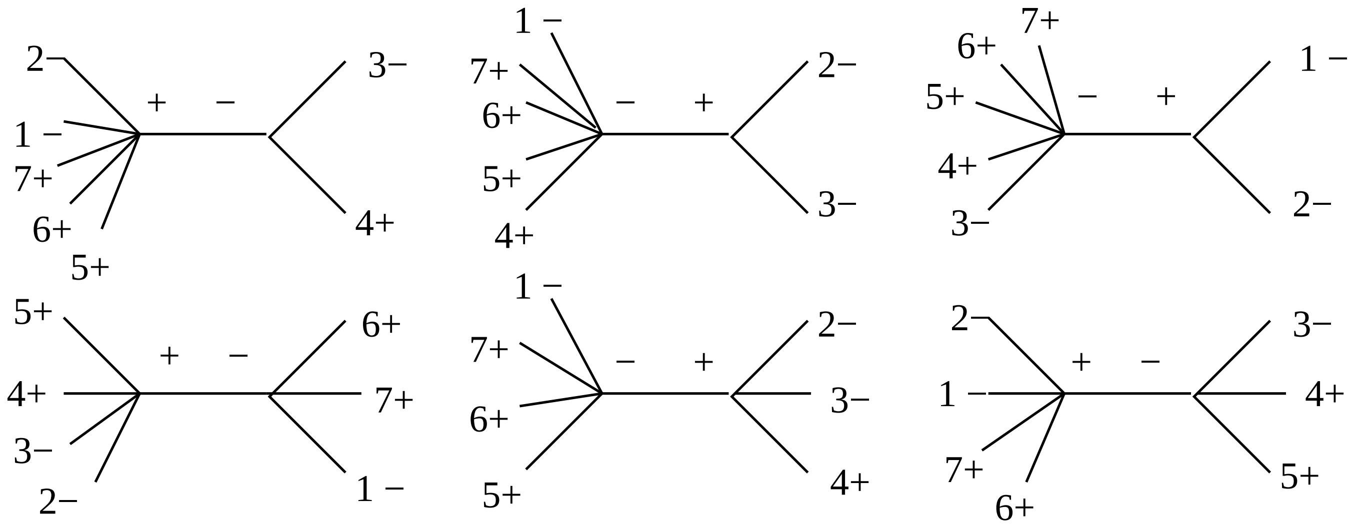
$$+ \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}$$

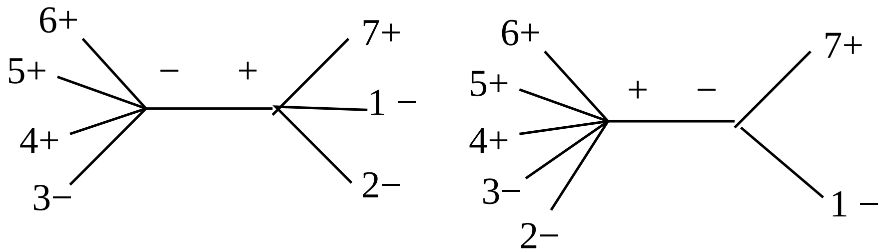


$$\langle 1|\,2+3\,|4\rangle \equiv \left\langle 1^- \middle|\, \not{k}_2 + \not{k}_3 \,\middle| 4^- \right\rangle$$

q arbitrary but null

Trivial to generalize to n-points. Consider 7 point case





Only 2 new diagrams.

For $- - - + + + \cdots +$ number of diagrams grow as $2(n-3)$.

It will be very interesting to explore the full consequences.

Summary

- Amplitudes with arbitrary numbers of legs – hidden dualities and symmetries.
- Gravity $\sim$ (gauge theory) $\times$ (gauge theory).
- Recycling is good.
- Deeper theoretical understanding $\rightarrow$ more efficient calculation – twistor space is the latest example.
- There is clearly much more structure to uncover in gauge theory and gravity perturbative expansions.