Anyonic approach to quantum computation

Players

Anyons

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q-bit

R. o. W.

Systems

FQHE ?
Cuprates above T_c ?
[redacted] ?

Math

Liquid NMR
solid state NMR
electron spin
quantum dot
optical cavity
ion trap
⋮

What are "any-ons" ?

boson $\quad \psi_1 \otimes \psi_2 = \psi_2 \otimes \psi_1$

fermion $\quad \psi_1 \otimes \psi_2 = -\psi_2 \otimes \psi_1$

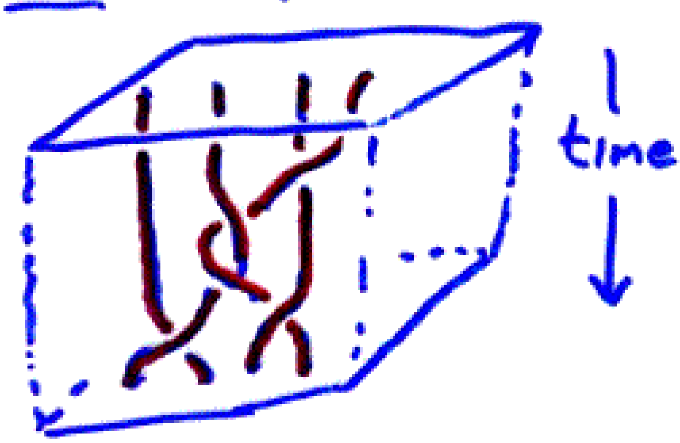
exotic statistics

anyon (abelian) $\quad \psi_1 \otimes \psi_2 = \omega\, \psi_2 \otimes \psi_1 \quad |\omega| = 1$

anyon (nonabelian) $\quad \psi_1 \otimes \psi_2 = U\, \psi_2 \otimes \psi_1$,

U unitary

Exotic statistics is a 2-dim phenomena. It yields a braid group representation : $\rho : \text{Braid} \to U(h)$



h = "internal degrees of freedom"

Why try to compute with anyons?

Disadvantages

1. We're not absolutely sure the useful kind (nonabelian) exists in any physical system.
2. If they exist they must be grabbed and braided around each other at will. Several thousand twists needed to break RSA.

Advantages

Intrinsically fault tolerant: anyons are expected to have exponentially decaying tails so physical braiding should approximate $\rho : \text{Braid} \to U(h)$ up to $e^{-c(\text{length scale})}$.

No q-bit approach offers this error protection!

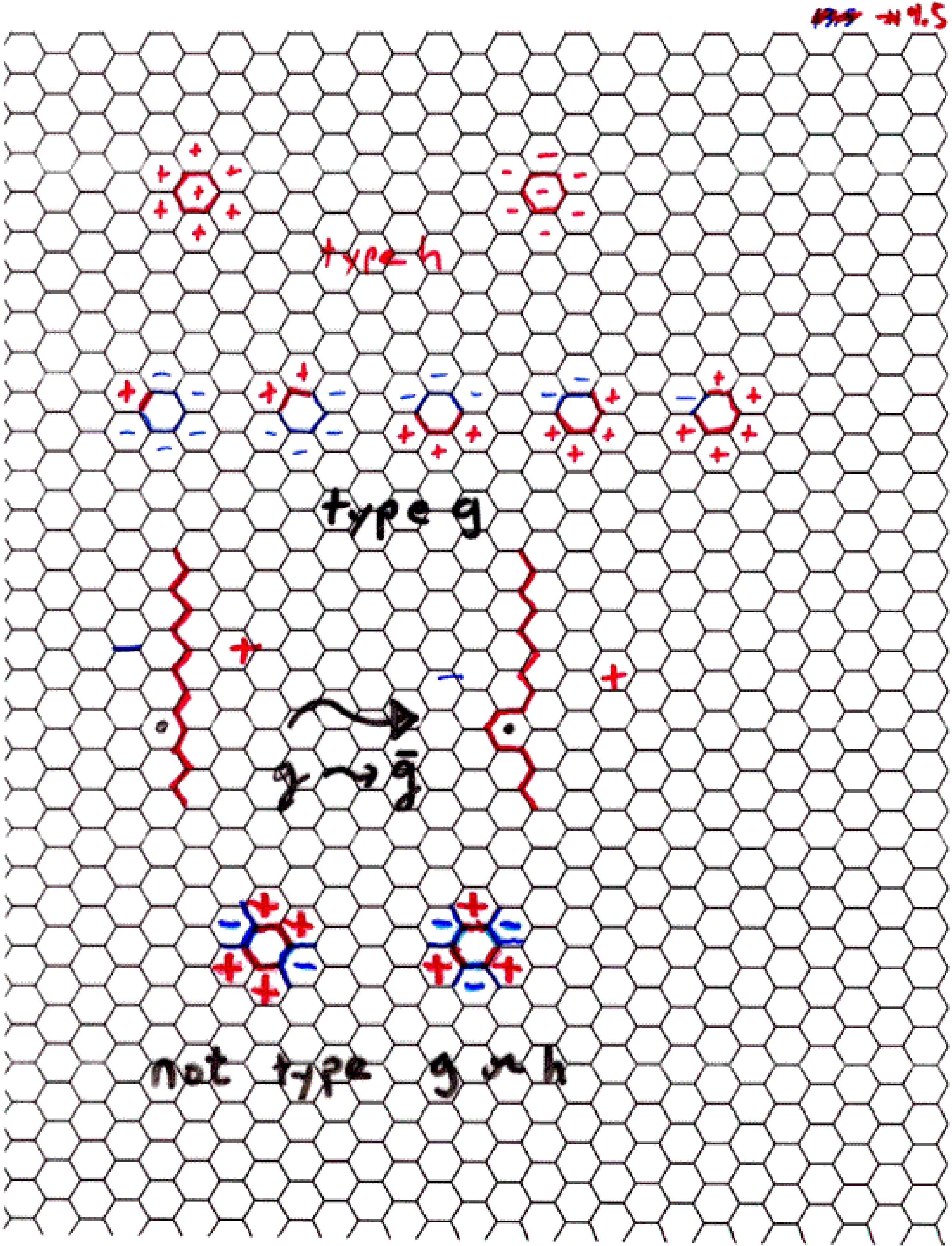
Status of Anyons ?

Abelian statistics (e.g. $\omega = e^{2\pi i/3}$) is widely believed* to occur in some quantum hall regimes, e.g. in the Laughlin state $\nu = 1/3$.

At $\nu = 5/2$ there is some numerical and mathematical evidence for the nonabelian model called CS4.

At $\nu = 8/5$ there is, perhaps shakier, evidence for the first fully interesting computationally complete, nonabelian model called CS5. The associated ρ: Braid $\to U(h)$ is dense, hence universal.

* Atleast in Sweden


type h
type g
not type g
h

Quantum Computer

$\mathcal{h}$ hilbert space

$\psi_0 \in \mathcal{h}$ initial vector

ρ representation into $U(\mathcal{h})$
(locality, density)

C classical compiler

ψ_f final vector $= \rho(\text{compiled word})\,\psi_0$

π observable (for "readout")

Can we find anyons outside FQHE ?

(And hope to shed extreme temperatures and chiral asymmetry?)

Above T_c cuprate superconductors become antiferromagnets which have been conjectured to support spin-charge separation of the electron.

The result would be anyonic quasi-particles – "chargon" + "spinon" with a nontrivial but not sufficiently nontrivial $\rho: \text{Braids} \to U(h)$
(Fisher - Senthil, Nayak - Shtengel)

Today I will describe a new anti-ferromagnetic spin model. Some version of it may already exist – waiting to be discovered – or might be engineered.

The model

The model can live on any surface with or without boundary (if there __is__ boundary then boundary conditions will be important)

But let us begin locally with Ising spins located on the 2-cells of a hexagon-tiling of the plane. Thus each hexagon h is assigned a single degree of freedom $\mathbb{C}^2_h = \langle |+\rangle, |-\rangle \rangle$ corresponding to "spin up" and "spin down".

The hamiltonian H_0 is a sum of local terms which operate on 7-clusters: a hexagon and its neighbors.

Up to an overall $|+\rangle$ to $|-\rangle$ flip

The domain wall γ between +

and − regions carries all the

information in a classical spin

configuration.

$d = \frac{1+\sqrt{5}}{2}$

H_0 is designed to "know" about

isotopy of γ and to "know" the

weight d of a small circle added to γ.

As will be explained, this means

H_0 "knows" the Even Temperley, Lieb category.

Under perturbation $H_0 \to H_\epsilon$ the

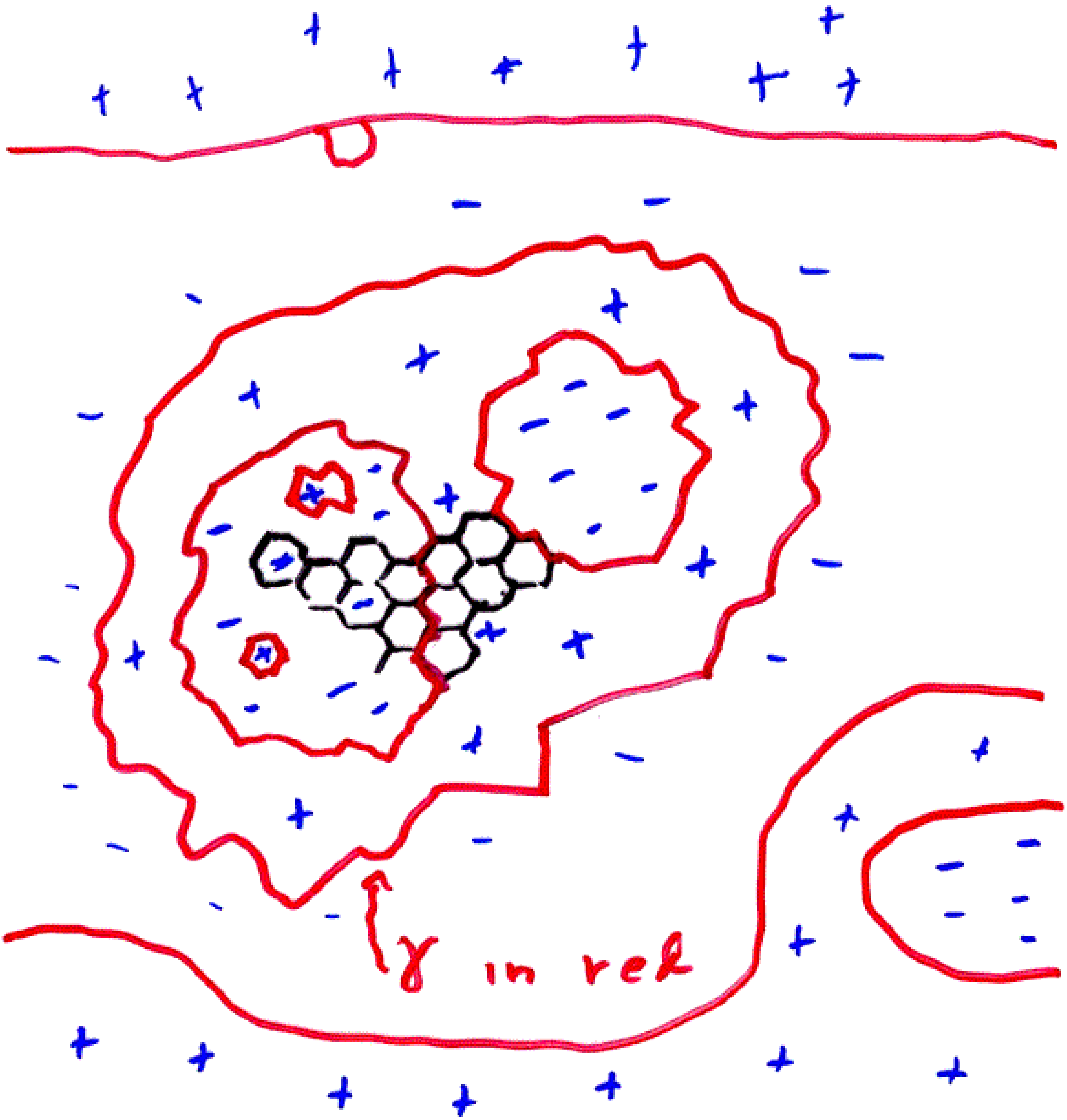
hamiltonian "learns" about a much more

interesting modular category, $\mathcal{D}(\mathcal{F}ib) = \mathcal{h}$,

which is the hilbert state space for

our quantum computer.

8.5



The definition of H_0

$\mathcal{H} = \bigotimes_h \mathbb{C}^2_h$ = space of spin = 1/2 particles living on hexagons (2-cells)

$\mathcal{H}$ = span (classical spin configurations s)

A pair (s,c) = (spin conf. , 2-cell) is type h if s assigns the same spin to c and all its neighbors.

A pair (s,c) is type g if $\partial^+ c$ and $\partial^- c$ are both homeomorphic to intervals. $\partial^\pm c$ is the interface of c the neighbors to which s assigns $|+\rangle$ / $|-\rangle$.

$$H_0 = \sum_{\text{type } h} |h - \tfrac{1}{d}\bar{h}\rangle\langle h - \tfrac{1}{d}\bar{h}| + \sum_{\text{type } g} |g - \bar{g}\rangle\langle g - \bar{g}|$$

$d = \frac{1+\sqrt{5}}{2}$ = "golden number"

bar $^-$ means flip spin on c.

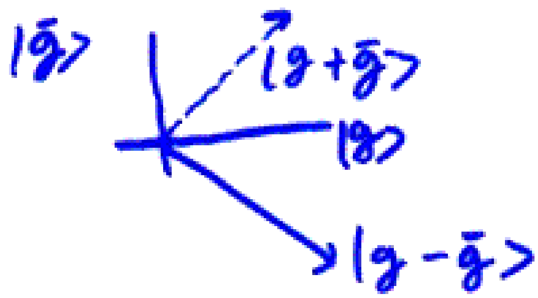
$$H_0 = \sum_h |h - \tfrac{1}{2} d\bar{h}\rangle\langle h - \tfrac{1}{2} d\bar{h}| + \qquad \mathrm{I}$$

$$\sum_g |g - \bar{g}\rangle\langle g - \bar{g}| \qquad \mathrm{II}$$

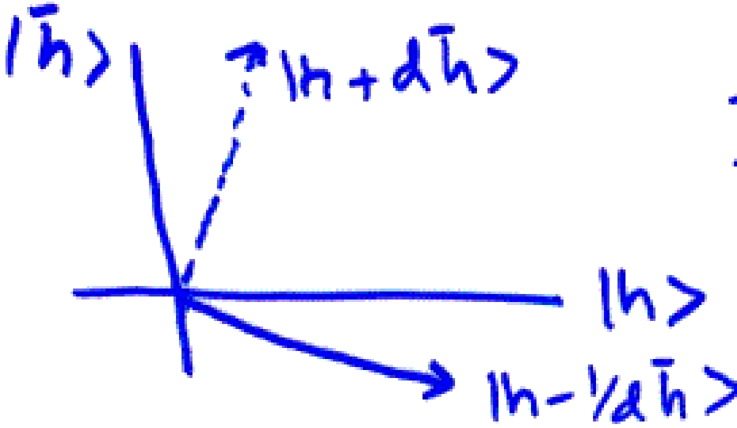
$$H_\varepsilon = H_0 + \varepsilon \sum_v \sigma_v^x \quad , \sigma^x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

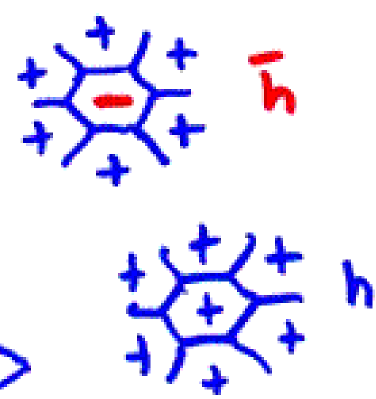
on $\mathbb{C}_v^2 = \langle +, - \rangle$

Term II means "isotopy democracy"



Term I means "H_0 likes circles"





To understand the highly degenerate ground state spaces G_0 and G_ε of H_0 and H_ε we need to explain the Temperly - lieb category TL_d and its extremely rigid algebraic structure.

Just as in ordinary algebra an ideal is a subset of morphisms internally closed under a formal + and externally closed under $\circ$ and $\otimes$

Thm (Goodman, Wenzl) The tensor category TL_d has a unique ideal $R_d \neq \{0\}, TL_d$. d special, $2\cos \pi/m$.

$G_0 \approx ETL_d$, and I believe G_ε must have structure of a quotient category:

$$G_\varepsilon \approx ETL_d / R_d = \mathcal{D}(\mathcal{F}ib)$$

locality

in algebra: ideal is locally determined
e.g. the even integers

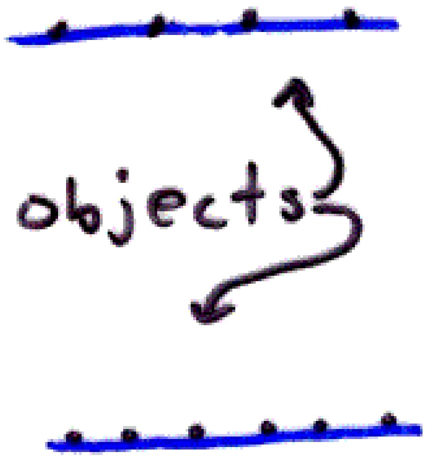
in perturbation theory: We may expect to find a stable phase defined by local projectors acting on a "crude" ground-state space.

e.g. within a Landau level, projection onto relative angular momentum states for each electron pair:

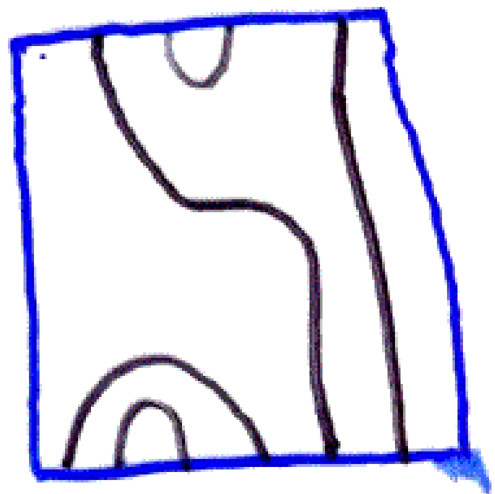
$$H = \sum_{k=0}^{\frac{q-3}{2}} \sum_{i<j} V P_{ij} ((2k+1) N_\phi)$$

defines Laughlin's odd denominator state space at filling fraction $\nu = 1/q$.

The tensor category TL_d

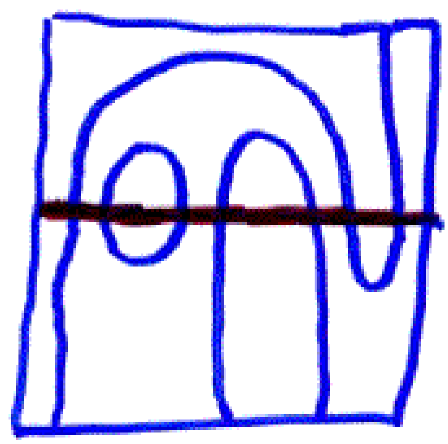


morphism

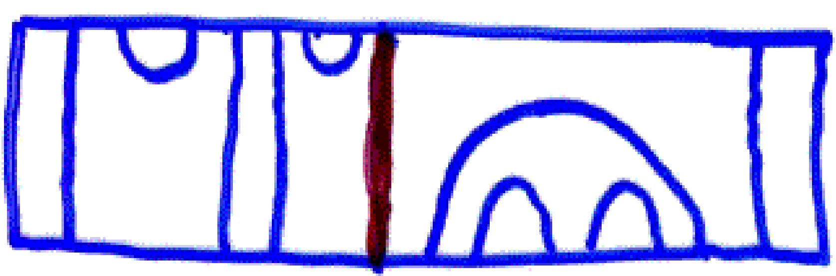


$+$ formal over $\mathbb{C}$

$\bullet$ vertical stacking:
closed circles count as d

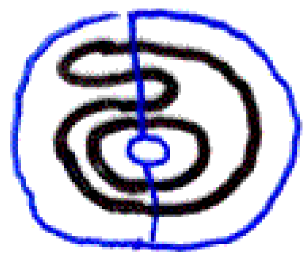


$\otimes$ horizontal stacking:



TL_d has a pairing

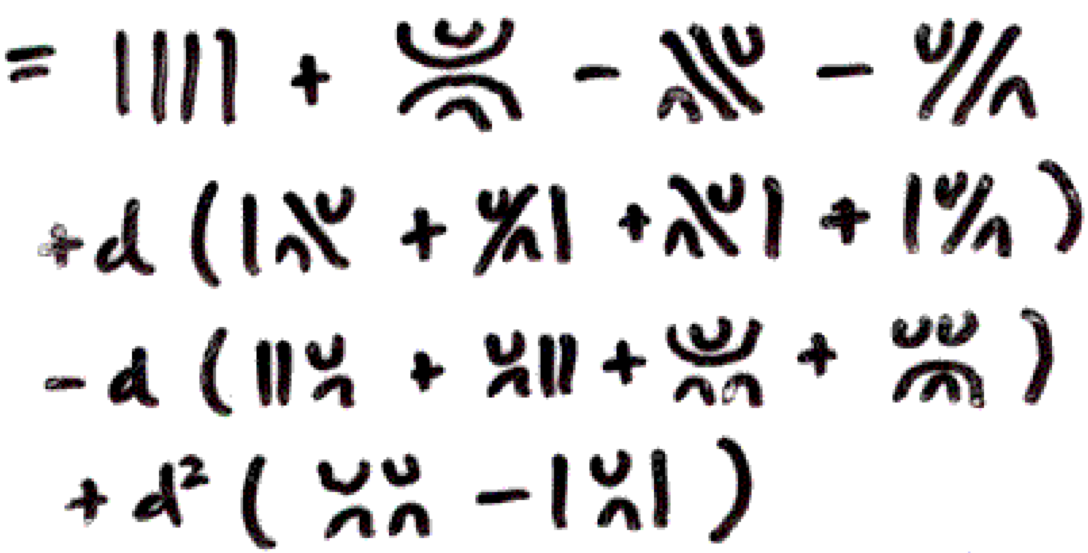
 $= d^{\#} = d^2$

$\#$ = no. circles  $= 2$

$\langle\, ,\, \rangle$ is pos. semidefinite

with radical $R_d = \langle P_4 \rangle$

P_4 = 4^{th} Jones-Wenzl projector



$\langle\ \rangle$ means closure under $\not{+}$ · $\otimes$

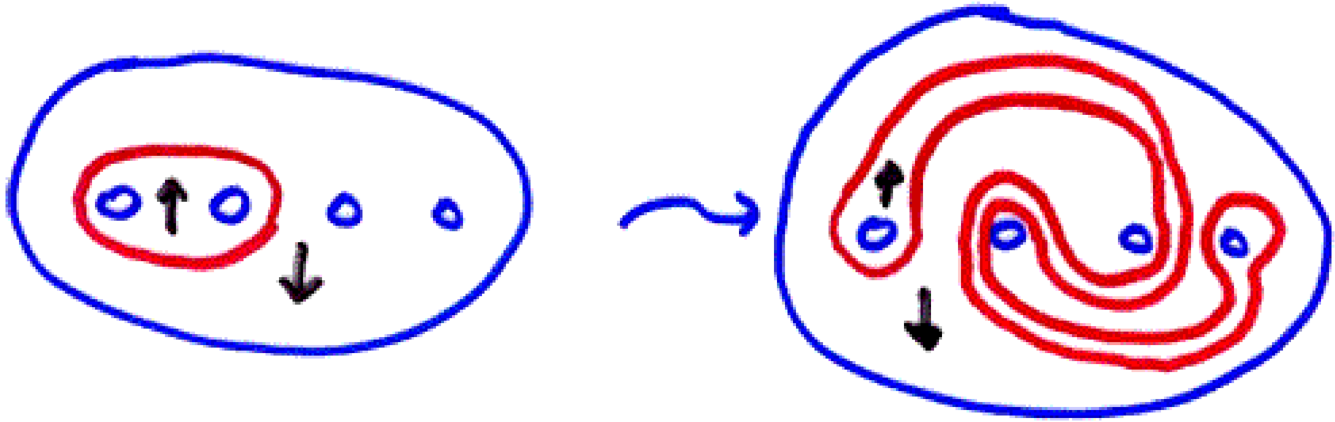
The whole discussion can be transfered to surface – with boundary or closed. The lines, for us, arise as the domain walls between "up" and "down" spins (located on vertices of a triangulation.)

We have : ETL_d^S =
span (generalized isotopy classes of domain walls)
"generalized" means trivial circle = d

And : $ETL_d^S / R_d = \mathcal{S}(\mathcal{F}ib)$

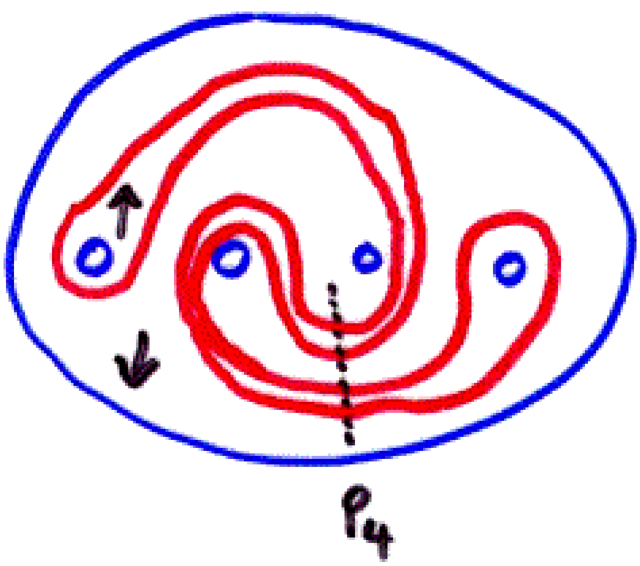
By [F., Larsen, Wang] A known modular functor with computationally complete Braid reps ρ.

Action of a braid on ETL_d^s



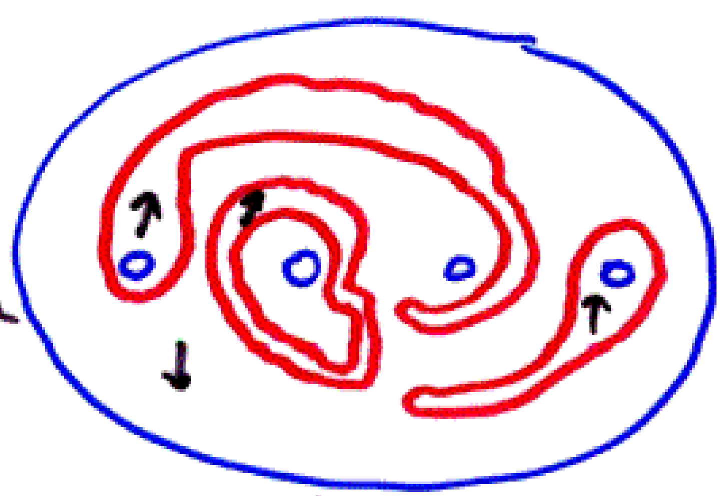
Action of this braid on ETL_d^s / R_d $= \mathcal{D}(\mathcal{F}, b)$

simplifies by applying ρ_4



$\longrightarrow$ 13 terms :

$-d$



$+ \cdots$

The ground state space G_0 of H_0 is a combinatorial model for the surface algebra ETL_d^S. Thus G_0 is <u>highly</u> degenerate. If a surface S has first Betti number $= b$ and is triangulated with v-vertices

$$\dim G_0 \approx \left(\frac{1+\sqrt{5}}{2}\right)^{2b} \cdot \mathrm{poly}(v),$$

($\deg(\mathrm{poly})$ is exponential in b)

The <u>quotient</u> ETL_d^S / R_d has dimension $\approx \left(\frac{1+\sqrt{5}}{2}\right)^{2b}$, independent of triangulation. This is our <u>candidate</u> <u>for G_ϵ</u> the perturbed ground state space.

Perturbation theory:

Study: $\Pi_{G_0} \circ \frac{d^k}{d\varepsilon^k} \exp(i\varepsilon V) \circ \mathrm{inc}_{G_0} : G_0 \to G_0$

If $V = \sum_i \sigma_i^x$ or $\sum_i |)(-\asymp\rangle_i \langle)(-\asymp|_i$

all k^{th}-order skein relations are terms of V^k.

In thermodynamic limit:

Ansatz I: G_ε is the joint null space of (strongly) local projectors acting on G_0

or

Ansatz II: G_ε is lowest eigenspace of a positive combination of (strongly) local projectors on G_0.

Arguing from I:

[GW] $\Longrightarrow$ order $k<3$ any local system of projectors projects in a complete set of directions: Any small skein spans ETL^S_d.

At $k=3$, $|P_4\rangle$ spans a proper subspace $R_d \subset G_0$

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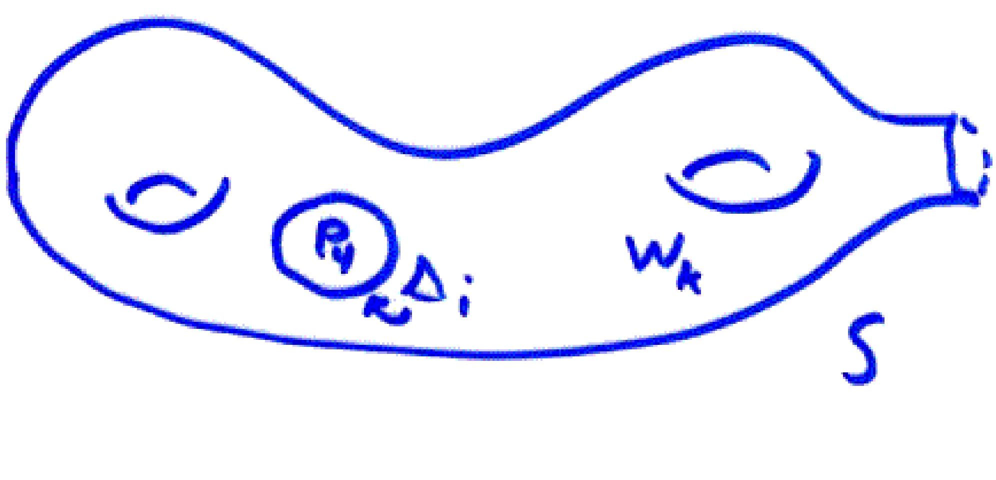
And R_d is the <u>only</u> [GW] nontrivial proper subspace of G_0 "=" ETL^S_d which is spanned by a (strongly) local family of vectors, e.g.

$\{(P_4)_{\Delta_i} \otimes (w_k)_{S \setminus \Delta_i}\}$, w_k arbitrary



So $G_0 / R_d =: h$ is <u>only</u> candidate for G_ε consistant with Ansatz I.

Furthermore, it may be possible using the quantum analogue of Markov chains to exhibit a hamiltonian* H' with ground state space $G_0/R_d = \Lambda$ and having a spectral gap in the thermodynamic limit, showing that there IS a stable phase to which the ansatz could apply.

*[F] FOCM v.1, p.183-204 (2001)

If the spectral gap is too hard to prove, do not worry, we will simply check it numerically on an anyonic (H_e) quantum computer.

A computationally complete rep.
$\rho : \text{Braid} \longrightarrow U(\mathcal{H})$ is believed
to arise from braiding "quasi-particle" excitations of a Quantum Hall electron fluid in the first Landau level at filling fraction: $\nu = 8/5$

FQHE is very tricky physics:

1. months to grow crystals
2. low temp. (mil. kelvin)
3. huge B-field (10-15 Tesla)

We follow: Fisher-Senthil and Nayak-Shtengel in looking for anyons in antiferromagnets which should have higher energy scale.

Also the conjectured (Read-Rezayi) link FQHE $\Longleftrightarrow$ Chern-Simons-Jones seems to need a quantum computer to check fully.