Harmony of Scattering Amplitudes: From Quantum Chromodynamics to Gravity

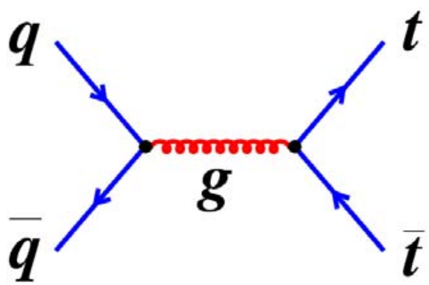
KITP Colloquium

December 10, 2008

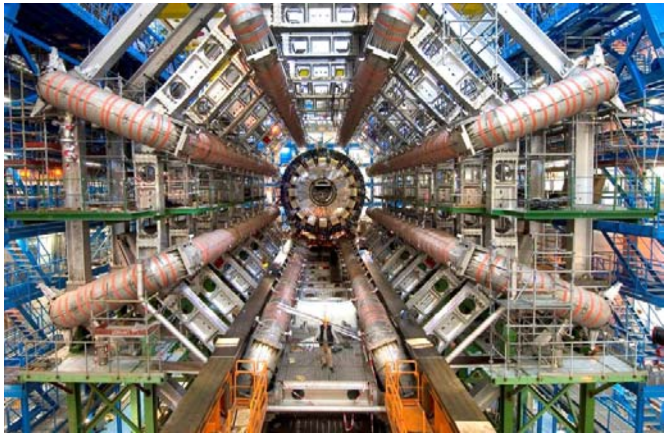
Zvi Bern, UCLA

Scattering amplitudes

Scattering particles is fundamental to our ability to unravel microscopic laws of nature.



Imminent arrival of the LHC raises importance of scattering amplitudes.

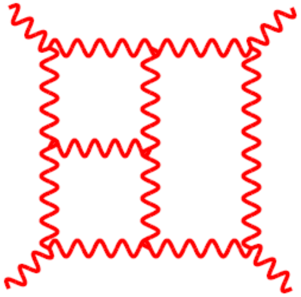


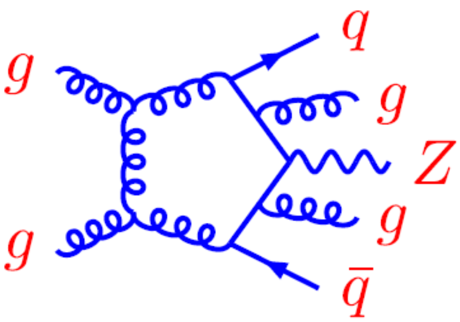
Here we discuss some theoretical developments on scattering in QCD, gravity and supersymmtric gauge theory

Outline

Will outline new developments in understanding scattering amplitudes. Surprising harmony.

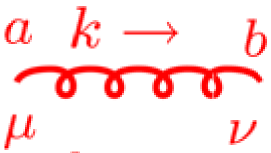
1. **Scattering from Feynman diagrams. Obscures harmony.**
2. **Modern on-shell methods for scattering.**
3. **QCD, Super-Yang-Mills theory and supergravity.**
4. **Applications:**
   - **— LHC Physics.**
   - **— AdS/CFT and N = 4 super-Yang-Mills.**
   - **— Reexamination of divergences in gravity theories.**



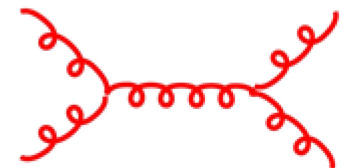


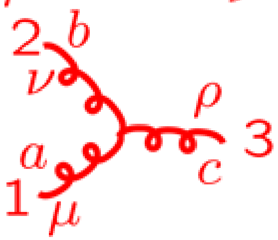
Scattering Amplitudes

Every graduate student in particle theory learns how to calculate scattering amplitudes via Feynman diagrams.



$$\frac{i}{k^2 + i\epsilon}$$





Coupling constant

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_2 - k_3)_\mu + \eta_{\rho\mu}(k_3 - k_1)_\nu)$$

color factor

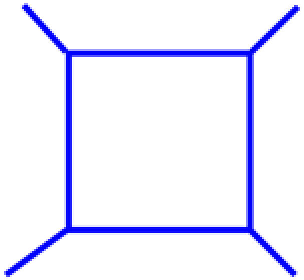
In principle this is a complete solution for small coupling

In practice not so easy:

- **Proper way to calculate in QCD? Asymptotic freedom, many scales, strong coupling, infrared safety, non-perturbative contributions, etc.**
- **Beyond the very simplest processes an explosion of complexity.**
- **Completely obscures the beauty and harmony.**

Example of difficulty

Consider a tensor integral:

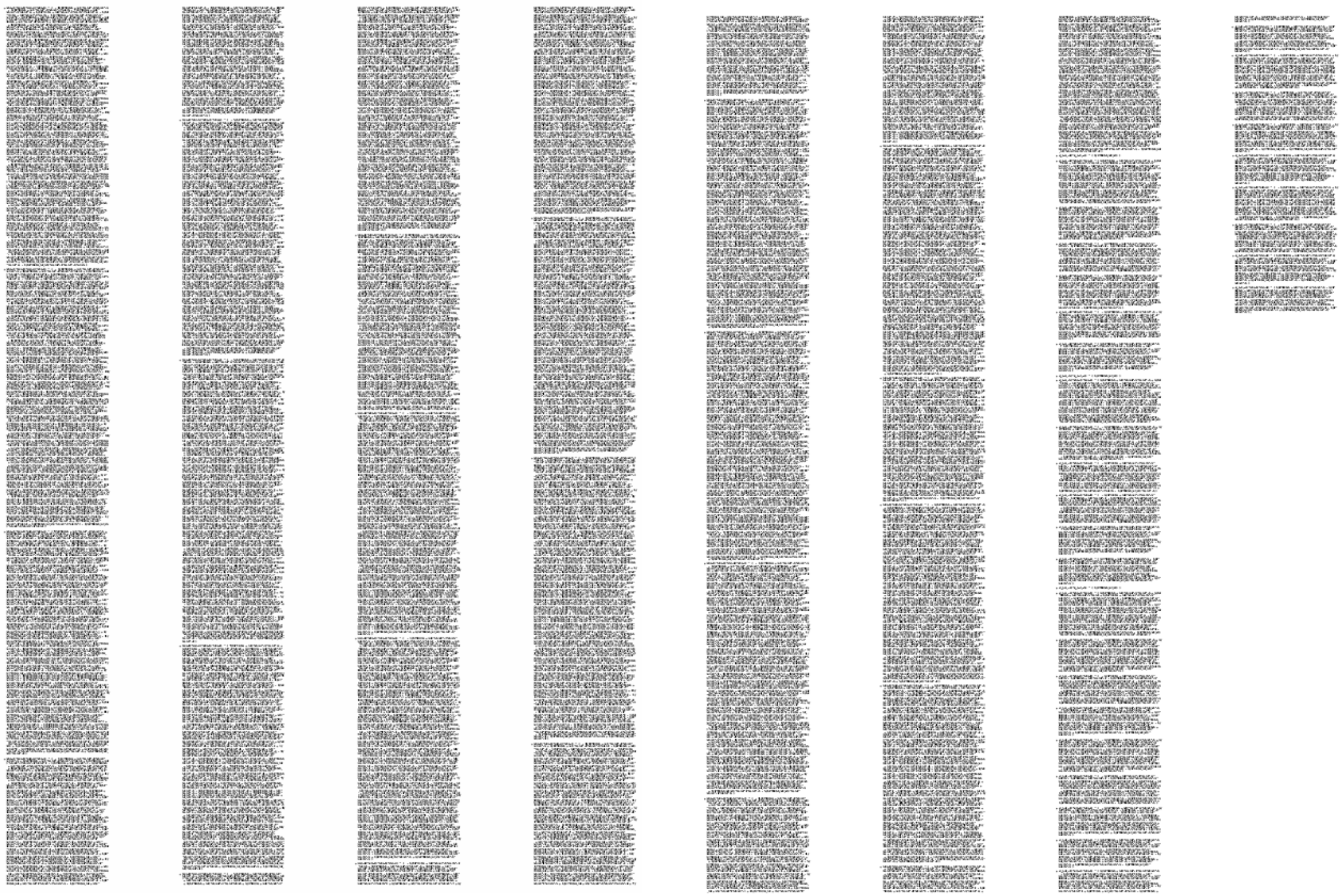


$$\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-\epsilon}} \frac{\ell^\mu \, \ell^\nu \, \ell^\rho \, \ell^\lambda}{\ell^2 \, (\ell - k_1)^2 \, (\ell - k_1 - k_2)^2 \, (\ell + k_4)^2}$$

Note: this is trivial on modern computer. Non-trivial for larger numbers of external particles.

Evaluate this integral via Passarino-Veltman reduction. Result is …

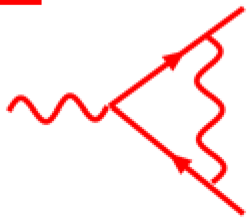
Result of performing the integration



Calculations explode for larger numbers of particles or loops. Clearly, there should be a better way

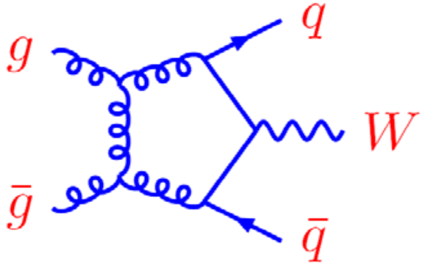
State-of-the-Art Loop Calculations

In 1948 Schwinger computed anomalous magnetic moment of the electron.



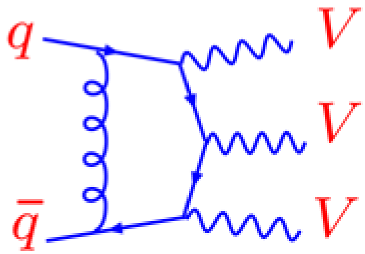
60 years later typical examples:

$pp \to W, Z + 2$ jets



ZB, Dixon, Kosower;
Dixon, Kunszt, Signer;
Campbell, Ellis (MCFM);
Febres Cordero, Reina, Wackeroth

$pp \to VVV \qquad V = Z, W$

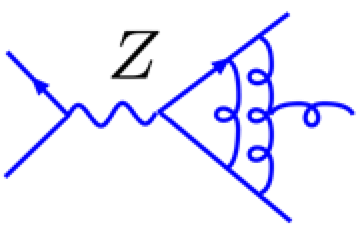


Lazopoulos, Petriello, Melnikov;
Binoth, Ossola, Papadopoulos, Pittau

No complete 6-point cross-section calculations in QCD, though serious progress described in this talk.

Two-loops: $e^+e^- \to 3$ jets state of the art.

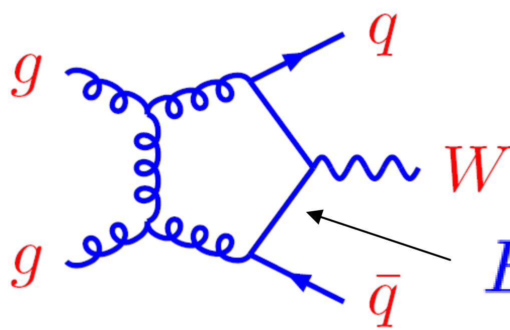
Gehrmann, Gehrmann-De Ridder, Glover, Heinrich; Weinzierl



Why are Feynman diagrams clumsy for high-loop or high-multiplicity processes?

- Vertices and propagators involve unphysical gauge-dependent off-shell states. An important origin of the complexity.

$$\int \frac{d^3\vec{p}\, dE}{(2\pi)^4}$$



Individual Feynman diagrams unphysical

$$E^2 - \vec{p}^{\,2} \neq m^2$$

Einstein's relation between momentum and energy violated in the loops. **Unphysical states! Not gauge invariant.**

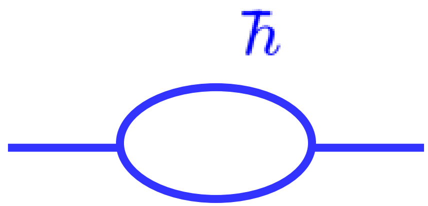
- **All steps should be in terms of gauge invariant on-shell physical states. On-shell formalism.**

ZB, Dixon, Dunbar, Kosower

Heisenberg

Feynman diagram loops violate on shellness because they encode the uncertainty principle.

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$



$$E^2 - \vec{p}^2 \neq m^2$$

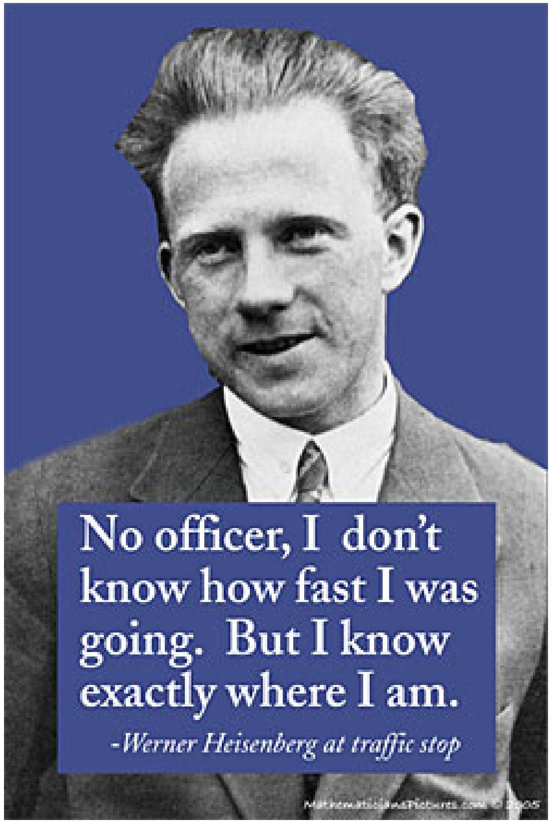
$$\int \frac{d^3\vec{p}\, dE}{(2\pi)^4}$$

You can create new particles even with insufficient energy as long as you destroy them quickly enough.

Theorem: Off-shellness or energy conservation violation is essential for getting the correct answer.

It looks like an on-shell formalism will fail to capture everything

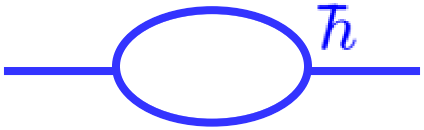
Sneaking Past Heisenberg



Favorite Theorem: **Theorems in physics are bound to be misleading or have major loopholes.**

What's the loophole here?

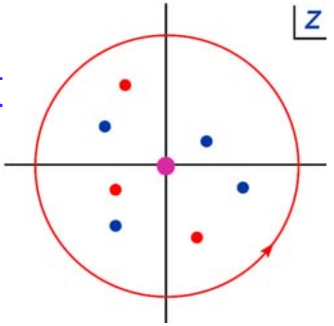
- **Want to reconstruct the complete amplitude using only on-shell physical information.**
- **Keep particles on-shell in intermediate steps of calculation, not in final results.**



On-Shell Recursion for Tree Amplitudes

Britto, Cachazo, Feng and Witten

Consider amplitude under complex shifts of the moment



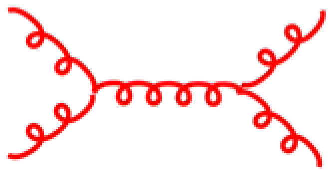
$$p_1^\mu(z) = p_1^\mu - z q^\mu \quad p_n^\mu(z) = p_n^\mu + z q^\mu \qquad q^2 = 0, \; p \cdot q = 0$$

$$(p_i^\mu(z))^2 = 0$$ complex momenta

If $A(z) \to 0, \quad z \to \infty$

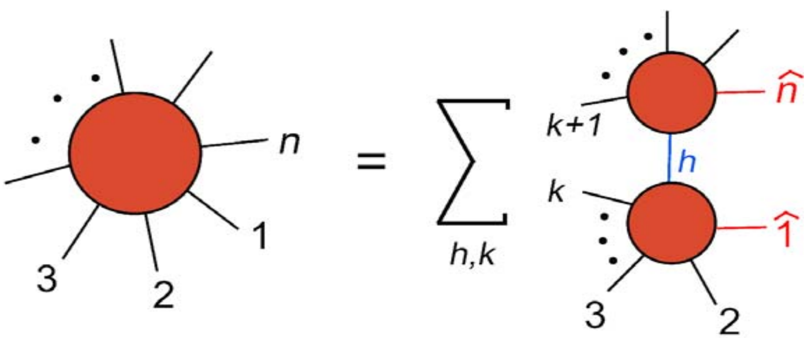
A(z) is amplitude with shifted momenta

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \mathrm{Res}_\alpha \frac{A(z)}{z}$$



$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

on-shell amplitude



Sum over residues gives the on-shell recursion relation

Poles in *z* come from kinematic poles in amplitude.

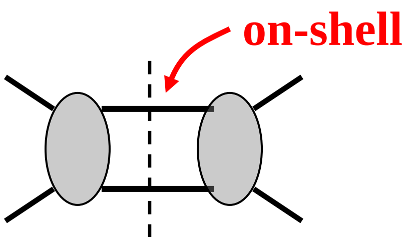
Same construction works in gravity

Brandhuber, Travaglini, Spence; Cachazo, Svrcek;
Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

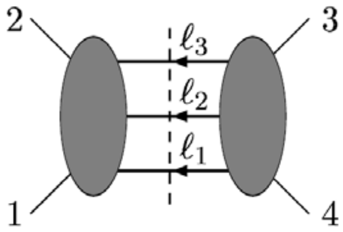
Bern, Dixon, Dunbar and Kosower

Modern Unitarity Method

Two-particle cut:

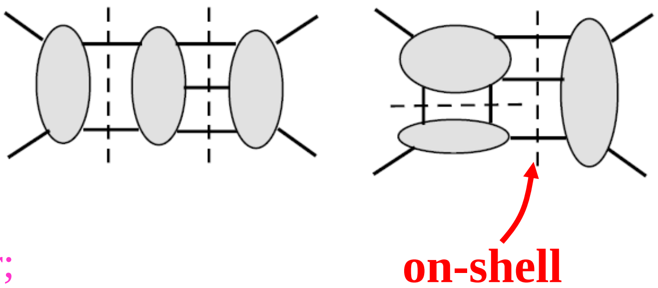


Three-particle cut:



Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Generalized unitarity as a practical tool.



Different cuts merged to give an expression with correct cuts in all channels.

Bern, Dixon and Kosower;
Britto, Cachazo, Feng

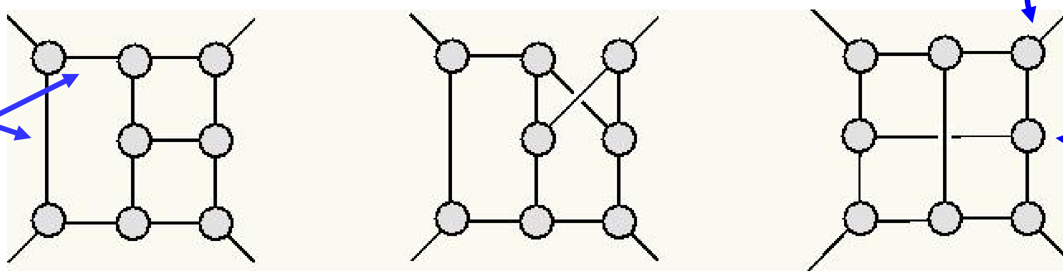
Reproduces Feynman diagrams except intermediate steps of calculation based on physical quantities not unphysical ones.

Method of Maximal Cuts

ZB, Carrasco, Johansson, Kosower

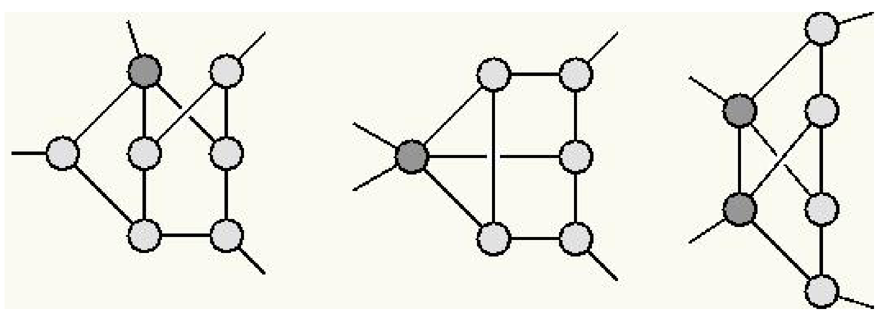
A refinement of unitarity method for constructing complete higher-loop amplitudes in any theory is "Method of Maximal Cuts". Systematic construction in any theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:

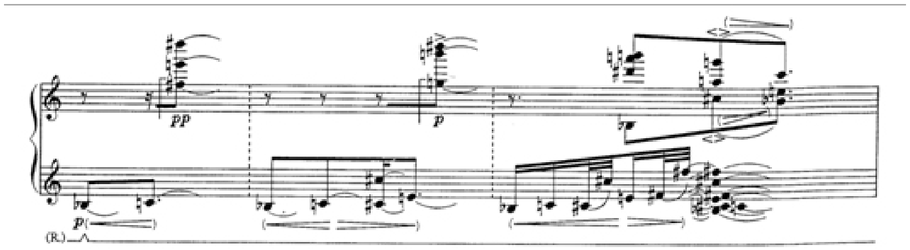


Fewer propagators placed on-shell.

Related to leading singularity method.

Cachazo and Skinner; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen

Examples of Harmony



Gravity vs Gauge Theory

Consider the gravity Lagrangian

flat metric

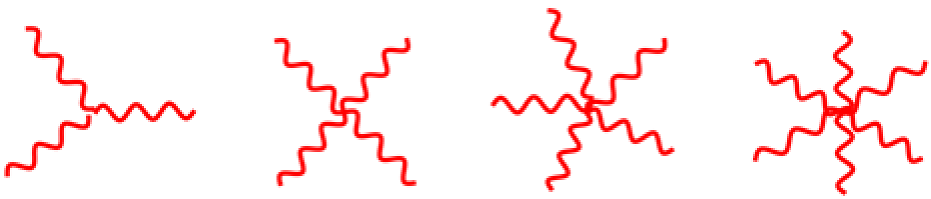
graviton field

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

Infinite number of complicated interactions

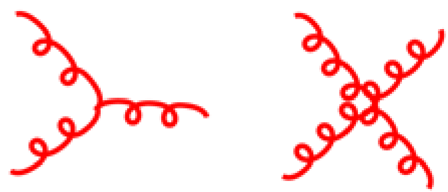


$+ \cdots$

terrible mess

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



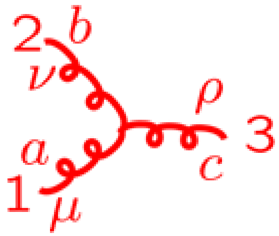
Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!

Three Vertices

Three gluon vertex:



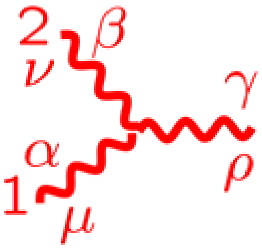
$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\begin{aligned}
&\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\
&+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\
&+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\
&+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

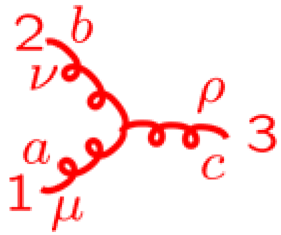
Naïve conclusion: Gravity is a nasty mess.

Simplicity of Gravity Amplitudes

On-shell **three vertices contains all information:**

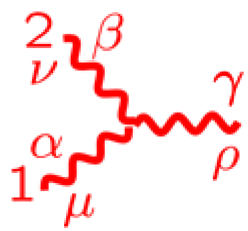
$$k_i^2 = 0$$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_1)_\rho + \text{cyclic})$$
$$\times\ (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.

- **BCFW on-shell recursion for tree amplitudes.**
  Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

- **Unitarity method for loops.**

ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

flat metric

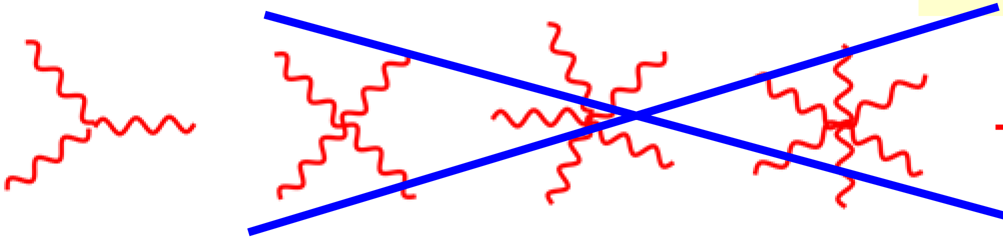
graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

Infinite number of irrelevant interactions!

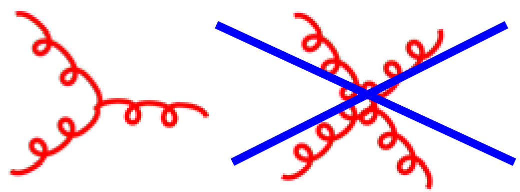


$+ \cdots$

Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



Only three-point interactions

no

Gravity seems ~~so much~~ more complicated than gauge theory.

Does ~~not~~ look harmonious!

KLT Relations

Even more remarkable relation between gauge and gravity amplitudes.

At *tree level* Kawai, Lewellen and Tye derived a relationship between closed and open string amplitudes.
In field theory limit, relationship is between gravity and gauge theory

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,, \qquad s_{ij} = (k_i + k_j)^2$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12} s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5)$$
$$+ s_{13} s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

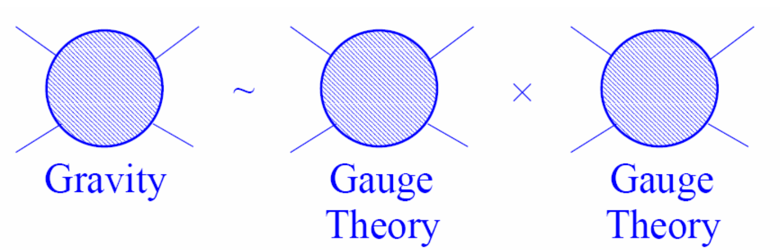
Gravity amplitude

Color stripped gauge theory amplitude

where we have stripped all coupling constants

$$\mathcal{A}_4^{\rm tree} = g^2 \sum\nolimits_{\rm non-cyclic} {\rm Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

Full gauge theory amplitude



Holds very generally.
See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories

Gravity and Gauge Theory Amplitudes

Berends, Giele, Kuijf; ZB, De Freitas, Wong

gravity

$$M_4^{\rm tree}(1_h^-, 2_h^-, 3_h^+, 4_h^+) = \left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\rm tree}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A_4^{\rm tree}(1_g^-, 2_g^-, 4_g^+, 3_g^+)$$

gauge theory

$$= \left(\frac{\kappa}{2}\right)^2 s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} \times \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,4\rangle\,\langle 4\,3\rangle\,\langle 3\,1\rangle}$$

$$\langle jl\rangle = \langle k_j^- | k_l^+\rangle = \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l) = \sqrt{2k_j\cdot k_l}\, e^{i\phi}$$

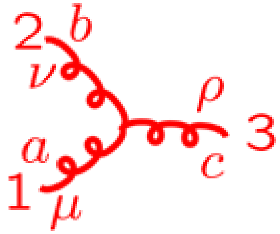
- **Agrees with result starting from Einstein Lagrangian**
- **Holds very generally for gravity theories.**

Harmony of Color and Kinematics

ZB, Carrasco, Johansson

coupling constant **color factor** **momentum dependent kinematic factor**

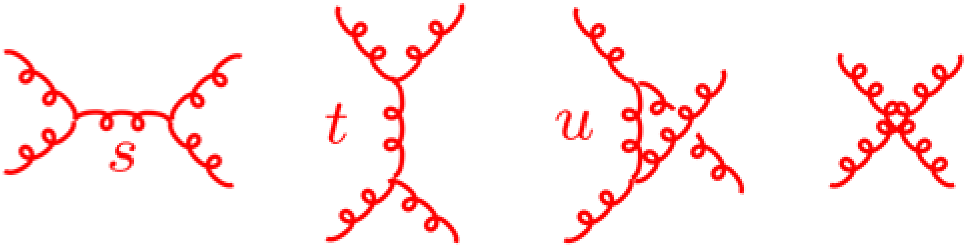
$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi identity

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$$



Use 1 = ***s/s* = *t/t* = *u/u***
to assign 4-point diagram
to others.

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

$$s = (k_1 + k_2)^2 \qquad u = (k_1 + k_3)^2$$
$$t = (k_1 + k_4)^2$$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics are singing same tune!

Harmony of Color and Kinematics

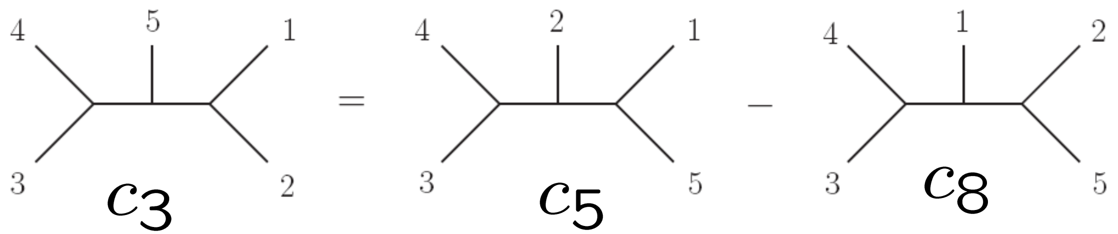
At higher points similar structure:

color factor

kinematic numerator factor

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{D_i}$$

Feynman propagators



$$c_3 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}\,, \qquad c_5 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}\,, \qquad c_8 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$c_3 - c_5 + c_8 = 0 \quad \Leftrightarrow \quad n_3 - n_5 + n_8 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.

- **Color and kinematics sing same tune!**
- **Nontrivial constraints on amplitudes.**

Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

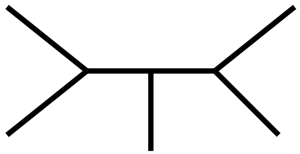
QCD:

$$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$$

sum over diagrams with only 3 vertices

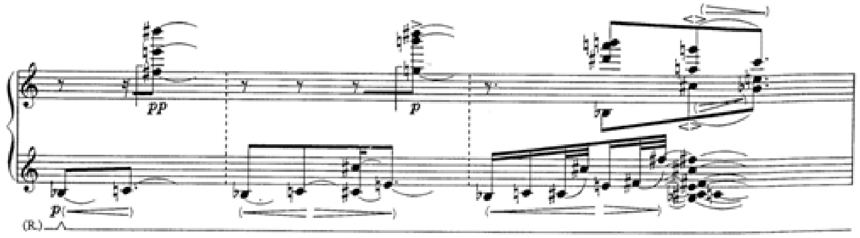
Einstein Gravity:

$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$



Claim: This is equivalent to KLT relations

Gravity and QCD kinematic numerators sing same tune!

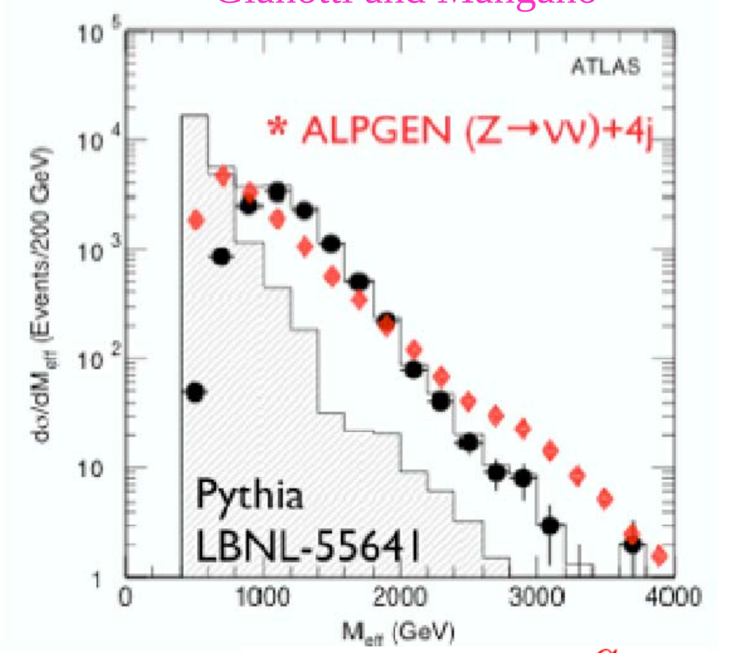


Cries out for a unified description of the sort given by string theory.

Applications to LHC Physics

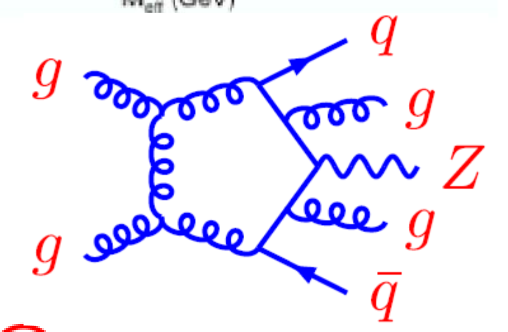
Example: Susy Search

Gianotti and Mangano



Early ATLAS TDR studies using PYTHIA overly optimistic.

- **ALPGEN is based on LO matrix elements and much better at modeling hard jets.**
- **What will disagreement between ALPGEN and LHC data mean for this plot?**
  Need NLO QCD to properly answer this.

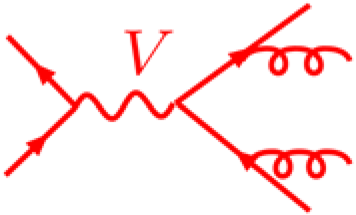


We need $pp \rightarrow Z + 4$ jets at NLO

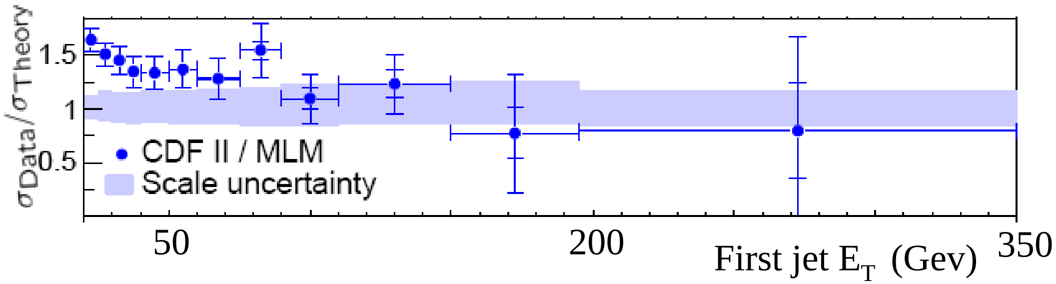
No complete 6-point NLO cross-section calculations in QCD, though serious progress described in this talk.

Example of Typical NLO Improvements

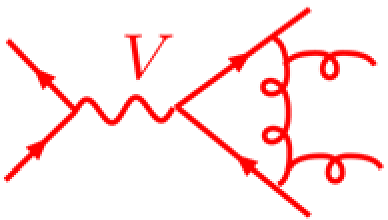
W + 2 jets at the Tevatron

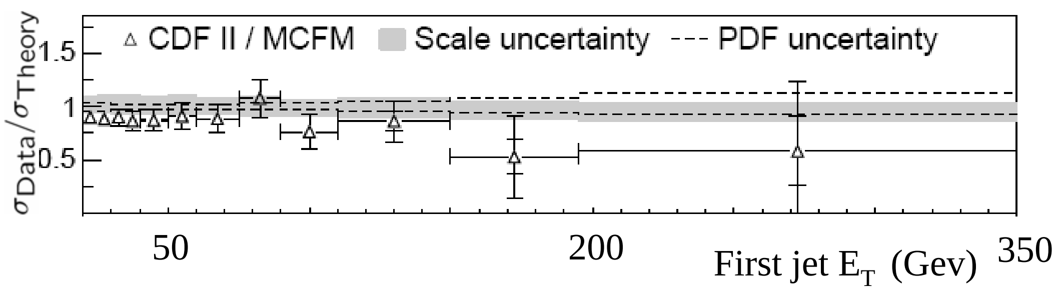


Note disagreement →



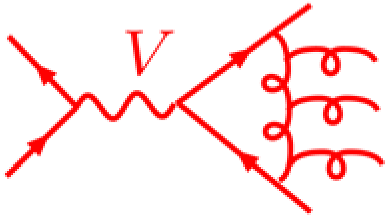
leading order + parton showering





NLO does better, smallest theoretical uncertainty

Want similar studies at the LHC also with extra jets.



Experimenter's NLO Wish List

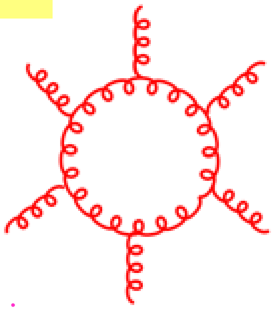
Run II Monte Carlo Workshop, April 2001

Single boson	Diboson	Triboson	Heavy flavour
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$W + b\bar{b} + \leq 3j$	$WW + b\bar{b} + \leq 3j$	$WWW + b\bar{b} + \leq 3j$	$t\bar{t} + \gamma + \leq 2j$
$W + c\bar{c} + \leq 3j$	$WW + c\bar{c} + \leq 3j$	$WWW + \gamma\gamma + \leq 3j$	$t\bar{t} + W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t} + Z + \leq 2j$
$Z + b\bar{b} + \leq 3j$	$ZZ + b\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t} + H + \leq 2j$
$Z + c\bar{c} + \leq 3j$	$ZZ + c\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma + b\bar{b} + \leq 3j$	$\gamma\gamma + b\bar{b} + \leq 3j$		
$\gamma + c\bar{c} + \leq 3j$	$\gamma\gamma + c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZ + b\bar{b} + \leq 3j$		
	$WZ + c\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		

Five-particle processes under good control with Feynman diagram based approaches.

Better ways needed to go beyond this.

Approaches for higher points



- **Traditional or numerical Feynman approaches.**

  Anastasiou. Andersen, Binoth, Ciccolini; Czakon, Daleo, Denner, Dittmaier, Ellis; Heinrich, Karg, Kauer; Giele, Glover, Guffanti, Lazopoulos, Melnikov, Nagy, Pilon, Roth, Passarino, Petriello, Sanguinetti, Schubert; Smillie, Soper, Reiter, Veltman, Wieders, Zanderighi, and many more.

- **On-shell methods: unitarity method, on-shell recursion**

  Anastasiou, Badger, Bedford, Berger, Bern, Bernicot, Brandhuber, Britto, Buchbinder, Cachazo, Del Duca, Dixon, Dunbar, Ellis, Feng, Febres Cordero, Forde, Giele, Glover, Guillet, Ita, Kilgore, Kosower, Kunszt; Mastrolia; Maitre, Melnikov, Spence, Travaglini; Ossola, Papadopoulos, Pittau, Risager, Yang; Zanderighi, etc

- **Most physics results have been from Feynman diagrams.**
  — two notable exceptions $pp \to W + 2$ jets and $pp \to VVV$

  ZB, Kosower, Dixon, Weinzierl; Ossola, Papadopoulos, Pittau

- **Most people working on this are instead now pursuing on-shell methods because of demonstrated excellent scaling with number of external particles. See recent LoopFest conference.**

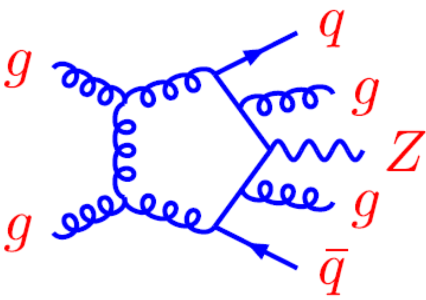
  http://electron.physics.buffalo.edu/loopfest7

BlackHat: An automated implementation of on-shell methods for one-loop amplitudes

Berger, ZB, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre

BlackHat is an automated C++ package for numerically computing one-loop matrix elements with 6 or more external particles.

- Input is numerical **on-shell** tree-level amplitudes.
- Output is numerical on-shell one-loop amplitudes.



BlackHat incorporates ideas discussed above to achieve the speed and stability required for LHC phenomenology at NLO.

Two other similar packages under construction

— **CutTools** Ossola, Papadopoulos, Pittau

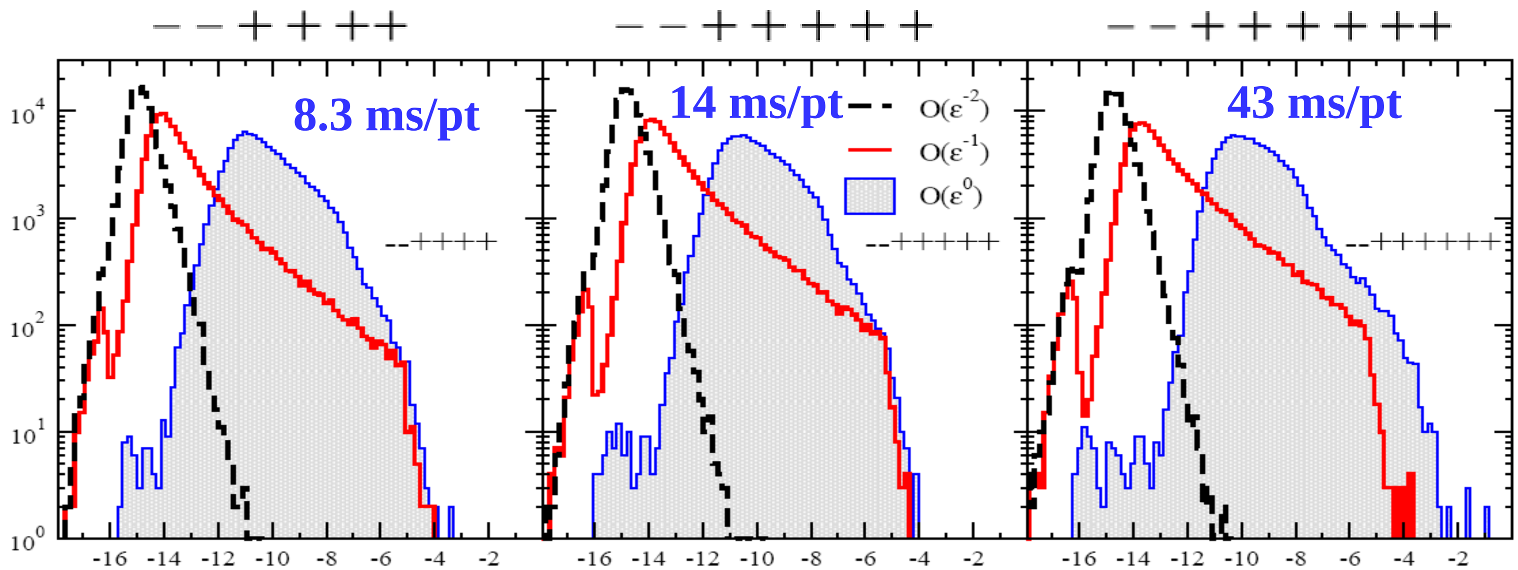
— **Rocket** Ellis, Giele, Kunszt, Melnikov, Zanderighi

Scaling with number of legs

Berger, ZB, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre

Extremely mild scaling with number of legs

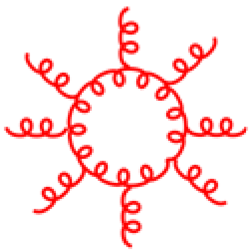
2.33 GHz Xeon



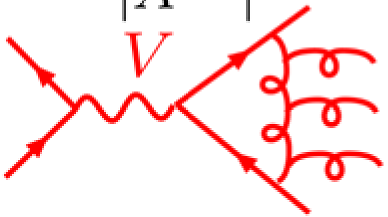
6 gluons **7 gluons** **8 gluons**

$$\text{relative precision} = \log_{10}\left(\frac{|A^{\text{num}} - A^{\text{ref}}|}{|A^{\text{ref}}|}\right)$$

amusing count for 8 gluons



+ 3,017,489 Feynman diagrams



vector bosons under control

More to be done to get physics

Applications to AdS/CFT

N = 4 Super-Yang-Mills to All Loops

Since ‘t Hooft’s paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. *N* = 4 sYM is much more promising.

- **Special theory because of AdS/CFT correspondence:**
- **Maximally supersymmetric — boson/fermion symmetry**
- **Simplicity both at strong and weak coupling.**

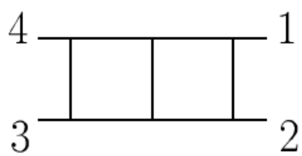
Remarkable relation

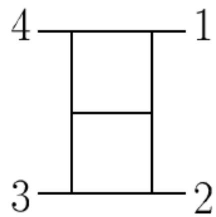
scattering at strong coupling in *N* = 4 sYM $\longleftrightarrow$ classical string theory in AdS space

To make this link need to evaluate *N*=4 super-Yang-Mills amplitudes to *all* loop orders. Seems impossible even with modern methods.

Loop Iteration of the $N = 4$ Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

$-st\,A_4^{\mathrm{tree}} \Big\{ s$ 

$+ \; t$ 

$\Big\}$

ZB, Rozowsky, Yan

$D = 4 - 2\epsilon$

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{\text{1-loop}}(s,t)|_{\epsilon \to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

$$M_4^{\text{loop}} = A_4^{\text{loop}} / A_4^{\text{tree}} \qquad f(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$$

Anastasiou, ZB, Dixon, Kosower

$f(\epsilon)$ is universal function related to IR singularities

Two-loop four-point planar amplitude is an iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

ZB, Dixon, Smirnov

All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?

constant independent of kinematics.

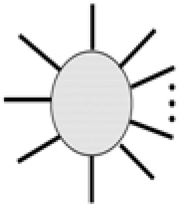
$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude



"BDS conjecture"

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

$$F_4^{\text{1-loop}} = \frac{1}{2}\ln^2\left(\frac{t}{s}\right) + 4\zeta_2$$

- **To make this guess used strong constraint constraints from analytic properties of amplitudes.**

Gives a definite prediction for *all* values of coupling given BES integral equation for the cusp anomalous dimension.

Beisert, Eden, Staudacher

Alday and Maldacena Strong Coupling

ZB, Dixon, Smirnov

For MHV amplitudes:

$$\mathcal{A}_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_4^{\text{1-loop}} + C\right]$$

constant independent of kinematics.

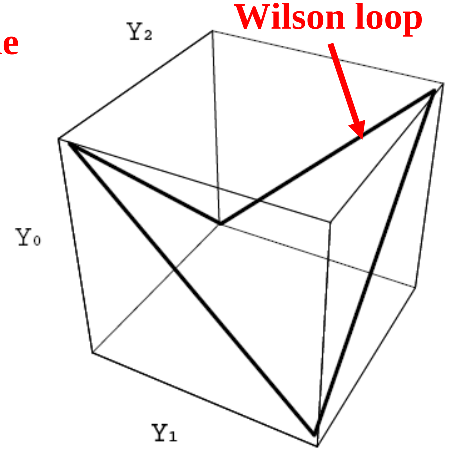
all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude

In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at *strong coupling* from an AdS string theory computation. Minimal surface calculation—like a soap bubble.



- **Identification of new symmetry: "dual conformal symmetry"**
- **Link to integrability. Infinite number of conserved charges**

Drummond, Henn, Korchemsky, Sokatchev ;Berkovits and Maldacena; Beisert, Ricci, Tseytlin, Wolf
Brandhuber, Heslop, Travaglini

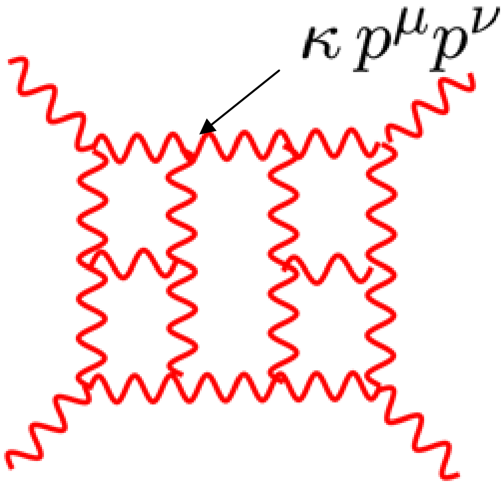
Unfortunately, trouble at 6 and higher points.

Alday and Maldacena; ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich

Applications to Quantum Gravity

Is a UV finite theory of gravity possible?

$\kappa = \sqrt{32\pi G_N}$ ← Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Reasons to focus on *N* = 8 maximal supergravity:

- **With more susy suspect better UV properties.** Cremmer and Julia
- **High symmetry implies technical simplicity—may even be "simplest" quantum field theory** Arkani-Hamed, Cachazo, Kaplan

Finiteness of Point-Like Gravity Theory?

We are interested in UV finiteness of $N = 8$ supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

- **Here we only focus on order-by-order UV finiteness.**
- **Non-perturbative issues and viable models of Nature are *not* the goal for now.**

Opinions from the 80's

If certain patterns that emerge should persist in the higher orders of perturbation theory, then … $N = 8$ supergravity in four dimensions would have ultraviolet divergences starting at three loops.

Green, Schwarz, Brink, (1982)

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

The idea that *all* supergravity theories diverge at 3 loops has been widely accepted wisdom for over 20 years

There are a number of very good reasons to reanalyze this.

Non-trivial one-loop cancellations: no triangle & bubble integrals

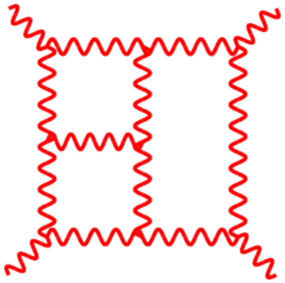
ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Dunbar , Ita, Perkins, Risager; Green, Vanhove, Russo; Bjerrum-Bohr Vanhove; Arkani-Hamed, Cachazo, Kaplan

Unitarity method implies higher-loop cancellations.

ZB, Dixon, Roiban

Feynman Diagrams for Gravity

Suppose we wanted to check superspace power counting proposal of 3 loop divergence.



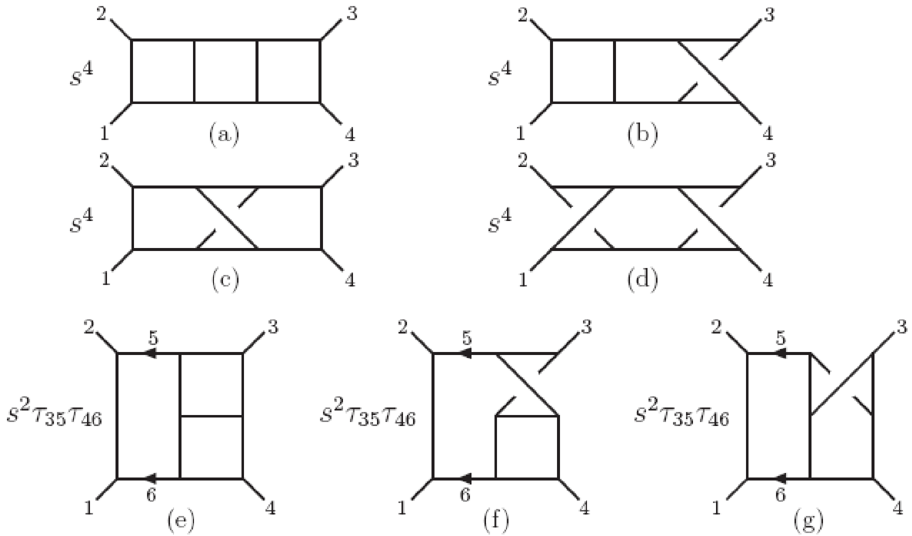
If we attack this directly get $\sim 10^{20}$ **terms in diagram. The alegebraic explosion is a reason why this hasn't been evaluated using Feynman diagrams.**

Counted number of terms in one diagram from expanding vertices and propagators, not nunber of diagrams or the algebraic explosion trying to reduce to integral basis.

Complete Three Loop Result

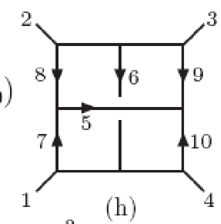
ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

Obtained via maximal cut method:

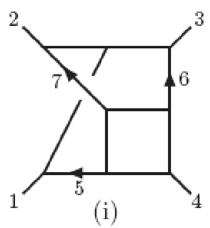


$$\tau_{ij} = 2k_i \cdot k_j$$

$$\begin{aligned}&(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2\\&+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\&+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\&+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})\end{aligned}$$



$$\begin{aligned}&(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})\\&-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})\\&+l_5^2s^2t+l_6^2st^2-\tfrac{1}{3}l_7^2stu\end{aligned}$$



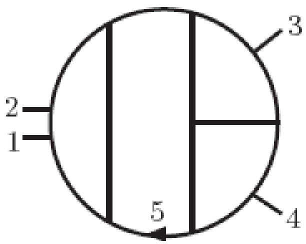
Three-loop is not only ultraviolet finite it is "superfinite"—cancellations beyond those needed for finiteness!

$\mathcal{N}$ = 8 Four-Loop Calculation in Progress

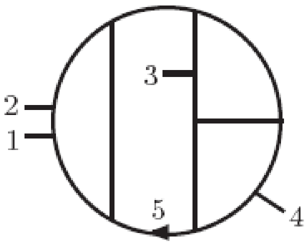
ZB, Carrasco, Dixon, Johansson, Roiban

Some $\mathcal{N}$ =4 YM contributions:

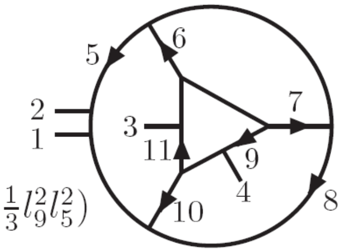
$s_{12}^2 s_{45}$



$s_{12}^2 s_{45}$



$$s_{12}^2 s_{98} - s_{12} s_{35} s_{67} + \tfrac{1}{3} l_9^2 s_{12}(s_{35} - s_{12}) + s_{12}(l_5^2 s_{4,10} - l_5^2 l_{11}^2 - \tfrac{1}{3} l_9^2 l_5^2)$$



50 distinct planar and non-planar diagrammatic topologies

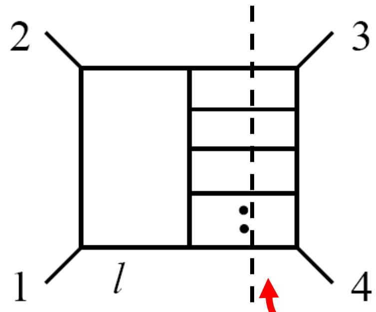
$\mathcal{N}$ = 4 super-Yang-Mills case is complete!
$\mathcal{N}$ = 8 supergravity case still in progress.

Four-loops will teach us a lot – bottles of wine to be exchanged:

1. **Direct challenge to simplest superspace explanations.**
2. **Proof of finiteness will likely need insights gathered from this calculation.**

L-Loop Cancellations

ZB, Dixon, Roiban



- We can probe infinite loop orders by looking at a limited class of cuts.
- Probes reveal superfiniteness: finite for $D < \frac{6}{L} + 4$
- *Not* a proof of finiteness because you would need to check *all* cuts.
- Improved behavior can be traced back to good behavior of tree-level amplitudes under large complex shifts of momenta.

ZB, Carrasco, Forde, Ita, Johansson; Arkani-Hamed, Cachazo, Kaplan

Summary

- On-shell methods offer a powerful alternative to Feynman diagrams.
- Remarkable structures in scattering amplitudes:
  — color $\longleftrightarrow$ kinematics.
  — gravity ~ (gauge theory)^2
- Remarkable harmony between gravity and gauge theory scattering amplitudes.

Applications:

- NLO QCD for the LHC: Amplitudes under control, physics on its way.
- $N = 4$ *S*upersymmetric gauge theory: New venue opened for studying AdS/CFT.
- Quantum gravity: Is a point-like UV finite theory possible? New evidence suggests it is but proof is a challenge.

Some amusement

YouTube: Search “Big Bang DMV”, first hit

Extra

Basic Strategy

ZB, Dixon, Dunbar, Perelstein and Rozowsky (1998)

$N = 4$ Super-Yang-Mills Tree Amplitudes $\xrightarrow{\text{KLT}}$ **$N = 8$ Supergravity Tree Amplitudes** $\xrightarrow{\text{Unitarity}}$ **$N = 8$ Supergravity Loop Amplitudes** $\longrightarrow$ **Divergences**

- Kawai-Lewellen-Tye relations: sum of products of gauge theory tree amplitudes gives gravity tree amplitudes.
- Modern unitarity method: efficient formalism for perturbatively quantizing gauge and gravity theories. Loop amplitudes from tree amplitudes.

ZB, Dixon, Dunbar, Kosower (1994)

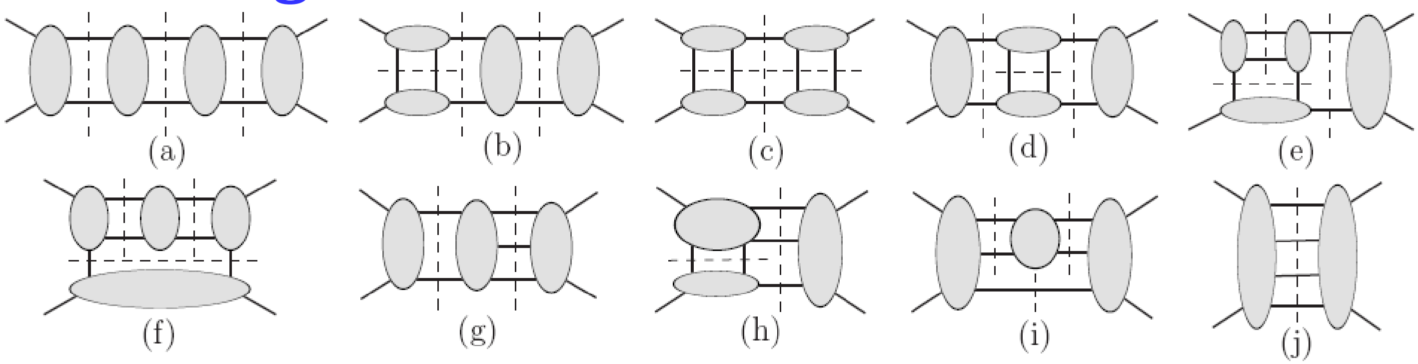
Key features of this approach:

- **Gravity calculations mapped into much simpler gauge theory calculations.**
- **Only on-shell states appear.**

Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



reduces everything to product of tree amplitudes

For cut (g) have:

$$\sum_{\mathcal{N}=8 \text{ states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use KLT

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12} A_4^{\text{tree}}(1,2,l_3,l_1) A_4^{\text{tree}}(2,1,l_3,l_1)$$

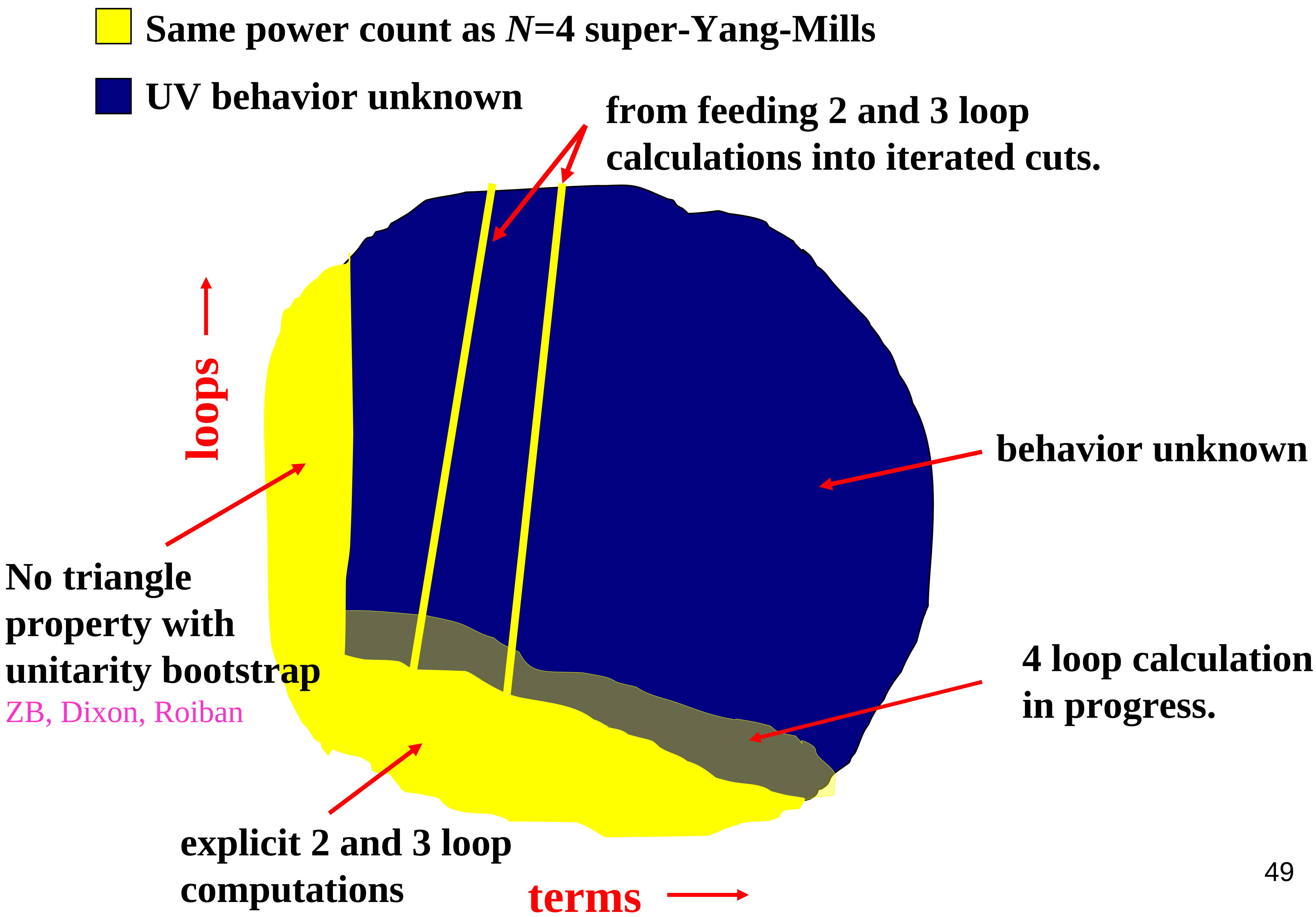
$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1 q_1} s_{l_3 q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\}\,,$$

supergravity

super-Yang-Mills

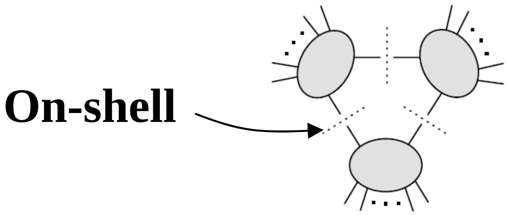
$\mathcal{N}$ = 8 supergravity cuts are sums of products of $\mathcal{N}$ = 4 super-Yang-Mills cuts

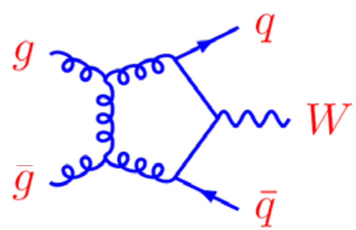
Schematic Illustration of Status



Some Developments

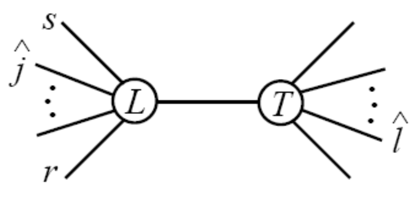
- **Generalized cuts – used to produce one-loop matrix elements for $pp \to W, Z + 2$ jets. Used in MCFM** ZB, Dixon, Kosower





- **Realization of the remarkable power of complex momenta in generalized cuts. Inspired by Witten's twistor string paper. Very important.** Britto, Cachazo, Feng; Britto et al series of papers.

- ***D*-dimensional unitarity to capture rational pieces of loops.**
  ZB, Morgan; ZB, Dixon, Dunbar, Kosower, ZB, Dixon, Kosower;
  Anastasiou, Britto, Feng, Kunszt, Mastrolia; Giele, Kunszt, Melnikov; Britto and Feng;
  Giele and Zanderighi; Ellis, Giele, Kunszt, Melnikov; Badger



- **On-shell recursion for loops (based on BCFW)**
  Berger, ZB, Dixon, Forde, Kosower; + Febres Cordero, Ita, Maitre

- **Efficient on-shell reduction of integrals.** Ossola, Papadopoulos, Pittau; Forde

Where are the N = 8 Divergences?

Depends on who you ask and when you ask.

Howe and Lindstrom (1981)
Green, Schwarz and Brink (1982)
Howe and Stelle (1989)
Marcus and Sagnotti (1985)

3 loops: Conventional superspace power counting.

5 loops: Partial analysis of unitarity cuts.

ZB, Dixon, Dunbar, Perelstein, and Rozowsky (1998)

If harmonic superspace with N = 6 susy manifest exists

Howe and Stelle (2003)

6 loops: *If* harmonic superspace with N = 7 susy manifest exists

Howe and Stelle (2003)

7 loops: *If* a superspace with N = 8 susy manifest were to exist.

Grisaru and Siegel (1982)

8 loops: Explicit identification of potential susy invariant counterterm with full non-linear susy.

Kallosh; Howe and Lindstrom (1981)

9 loops: *Assume* Berkovits' superstring non-renormalization theorems can be naively carried over to N = 8 supergravity. Also need to extrapolate.

Green, Vanhove, Russo (2006)

Superspace gets here with additional speculations.

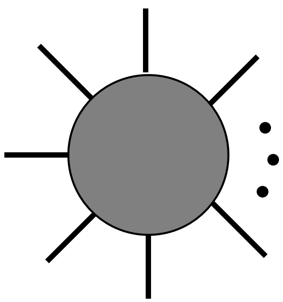
Stelle (2006)

Note: none of these are based on demonstrating a divergence. They are based on arguing susy protection runs out after some point.

Origin of Cancellations?

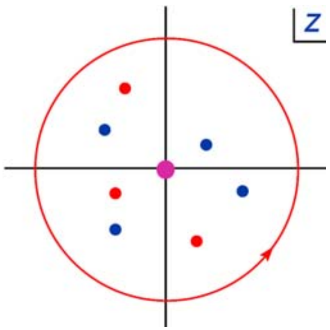
There does not appear to be a supersymmetry explanation for all-loop cancellations.

If it is *not* supersymmetry what might it be?



$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle \qquad A^{\text{tree}}(z) \longrightarrow 0$$

$$k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle, \qquad z \longrightarrow \infty$$



This property useful for constructing BCFW recursion relations for gravity .

Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Benincasa, Boucher-Veronneau , Cachazo; Arkani-Hamed, Kaplan; Hall

This same property appears to be directly related to the novel cancellations observed in the loops.

ZB, Carrasco, Forde, Ita, Johansson; Arkani-Hamed, Cachazo, Kaplan

Can we prove perturbative finiteness of *N* = 8 supergravity?

Time will tell…