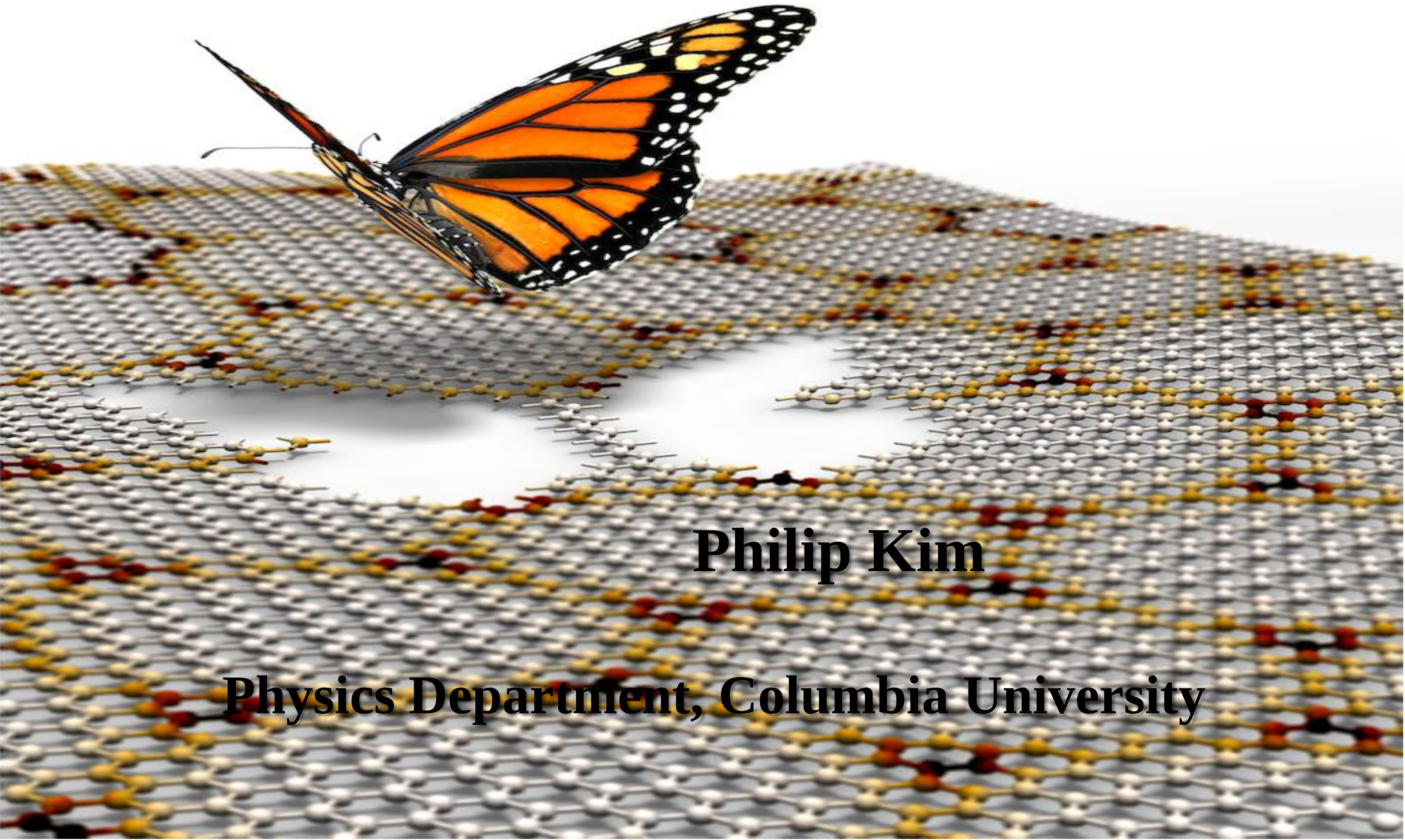
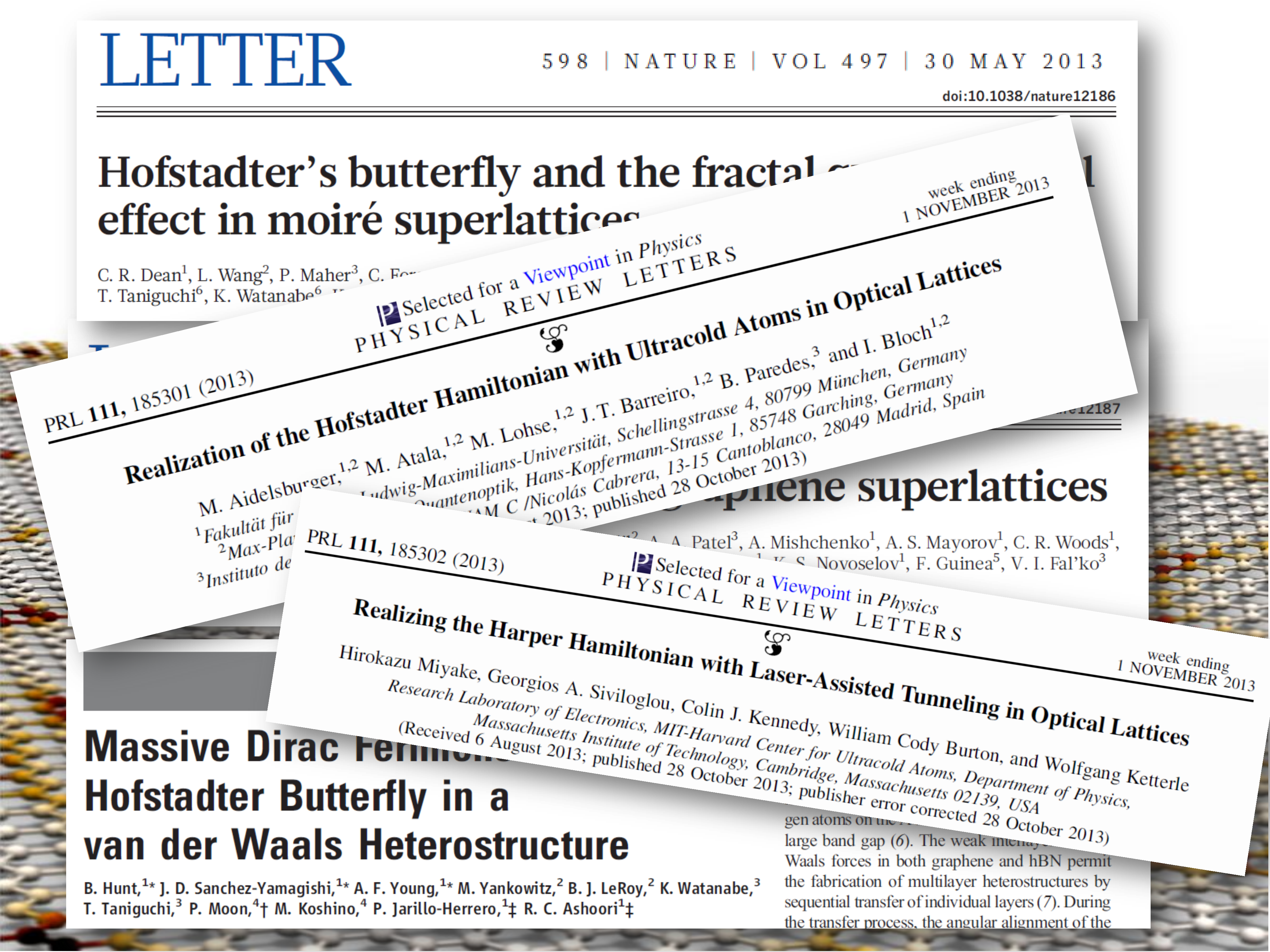
Bloch, Landau, and Dirac: Hofstadter's Butterfly in Graphene



Philip Kim

Physics Department, Columbia University

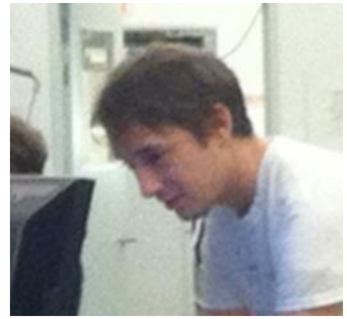


LETTER
598 | NATURE | VOL 497 | 30 MAY 2013
doi:10.1038/nature12186
Hofstadter's butterfly and the fractal
effect in moiré superlattices
C. R. Dean[1], L. Wang[2], P. Maher[3], C. For
T. Taniguchi[6], K. Watanabe[6]
PRL 111, 185301 (2013)
Selected for a Viewpoint in Physics
PHYSICAL REVIEW LETTERS
week ending 1 NOVEMBER 2013
Realization of the Hofstadter Hamiltonian with Ultracold Atoms in Optical Lattices
M. Aidelsburger,[1,2] M. Atala,[1,2] M. Lohse,[1,2] J. T. Barreiro,[1,2] B. Paredes,[3] and I. Bloch[1,2]
[1]Fakultät für Physik, Ludwig-Maximilians-Universität, Schellingstrasse 4, 80799 München, Germany
[2]Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, 85748 Garching, Germany
[3]Instituto de Física Teórica CSIC/UAM C /Nicolás Cabrera, 13-15 Cantoblanco, 28049 Madrid, Spain
2013; published 28 October 2013)
nature12187
phene superlattices
A. Patel[3], A. Mishchenko[1], A. S. Mayorov[1], C. R. Woods[1],
K. S. Novoselov[1], F. Guinea[5], V. I. Fal'ko[3]
PRL 111, 185302 (2013)
Selected for a Viewpoint in Physics
PHYSICAL REVIEW LETTERS
week ending 1 NOVEMBER 2013
Realizing the Harper Hamiltonian with Laser-Assisted Tunneling in Optical Lattices
Hirokazu Miyake, Georgios A. Siviloglou, Colin J. Kennedy, William Cody Burton, and Wolfgang Ketterle
Research Laboratory of Electronics, MIT-Harvard Center for Ultracold Atoms, Department of Physics,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
(Received 6 August 2013; published 28 October 2013; publisher error corrected 28 October 2013)
Massive Dirac Fermions
Hofstadter Butterfly in a
van der Waals Heterostructure
B. Hunt,[1*] J. D. Sanchez-Yamagishi,[1*] A. F. Young,[1*] M. Yankowitz,[2] B. J. LeRoy,[2] K. Watanabe,[3]
T. Taniguchi,[3] P. Moon,[4†] M. Koshino,[4] P. Jarillo-Herrero,[1‡] R. C. Ashoori[1‡]
gen atoms on the
large band gap (6). The weak interlayer
Waals forces in both graphene and hBN permit
the fabrication of multilayer heterostructures by
sequential transfer of individual layers (7). During
the transfer process, the angular alignment of the

Acknowledgment

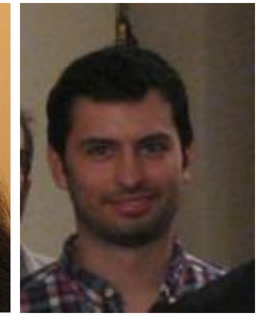
Prof. Cory Dean
(now at CUNY)

Lei Wang



Patrick Maher

Fereshte Ghahari



Carlos Forsythe

Prof. Jim Hone

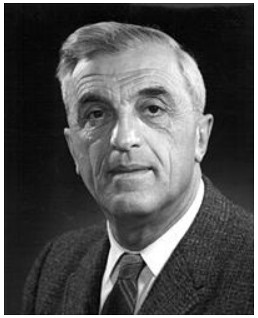
Prof. Ken Shepard

Theory: P. Moon & M. Koshino (Tohoku)

hBN samples: T. Taniguchi & K. Watanabe (NIMS)

Funding:

Bloch Waves: Periodic Structure & Band Filling



Felix Bloch

Zeitschrift für Physik, 52, 555 (1929)

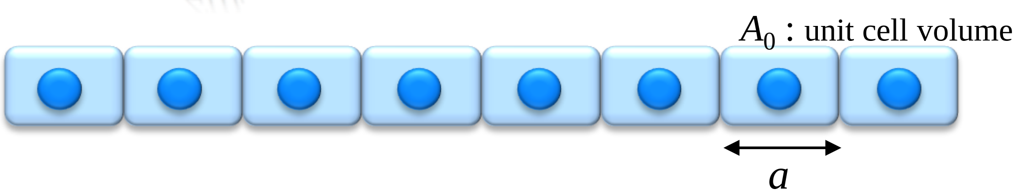
Über die Quantenmechanik der Elektronen in Kristallgittern.

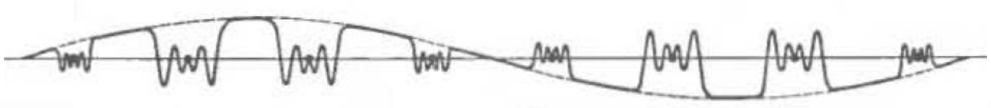
Von **Felix Bloch** in Leipzig.

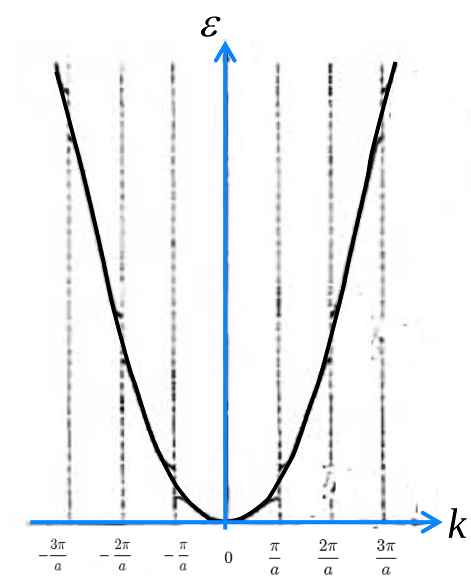
Mit 2 Abbildungen. (Eingegangen am 10. August 1928.)

Periodic Lattice

$$\tilde{H} = \frac{\tilde{p}^2}{2m} + U(x), \qquad U(x) = U(x+a)$$



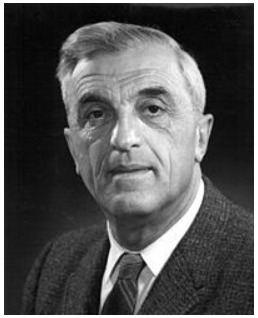




Block Waves:

$$\psi_{n,k}(x) = e^{ikx} u_{n,k}(x), \qquad u_{n,k}(x+a) = u_{n,k}(x)$$

Bloch Waves: Periodic Structure & Band Filling



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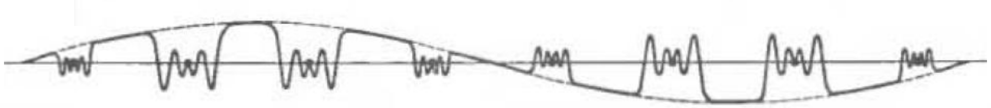

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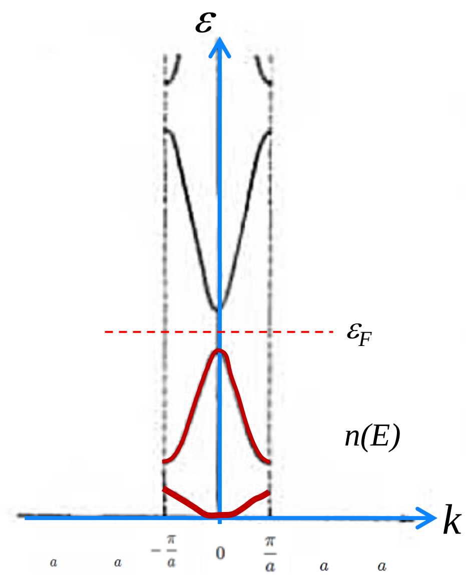
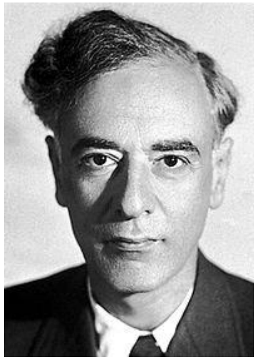


Band Filling factor

$$s = -A_0^2 \frac{\partial n(\varepsilon_F)}{\partial A_0}$$

MacDonald (1983)

Landau Levels: Quantization of Cyclotron Orbits



Lev Landau

Zeitschrift für Physik, 64, 629 (1930)

Diamagnetismus der Metalle.

Von **L. Landau**, zurzeit in Cambridge (England).

(Eingegangen am 25. Juli 1930.)

Free electron under magnetic field

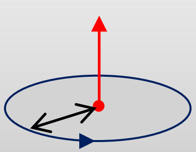
$$\tilde{H} = \frac{(\tilde{\mathrm{p}} - e\mathbf{A}/c)^2}{2m}$$

Energy and orbit are quantized:

$$\varepsilon_n = \hbar w_c(n + 1/2), \qquad w_c = eB/mc$$

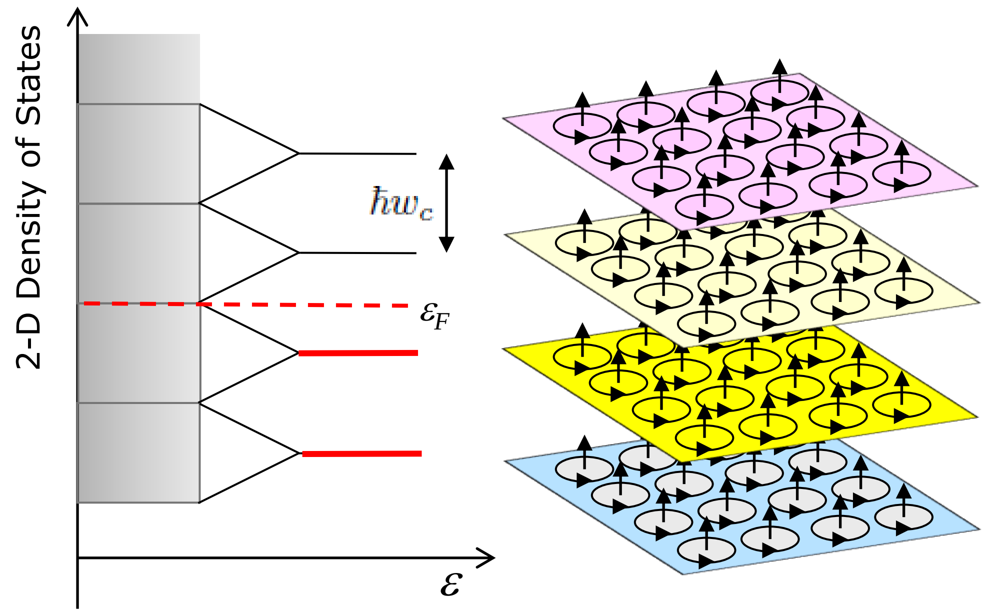
Each Landau orbit contains magnetic flux quanta

$$\phi_0 = \frac{hc}{e}$$



$$\ell_B = \sqrt{\hbar/eB}$$

2-dimensional electron systems



Massively degenerated energy level

Landau level filling fraction:

$$\nu = 2\pi\ell_B^2 n(\varepsilon_F)$$

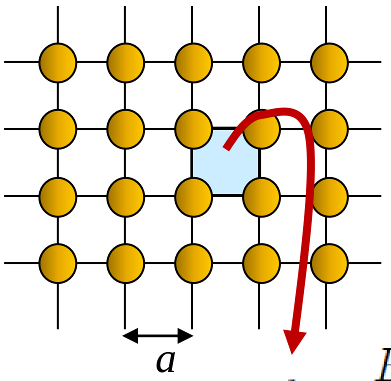
Harper's Equation: Competition of Two Length Scales

Proc. Phys. Soc. Lond. A 68 879 (1955)

The General Motion of Conduction Electrons in a Uniform Magnetic Field, with Application to the Diamagnetism of Metals

By P. G. HARPER†
Department of Mathematical Physics, University of Birmingham

Communicated by R. E. Peierls; MS. received 19th January 1955 and in amended form 27th April 1955



$$b = \frac{Ba^2}{\phi_0} = \frac{\phi}{\phi_0}$$

Tight binding on 2D Square lattice with magnetic field

$$\tilde{H} = \frac{(\tilde{\mathbf{p}} - e\mathbf{A}/c)^2}{2m} + U(\mathbf{r})$$

Two competing length scales:

a **: lattice periodicity**

l_B **: magnetic periodicity**

<u>Harper's Equation</u>

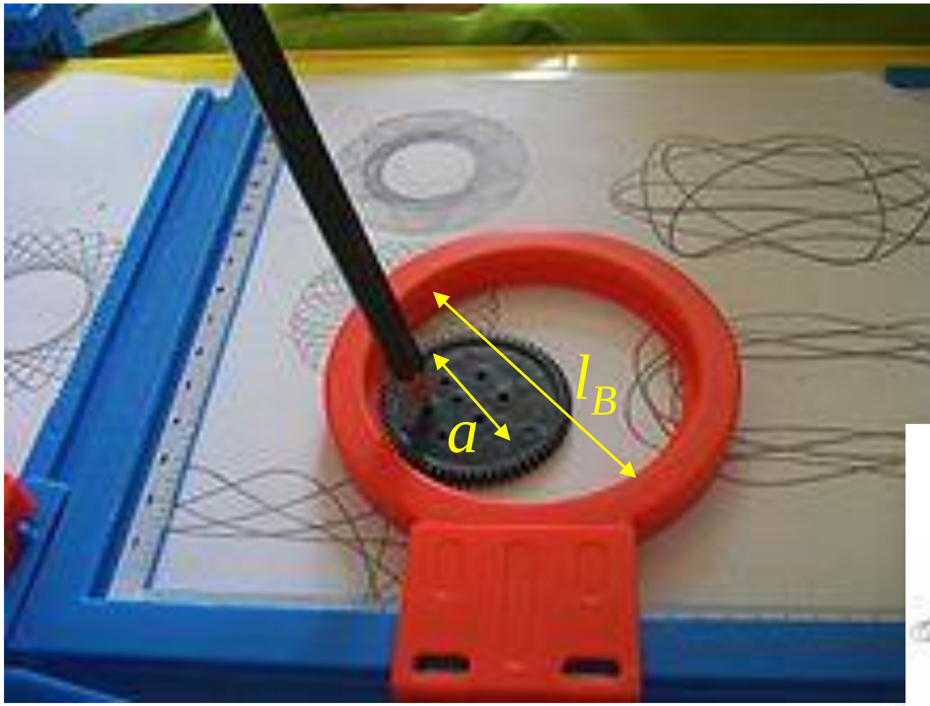
$$2\psi_l \cos(2\pi l b - \kappa) + \psi_{l+1} + \psi_{l-1} = E\psi_l$$

For $b \ll \mu^* H$, the broadening factor may be written approximately $\exp[-(bv\pi/\mu^* H)^2]$ and the broadening effect becomes additive to that due to collision as described by Dingle (1952 b).

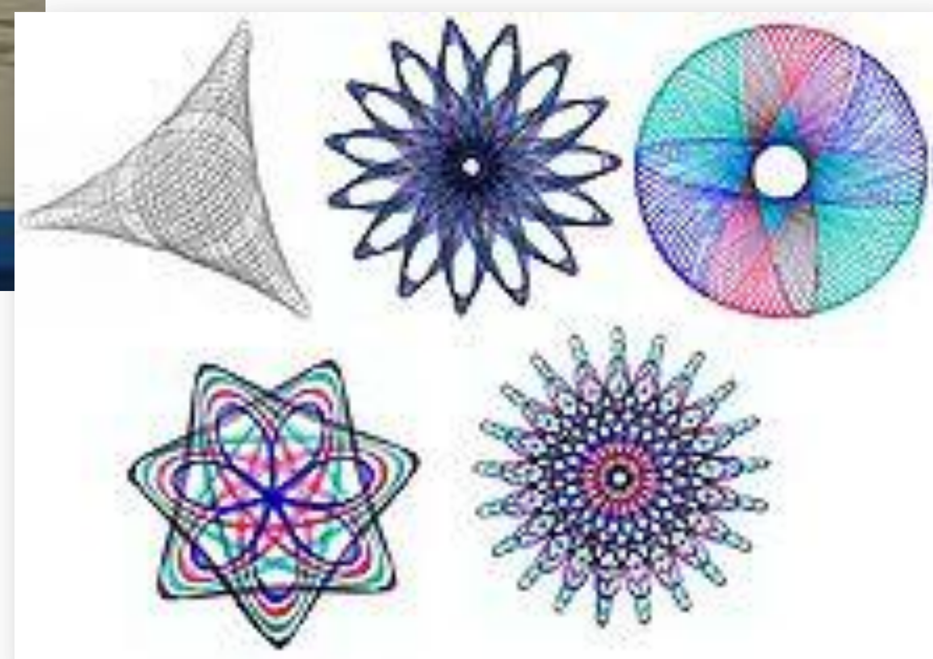
The level structure in the vicinity of an energy gap near a zone boundary is all-important for the de Haas–van Alphen effect. Unfortunately, this seems very difficult to determine in detail, even for a sinusoidal potential. Apart from the regularity already mentioned there seems little one can say. It is likely, however, that if periodicity exists, the period will give rise to effective mass parameters much smaller than one would otherwise expect. This is because the level structure will consist of irregular groups, regularly repeated. Since the period is large, the oscillatory period will also be large and the effective mass correspondingly smaller.

Commensuration / Incommensuration of Two Length Scales

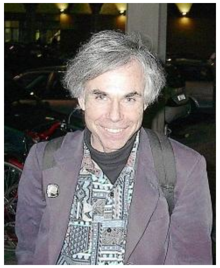
Spirograph



$$a \,/\, l_B = p/q$$



Hofstadter's Butterfly



PHYSICAL REVIEW B VOLUME 14, NUMBER 6 15 SEPTEMBER 1976

Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*

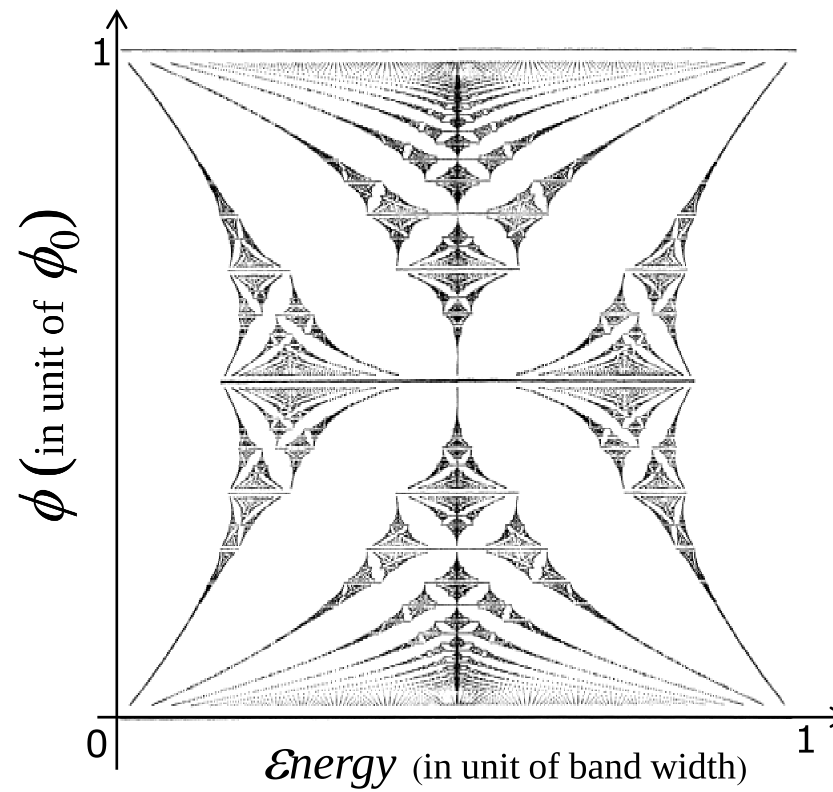
Douglas R. Hofstadter[†]
Physics Department, University of Oregon, Eugene, Oregon 97403
(Received 9 February 1976)

Harper's Equation

$$2\psi_l \cos\left(2\pi l b - \kappa\right) + \psi_{l+1} + \psi_{l-1} = E\psi_l$$

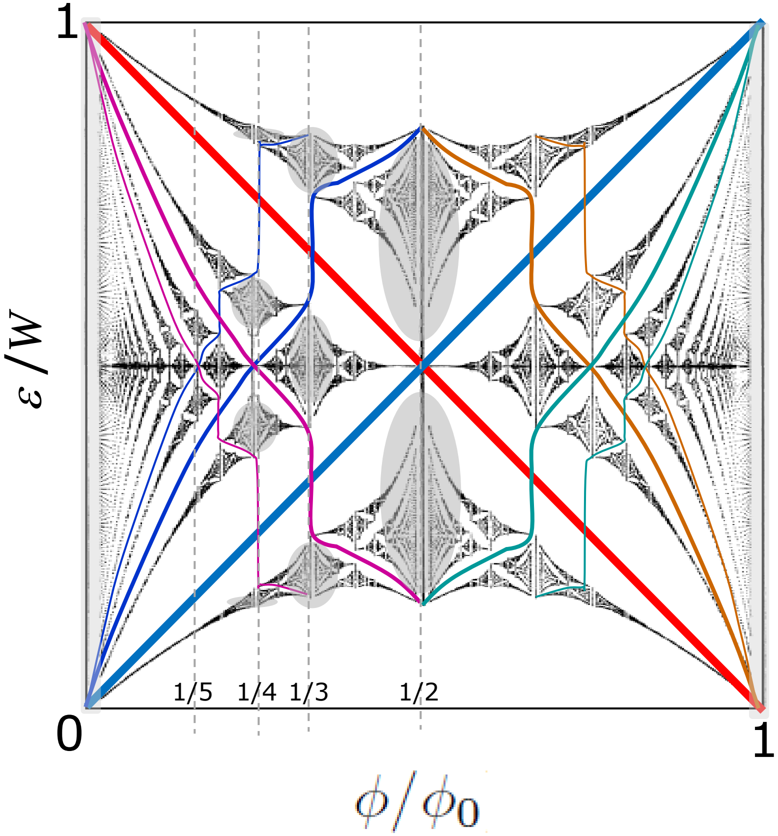
When $b=p/q$, where p, q are coprimes, each LL splits into ***q sub-bands that are p-fold degenerate***

Energy bands develop ***fractal structure*** when magnetic length is of order the periodic unit cell



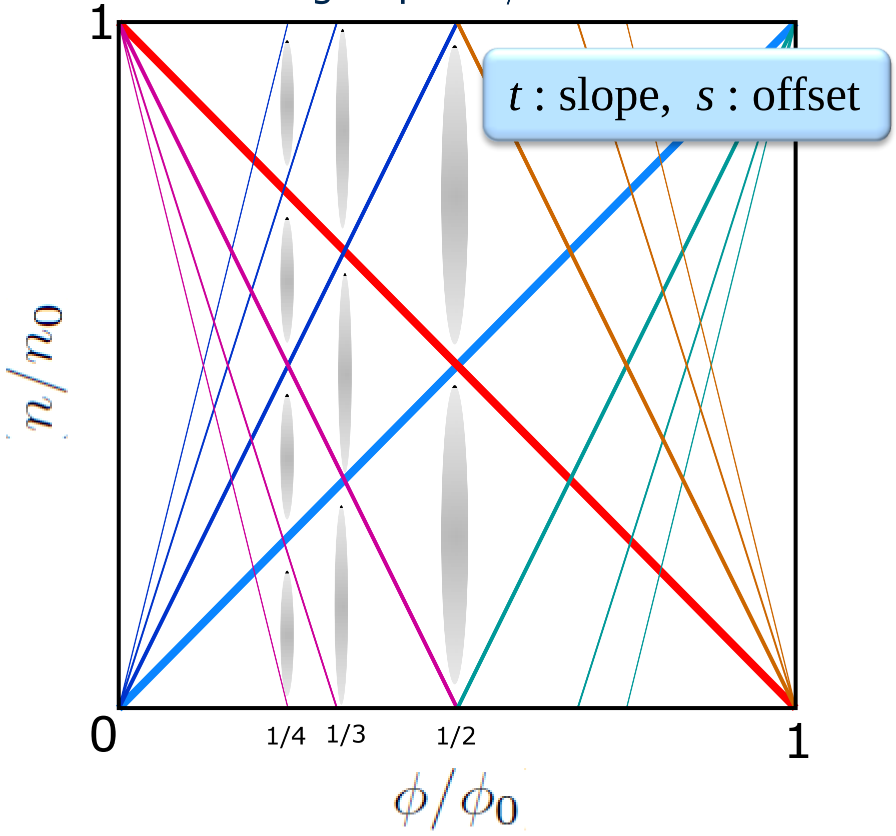
Energy Gaps in the Butterfly: Wannier Diagram

Hofstadter's Energy Spectrum



Wannier, *Phys. Status Solidi*. **88**, 757 (1978)

Tracing Gaps in ϕ and n



n_0: # of state per unit cell

ϕ : magnetic flux in unit cell

n : electron density

Diophantine equation for gaps

$$(n/n_0) = t(\phi/\phi_0) + s$$

$t, s \in \mathbb{Z}$

Streda Formula and TKNN Integers

What is the physical meaning of the integers *s* and *t* ?

Band Filling factor

$$s = -A_0^2 \frac{\partial n(\varepsilon_F)}{\partial A_0}$$

Quantum Hall Conductance

$$\sigma_{xy}^{Q} = ec \frac{\partial n(E)}{\partial B}\bigg|_{E=E_F} = \frac{e^2}{h} t$$

J. Phys. C: Solid State Phys., 15 (1982) L1299–L1303. Printed in Great Britain

LETTER TO THE EDITOR

Quantised Hall effect in a two-dimensional periodic potential

P Středa
Institute of Physics, Czechoslovak Academy of Sciences, 180 40 Praha 6, Na
Czechoslovakia

Received 6 October 1982

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

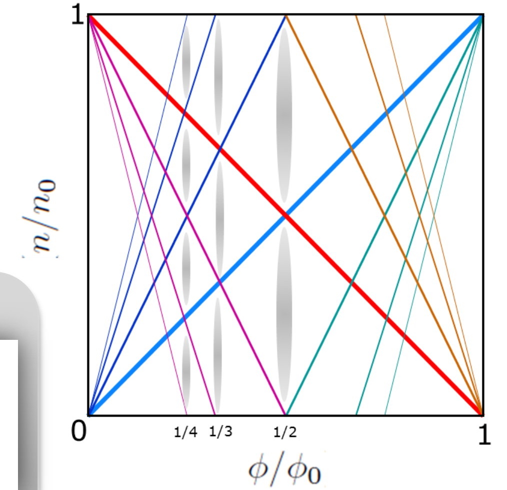
Quantized Hall Conductance in a Two-Dimensional Periodic Potential

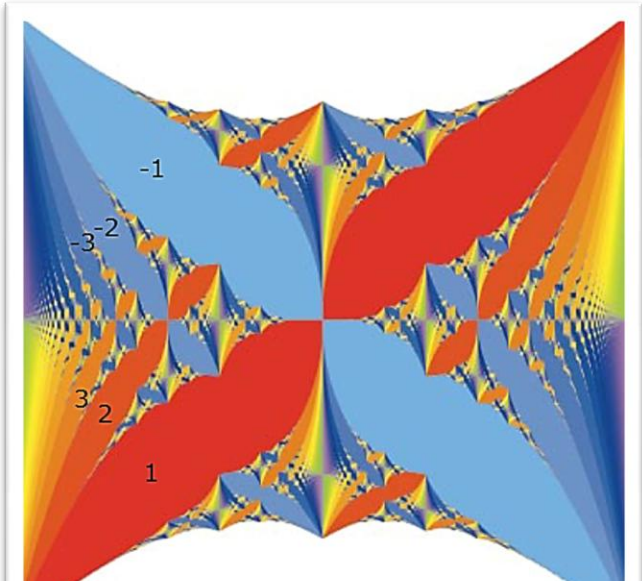
D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

$$\sigma_H = \frac{ie^2}{2\pi h} \sum \int d^2k \int d^2r \left(\frac{\partial u^*}{\partial k_1} \frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2} \frac{\partial u}{\partial k_1} \right)$$

$$= \frac{ie^2}{4\pi h} \sum \oint dk_j \int d^2r \left(u^* \frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j} u \right),$$

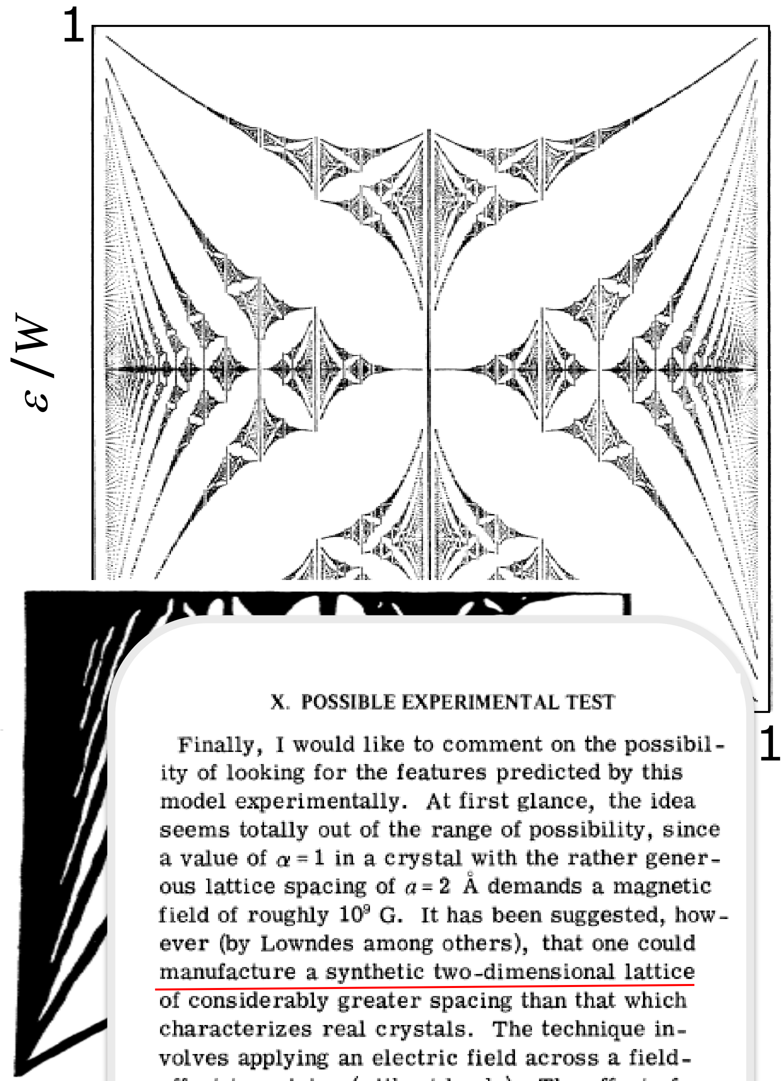
$$(n/n_0) = t(\phi/\phi_0) + s$$





Osadchy and Avron, J. Math. Phys. 42, 5665 (2001)

Experimental Challenges



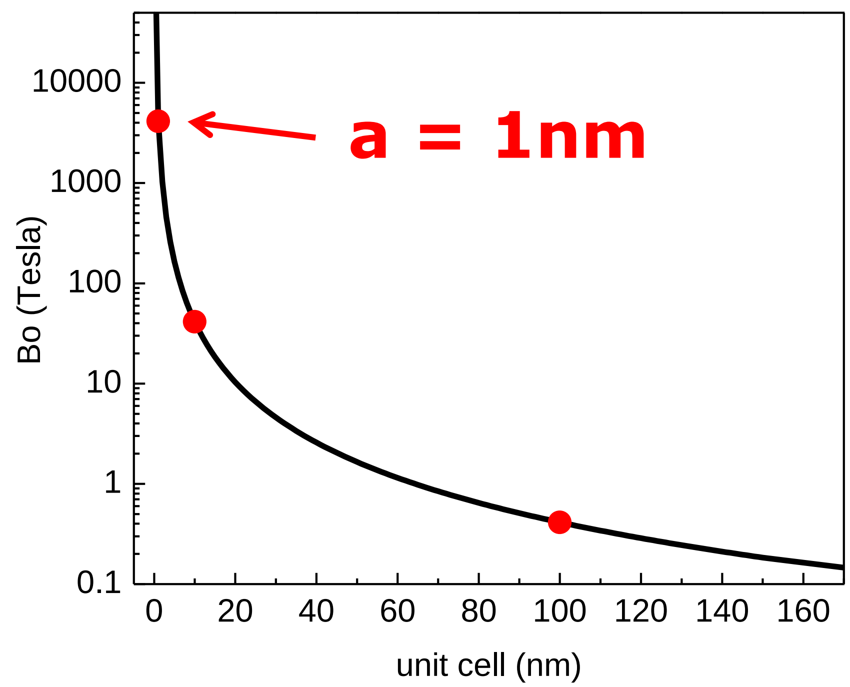
X. POSSIBLE EXPERIMENTAL TEST

Finally, I would like to comment on the possibility of looking for the features predicted by this model experimentally. At first glance, the idea seems totally out of the range of possibility, since a value of $\alpha = 1$ in a crystal with the rather generous lattice spacing of $a = 2$ Å demands a magnetic field of roughly 10^9 G. It has been suggested, however (by Lowndes among others), that one could manufacture a synthetic two-dimensional lattice of considerably greater spacing than that which characterizes real crystals. The technique involves applying an electric field across a field-effect transistor (without leads). The effect of

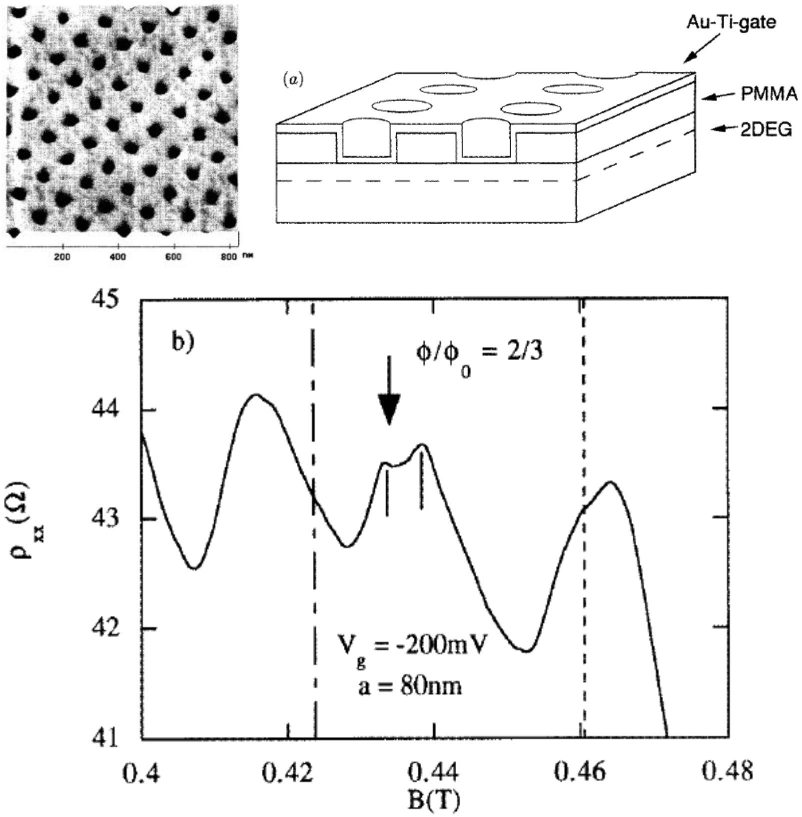
Hofstadter (1976)

Obvious technical challenge:

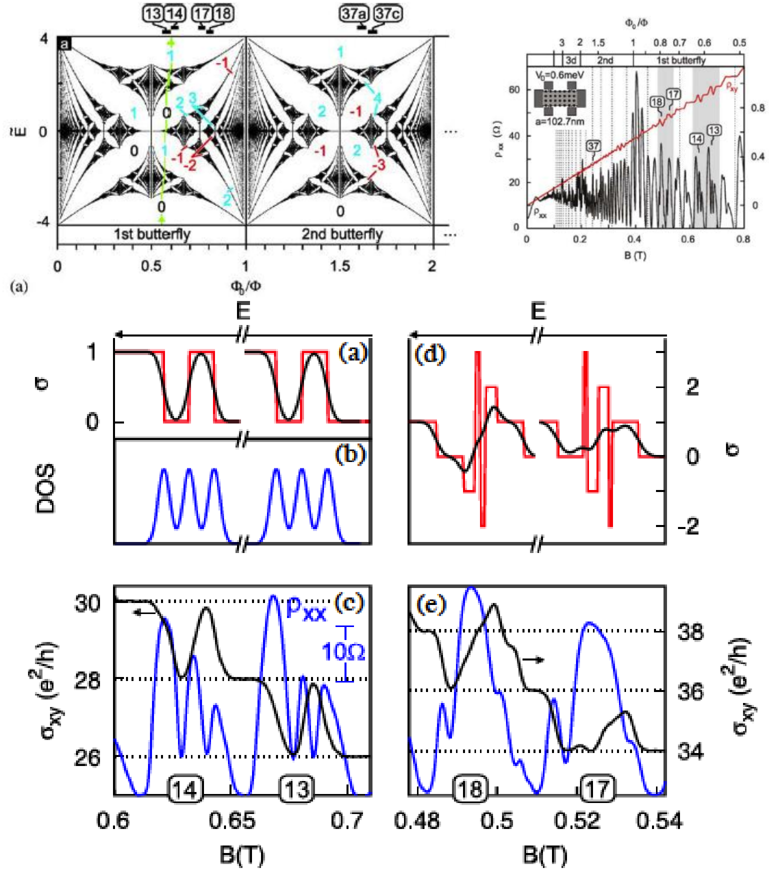
$$\frac{\phi}{\phi_o} = \frac{Ba^2}{h/e} \sim 1$$



Experimental Search For Butterfly



-Schlosser et al, Semicond. Sci. Technol. (1996)

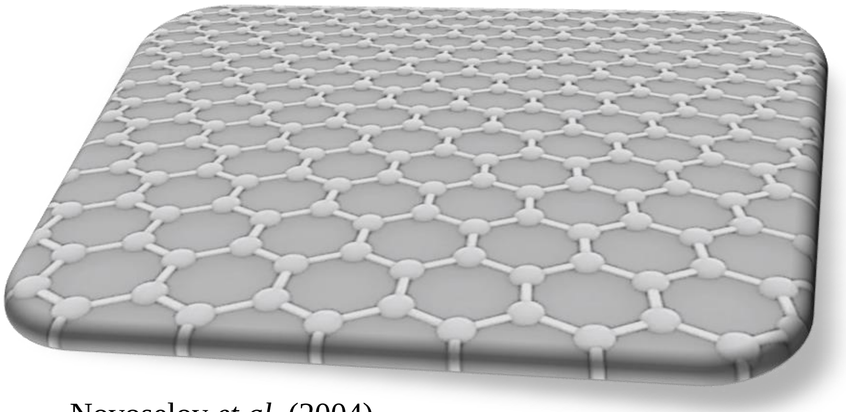


Albrecht et al, PRL. (2001);
Geisler et al, PRL (2004)

- Unit cell limited to ~40-100 nm
- limited field and density range accessible, weak perturbation
- Do not observe ‘fully quantized’ mingaps in fractal spectrum

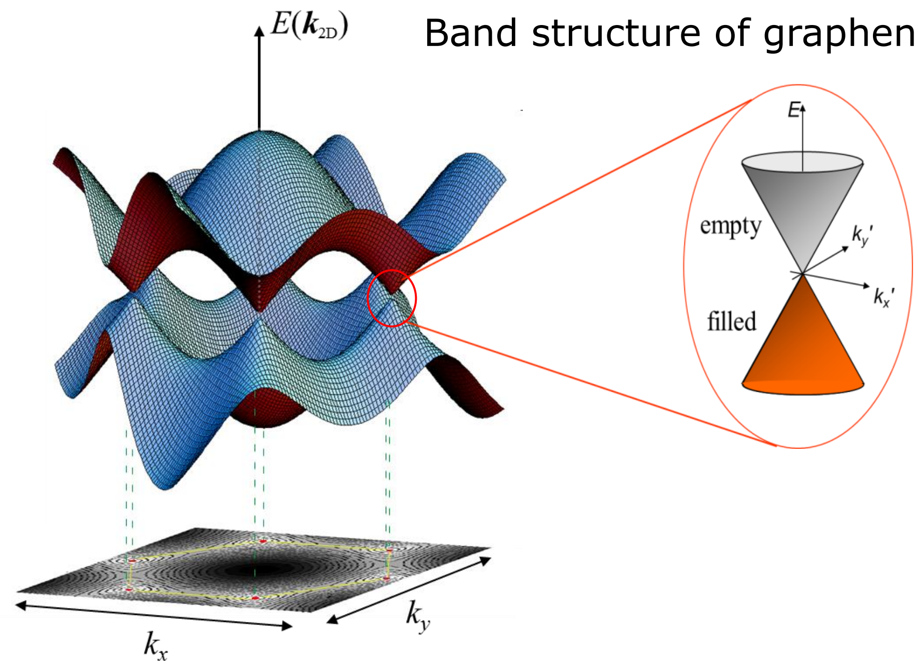
Electrons in Graphene: Effective Dirac Fermions

Graphene,
ultimate 2-d conducting system



Novoseloy *et al.* (2004)

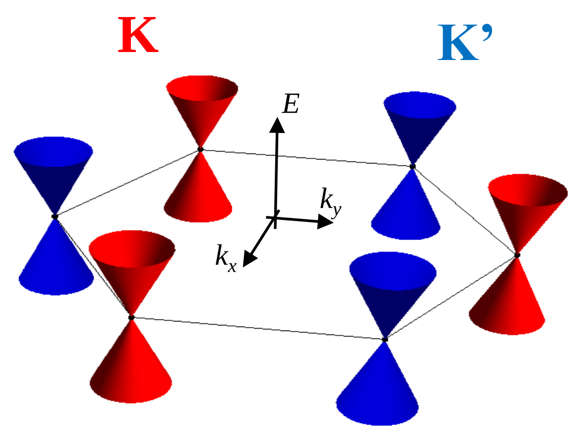
Band structure of graphene

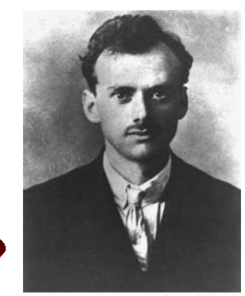


Effective Dirac Equations

$$H_{eff} = \pm \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix} = \pm \hbar v_F \vec{\sigma} \cdot \vec{k}_\perp$$

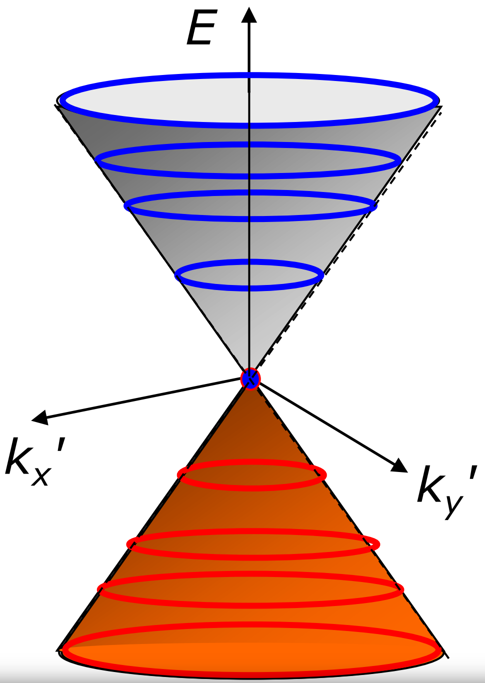
DiVincenzo and Mele, PRB (1984); Semenov, PRL (1984)

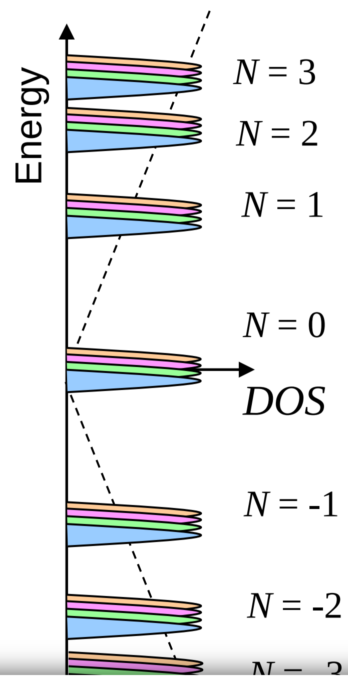


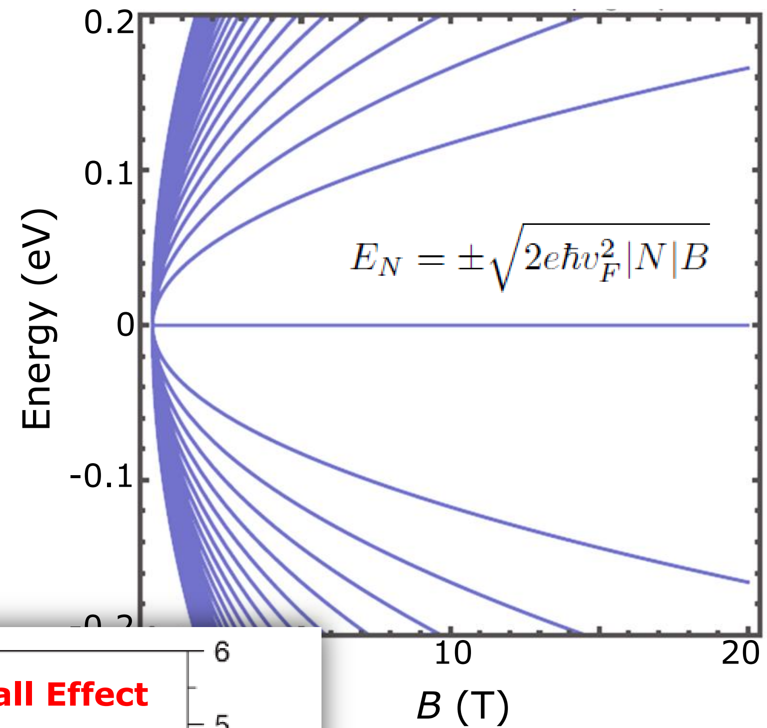


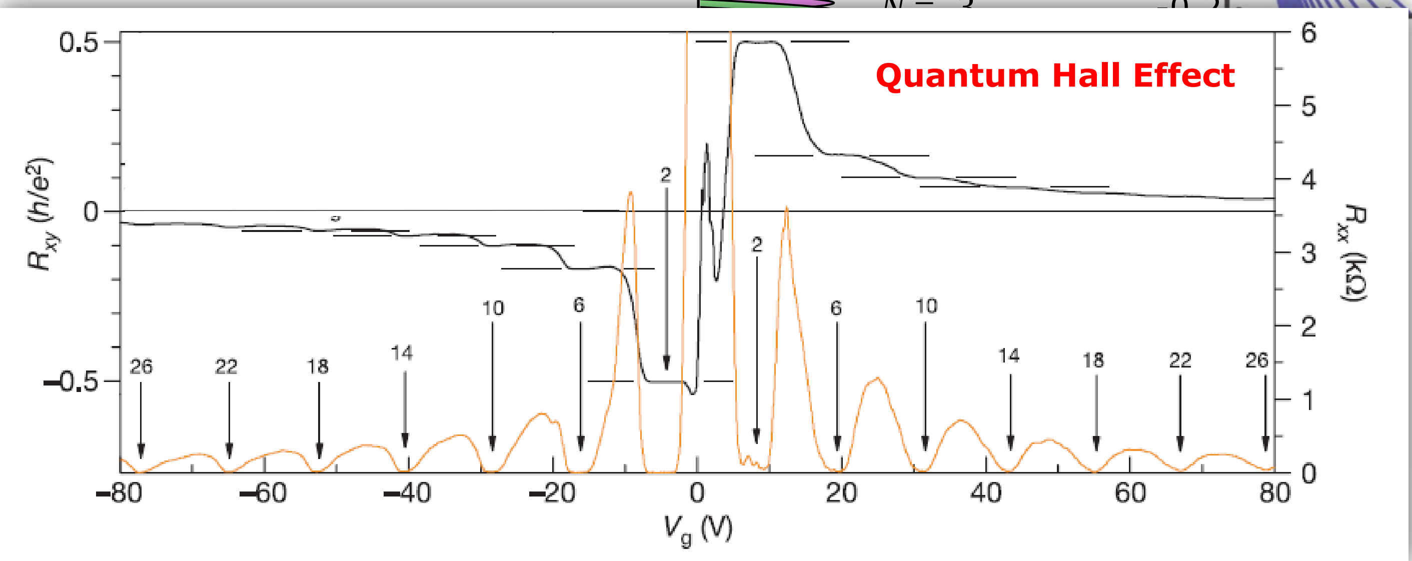
Paul Dirac

Graphene: Under Magnetic Fields









Quantization Condition

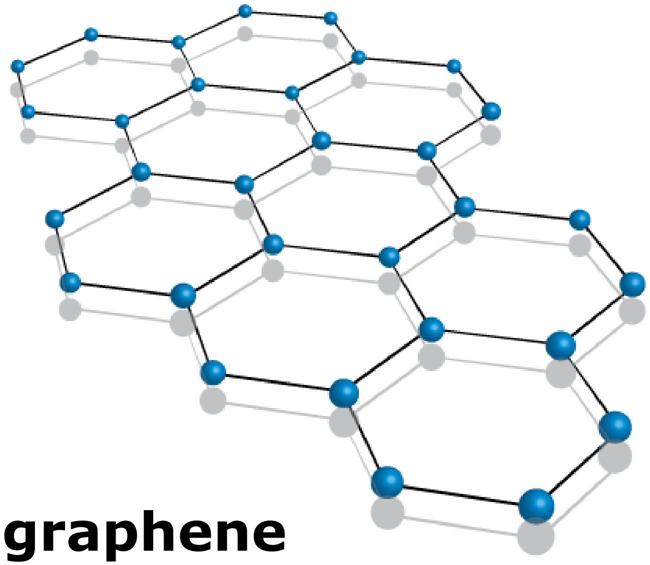
$$R_{xy}^{-1} = \frac{4e^2}{h}(N + \frac{1}{2})$$

$$\nu = \pm 2, \pm 6, \pm 10, \ldots$$

Novoselov et al (2005)
Zhang et al (2005)

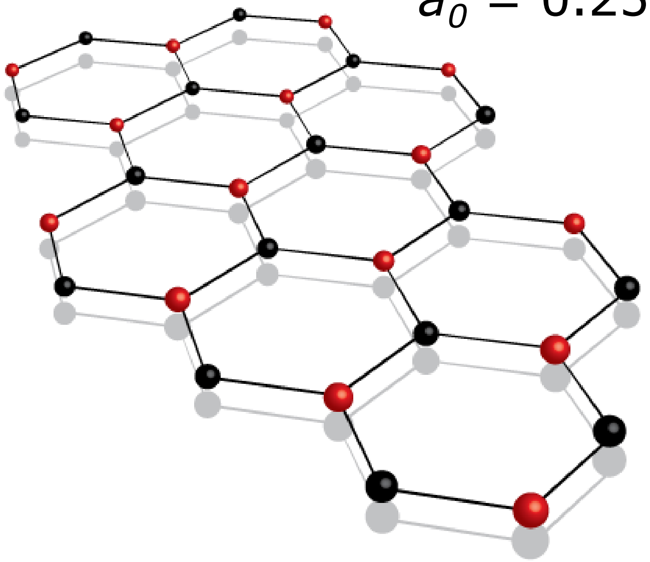
Hexa Boron Nitride: Polymorphic Graphene

a_0 = 0.246 nm



graphene

a_0 = 0.250 nm



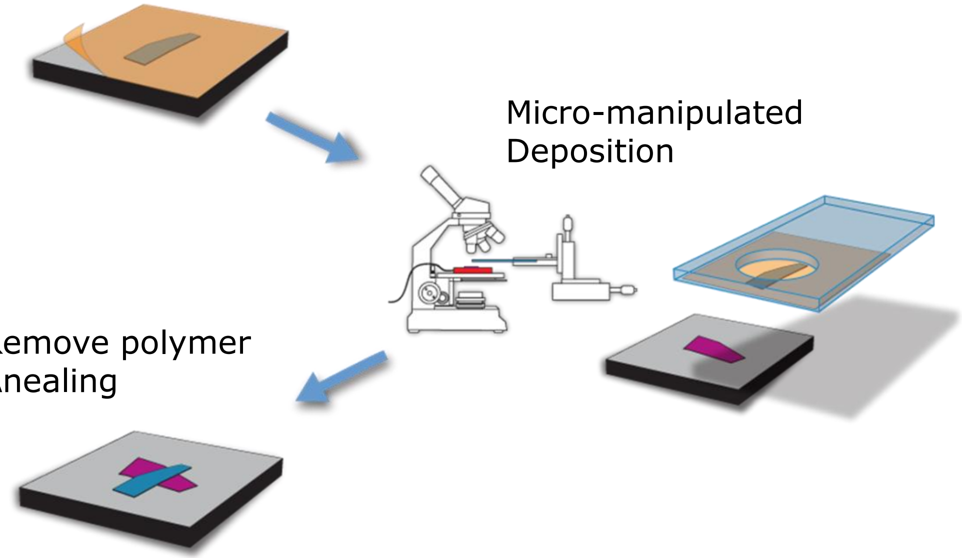
Boron Nitride

Comparison of h-BN and SiO_2

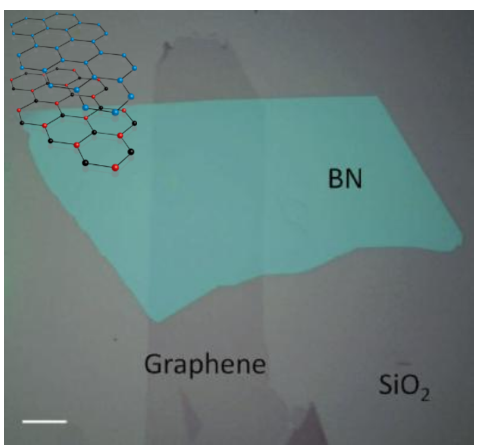
	Band Gap	Dielectric Constant	Optical Phonon Energy	Structure
BN	5.5 eV	~4	>150 meV	Layered crystal
SiO2	8.9 eV	3.9	59 meV	Amorphous

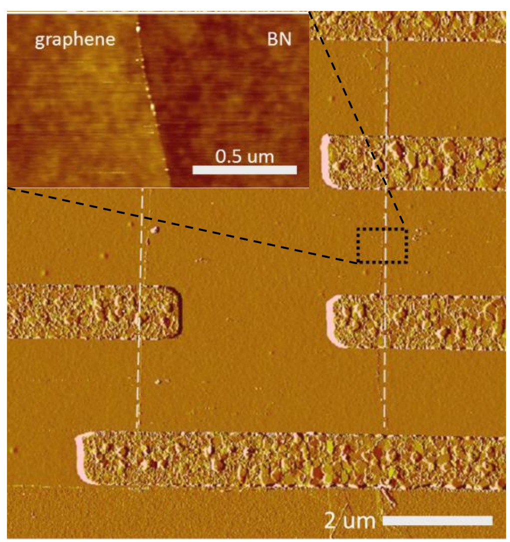
Stacking graphene on hBN

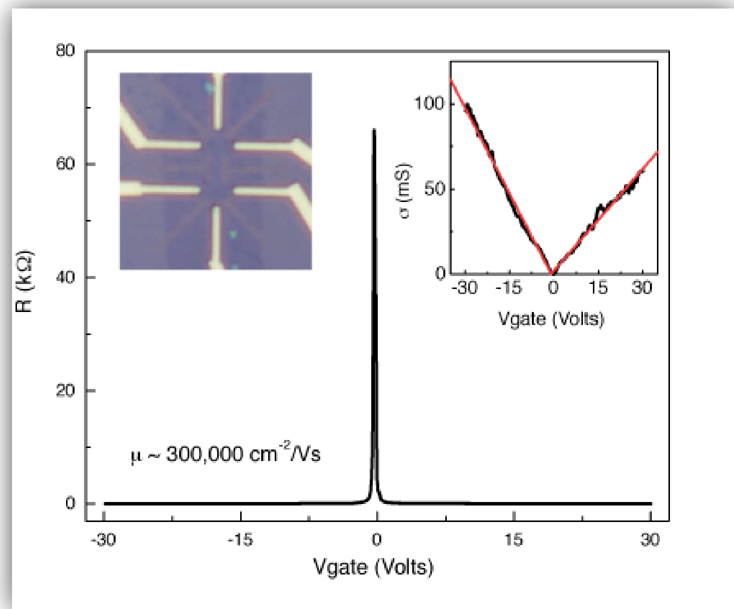
Dean et al. Nature Nano (2009)



- Co-lamination techniques
- Submicron size precision
- Atomically smooth interface

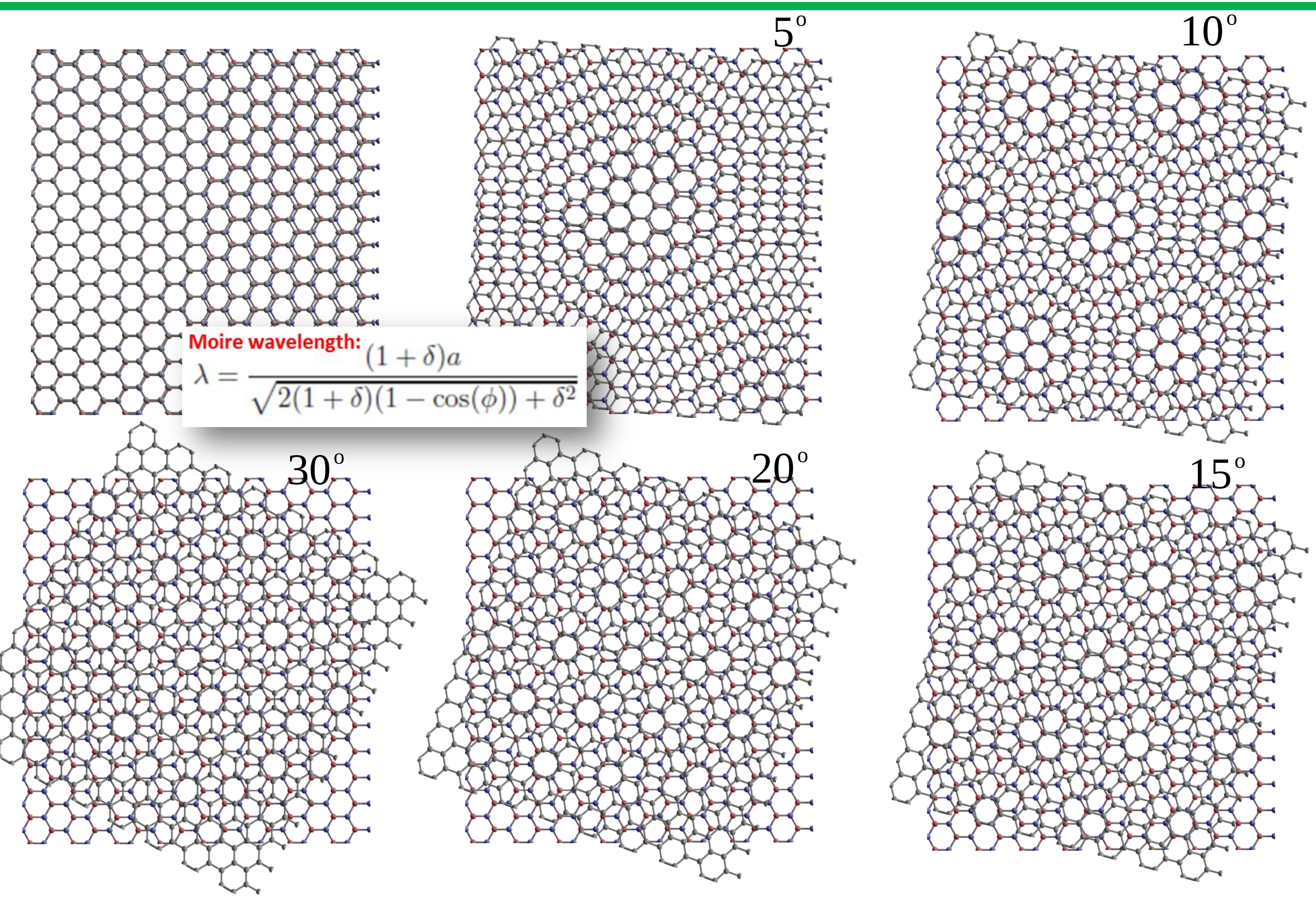






Mobility > 100,000 $cm^2V^{-1}s^{-1}$

Graphen/hBN Moire Pattern

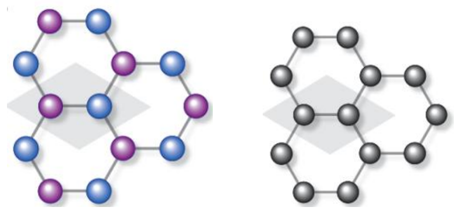


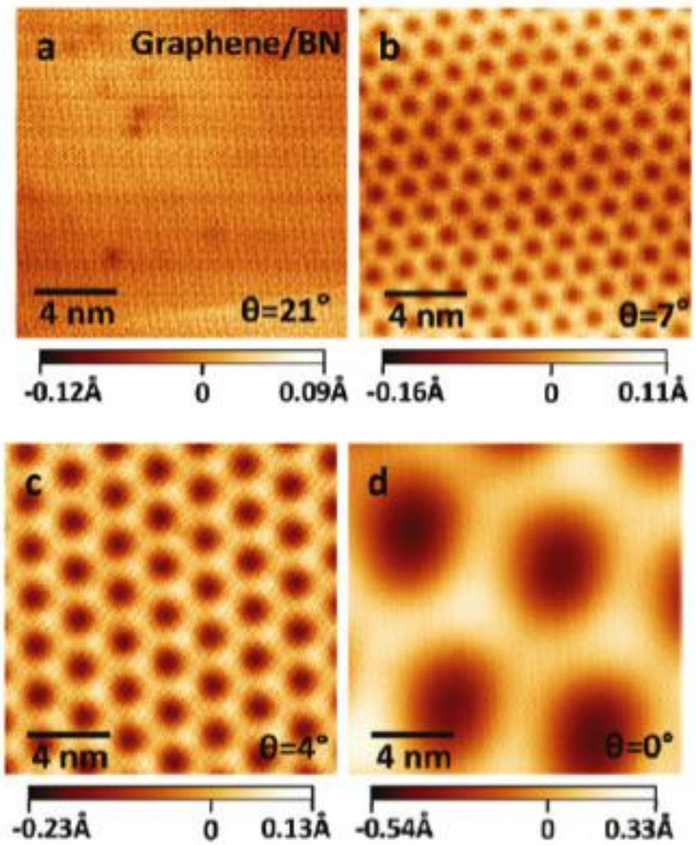
Moire wavelength:

$$\lambda = \frac{(1+\delta)a}{\sqrt{2(1+\delta)(1-\cos(\phi))+\delta^2}}$$

Moire pattern in Graphene on hBN: a new route to Hofstadter's butterfly?

Graphene on BN exhibits clear Moiré pattern





Xue et al, Nature Mater (2011);
Decker et al Nano Lett (2011)

LETTERS
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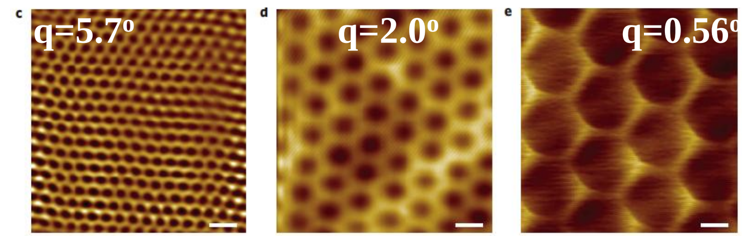
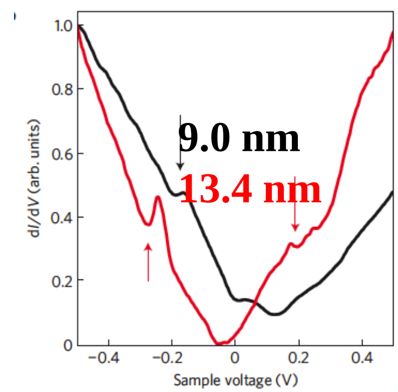
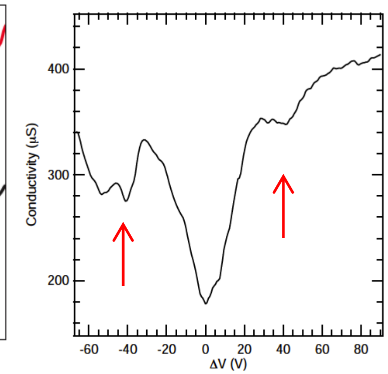
nature physics

Emergence of superlattice Dirac points in graphene on hexagonal boron nitride

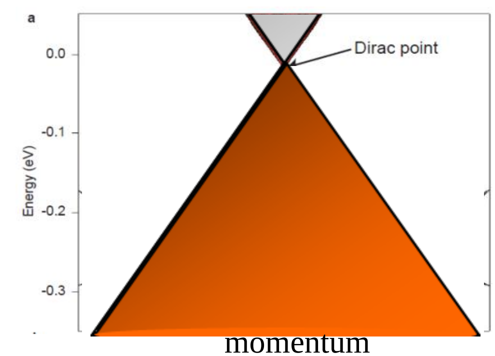
Matthew Yankowitz[1], Jiamin Xue[1], Daniel Cormode[1], Javier D. Sanchez-Yamagishi[2], K. Watanabe[3], T. Taniguchi[3], Pablo Jarillo-Herrero[2], Philippe Jacquod[1,4] and Brian J. LeRoy[1]*





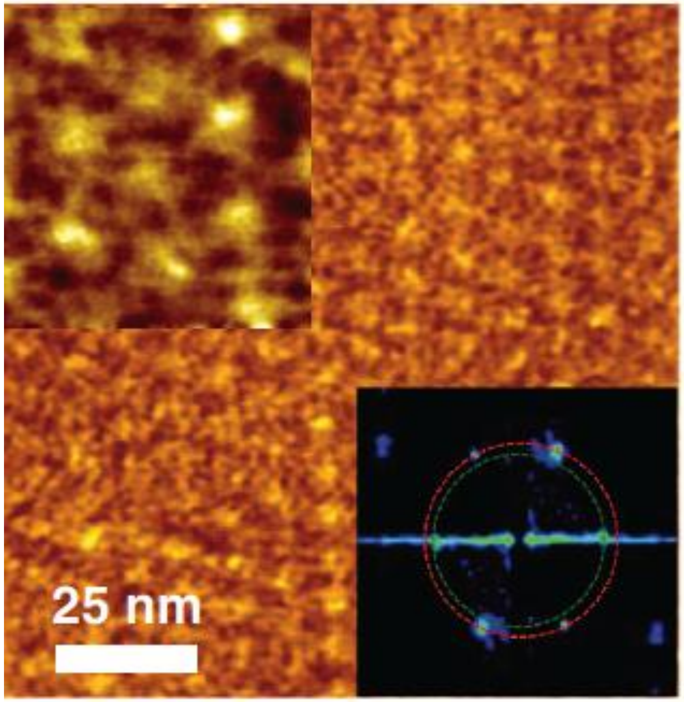


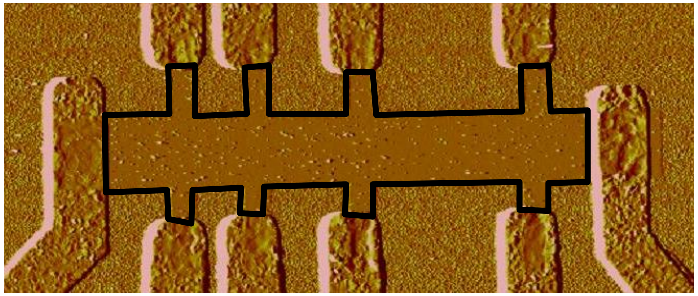
Minigap formation
near the Dirac point
due to Moire superlattice



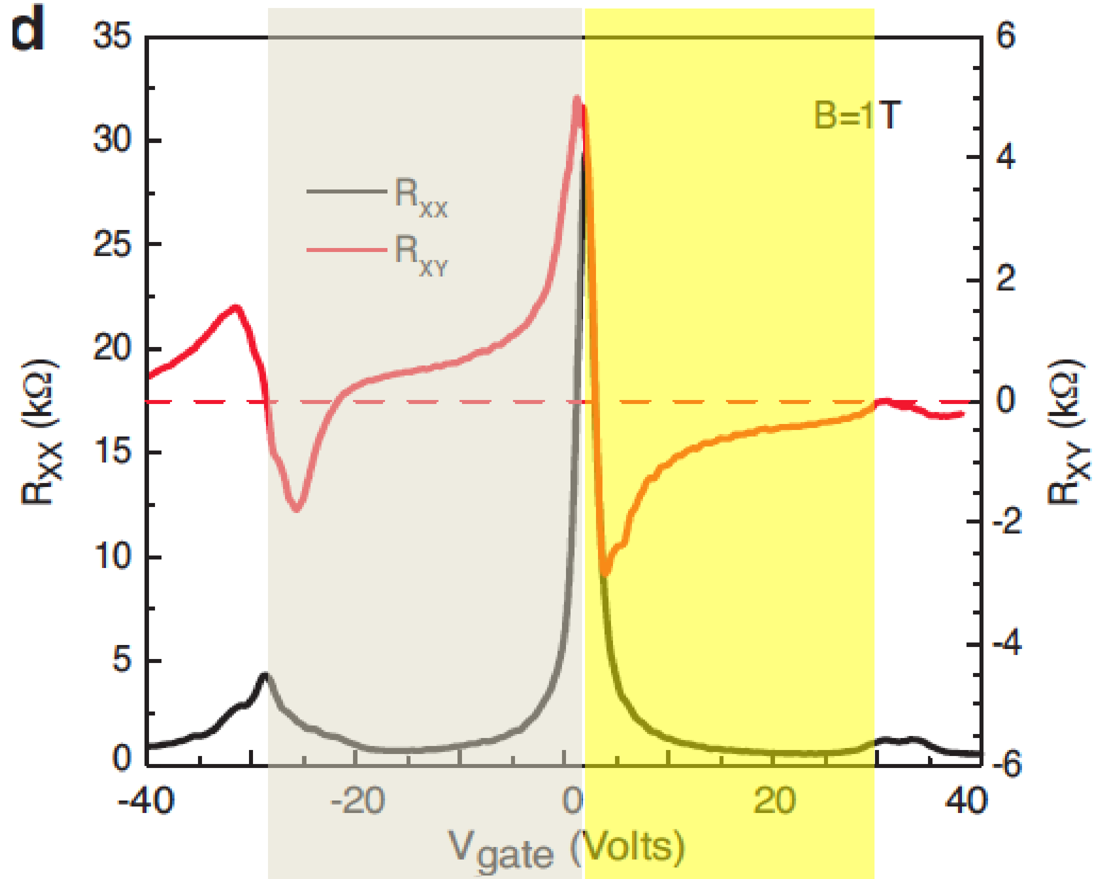
Transport Measurement Graphene with Moire Superlattice

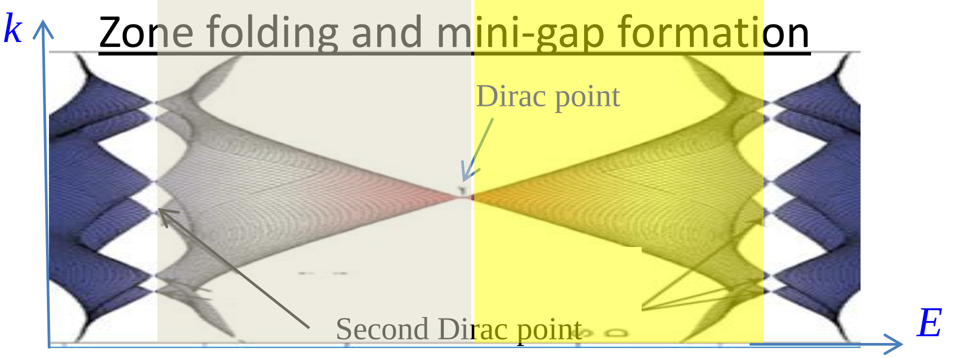
UHV AFM (Ishigami group)





Moiré λ_{AFM} = 15.5 nm



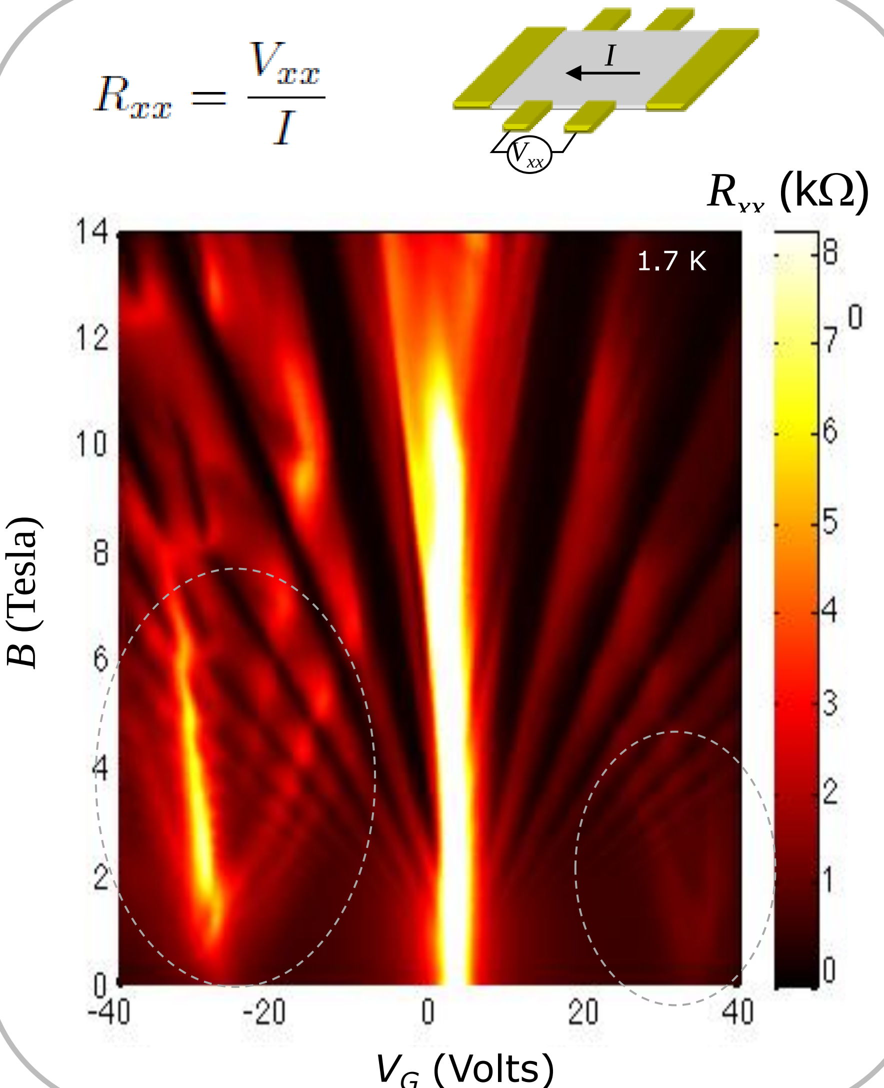


Abnormal Landau Fan Diagram in Bilayer on hBN

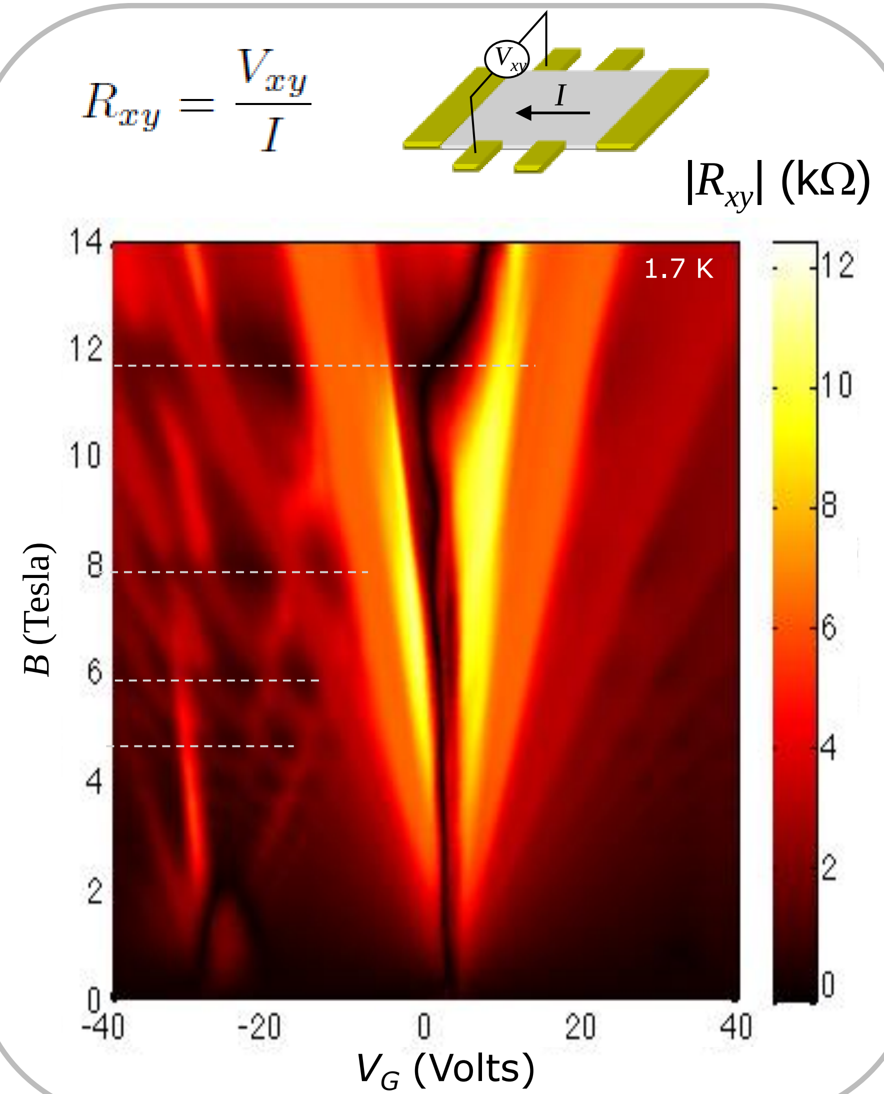
Special Samples with Large Moire Unit Cell

Low Magnetic field regime

$$R_{xx} = \frac{V_{xx}}{I}$$



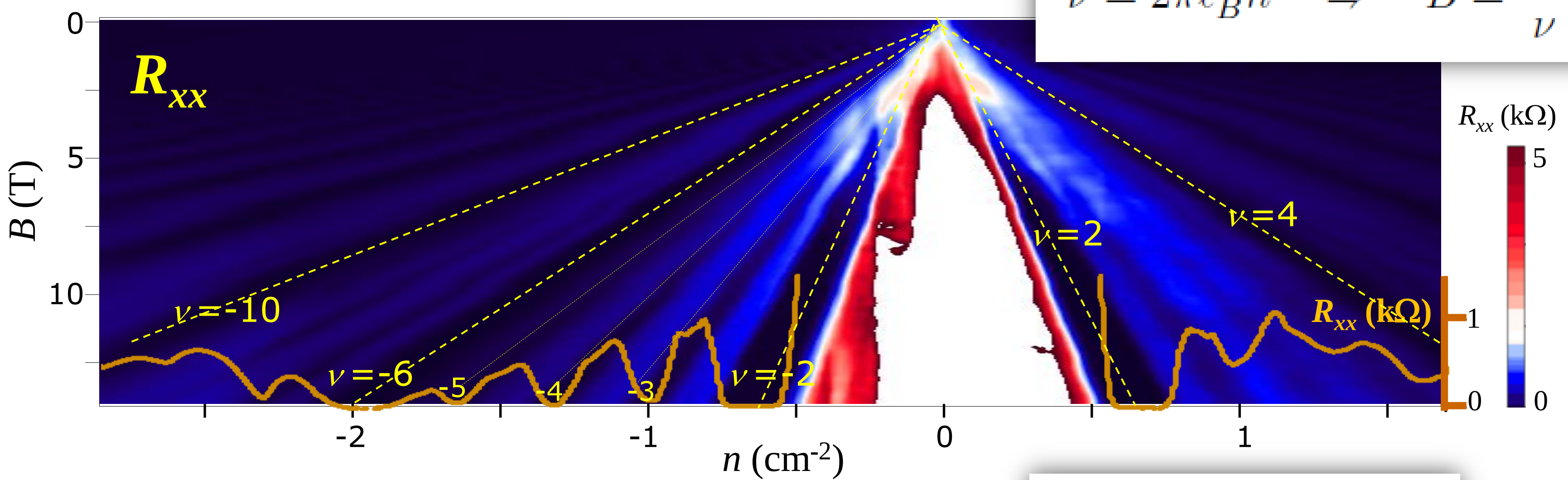
$$R_{xy} = \frac{V_{xy}}{I}$$



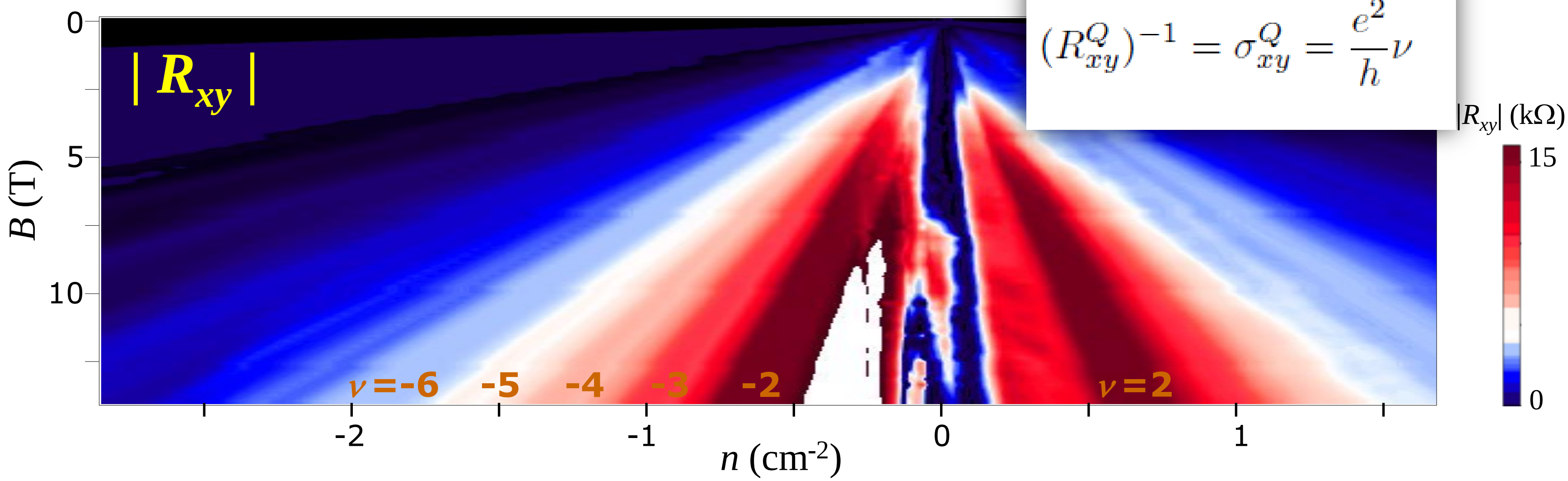
How to “Read” *Normal* Landau Fan Diagram?

Landau Fan Diagram for “typical” graphene

$$\nu = 2\pi \ell_B^2 n \quad \Rightarrow \quad B = \frac{\phi_0}{\nu} n$$



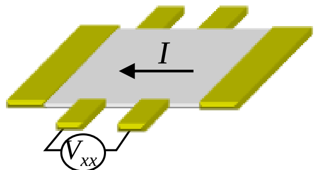
$$(R_{xy}^Q)^{-1} = \sigma_{xy}^Q = \frac{e^2}{h}\nu$$

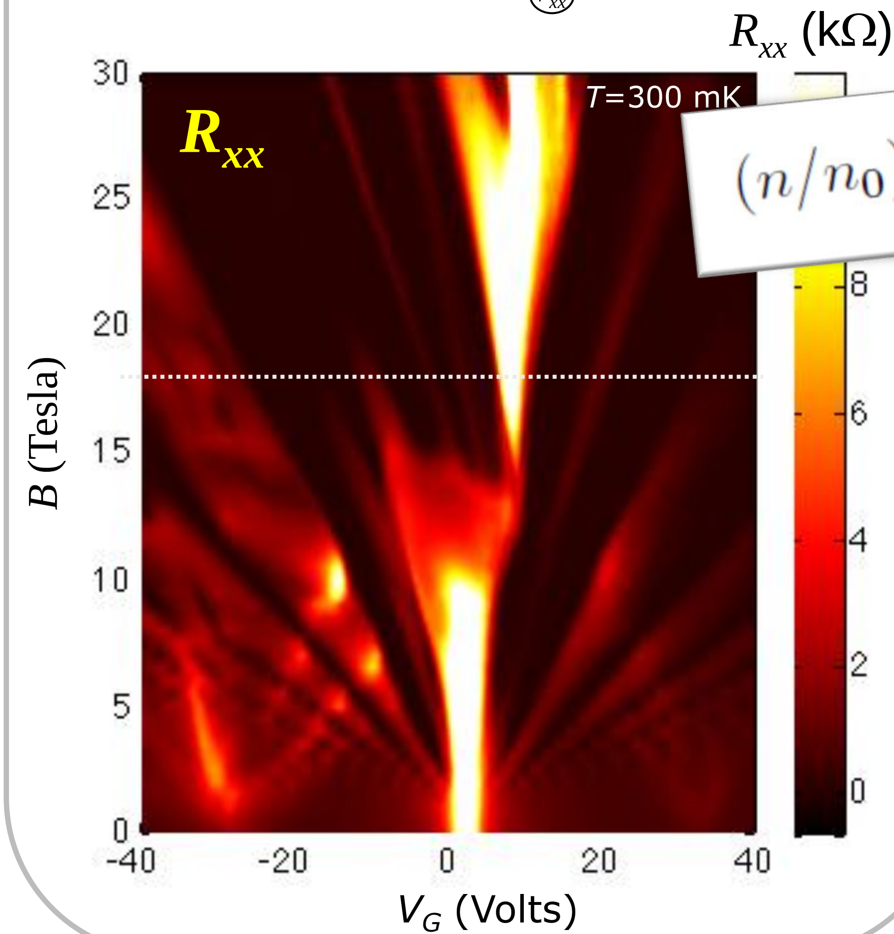


Quantum Hall Effect in Graphene Moire

Quantum Hall-like Transport

$$R_{xx} = \frac{V_{xx}}{I}$$



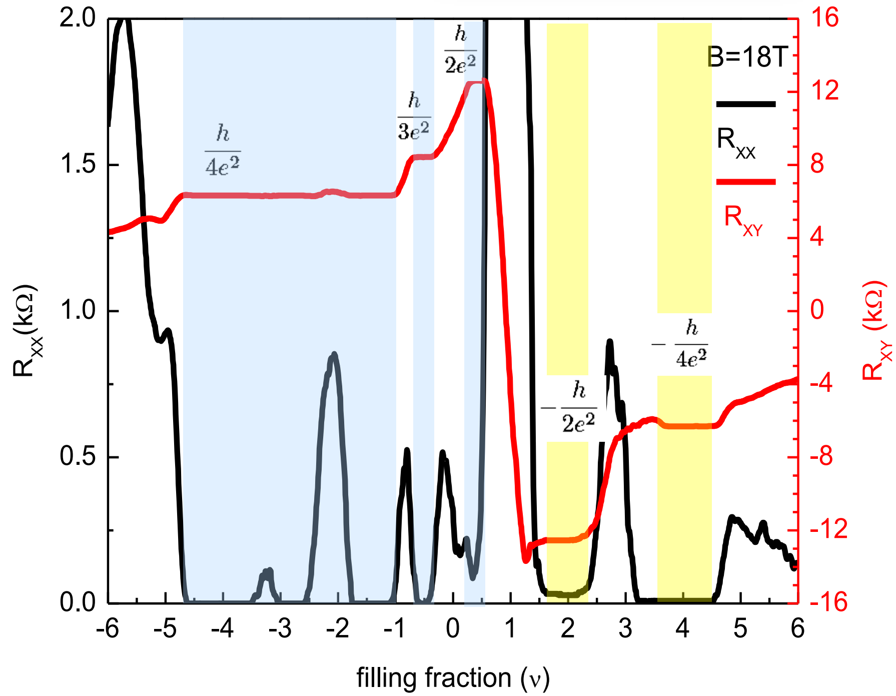


Landau level filling factor
$$\nu = \frac{\phi_0}{B} n$$

Quantum Hall conductance
$$R_{xy}^{-1} = \frac{e^2}{h} t$$

$$(n/n_0) = t(\phi/\phi_0) + s$$

$$\nu = t \quad \in \mathbb{Z}$$



Quantum Hall Effect with Two Integer Numbers

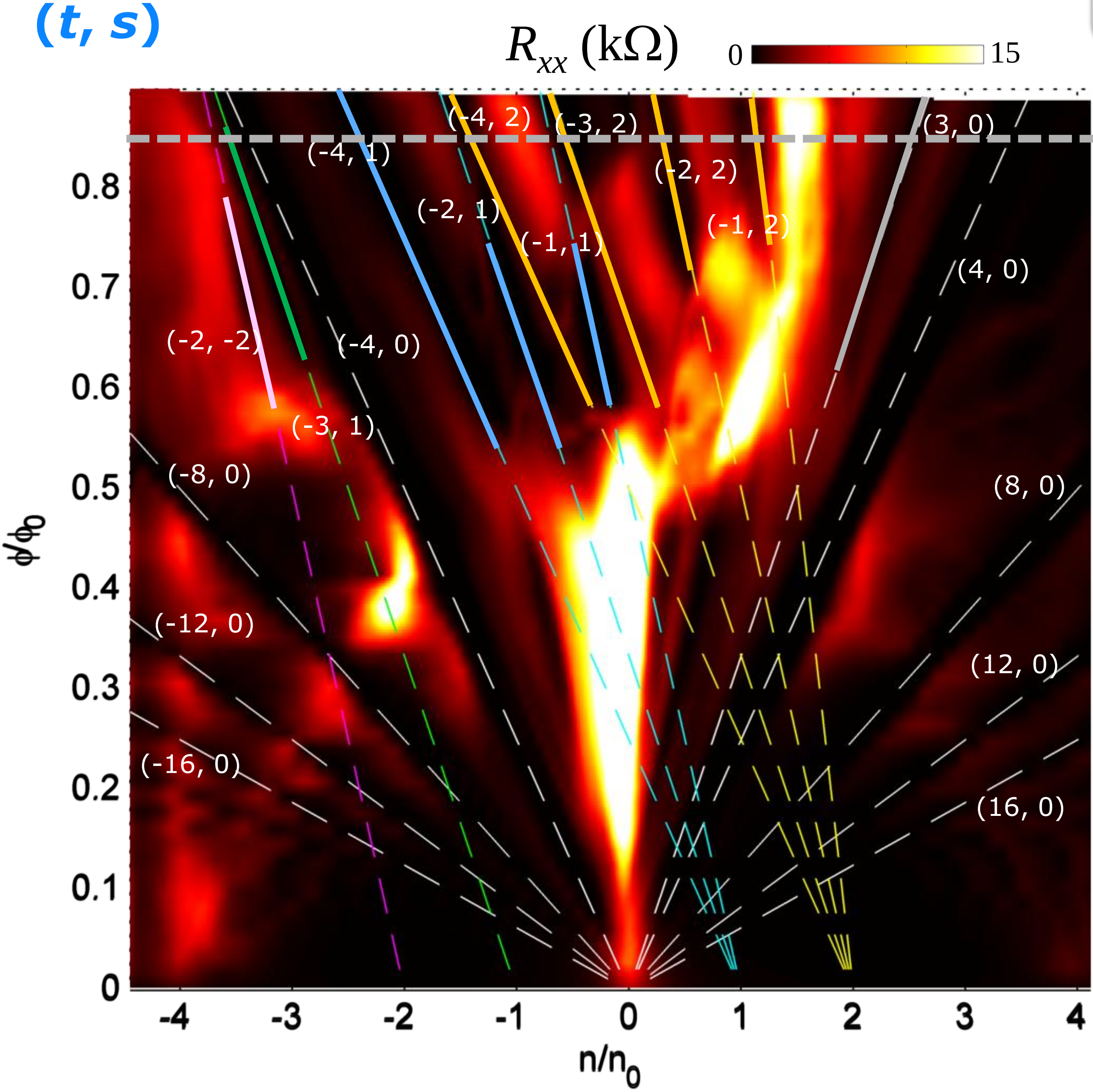
n/n_0: density per unit cell; ϕ : flux per unit cell

(t, s)

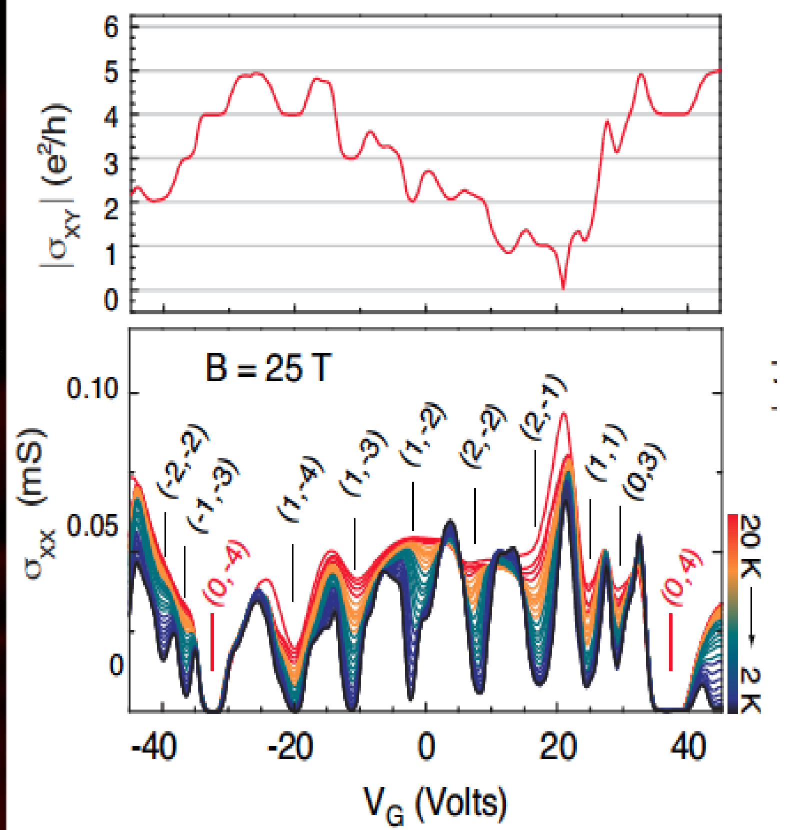
Diophantine equation for gaps

$$(n/n_0) = t(\phi/\phi_0) + s \qquad t, s \in \mathbb{Z}$$

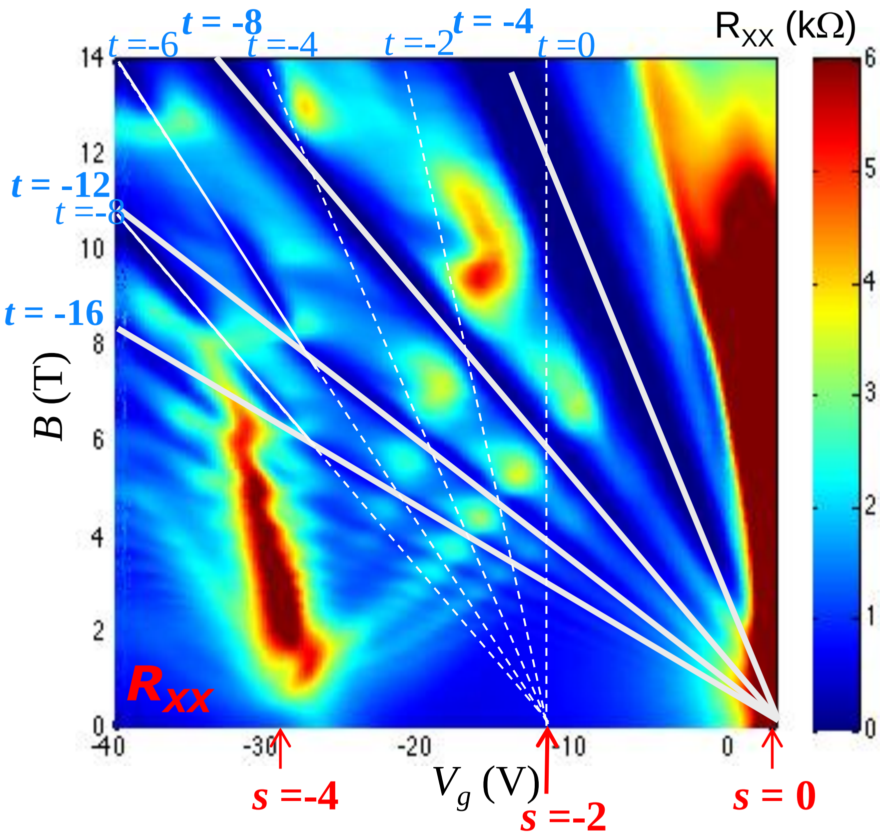
Wannier (1978)

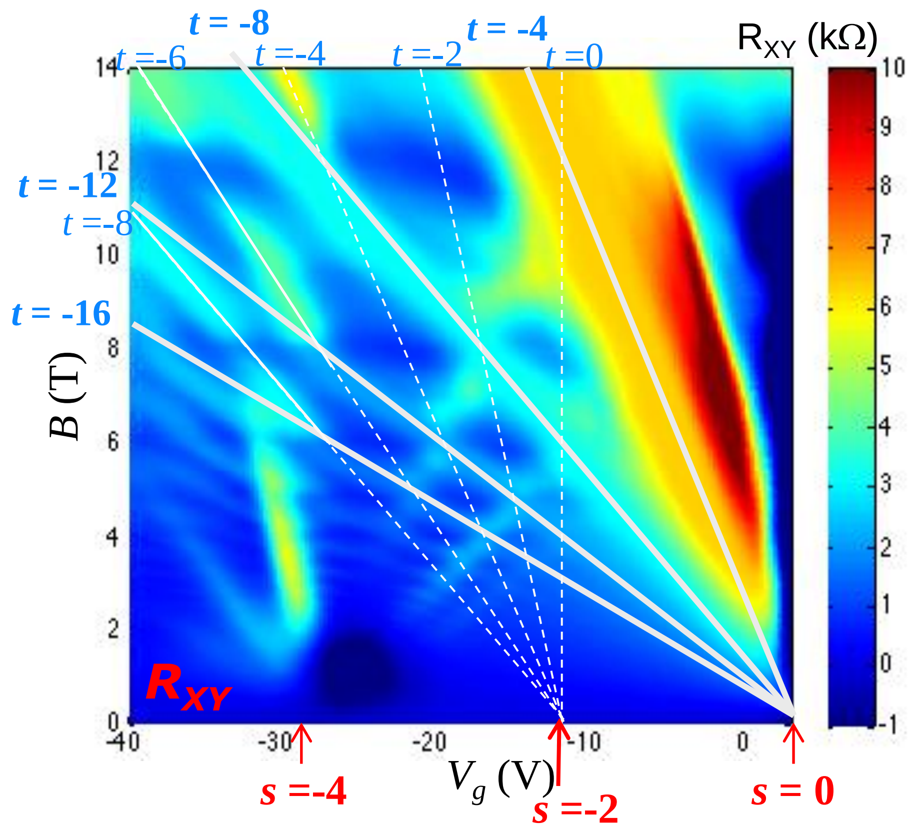


Quantization of σ_{xx} and σ_{xy}



Quantum Hall Effect with Two Integer Numbers

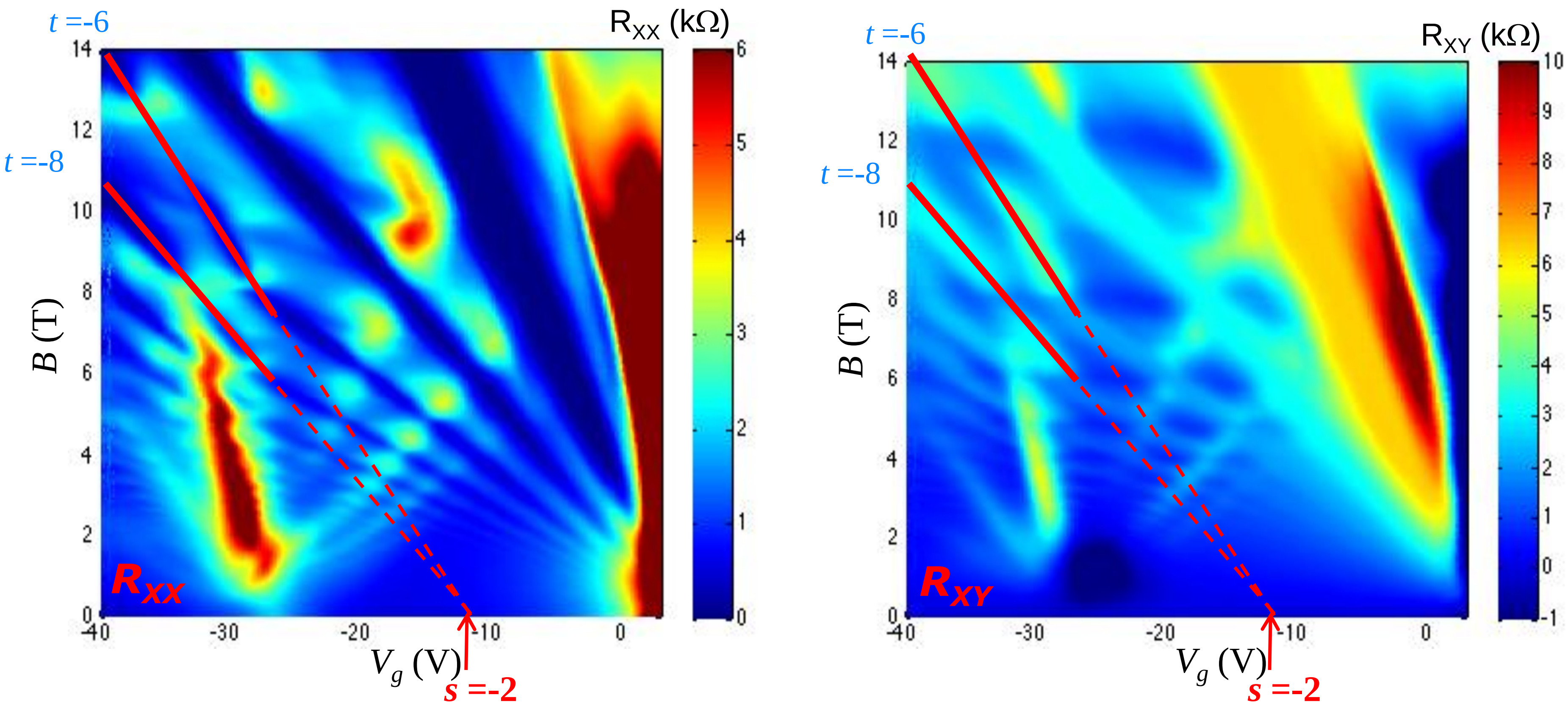




$$(n/n_0) = t(\phi/\phi_0) + s$$

$$R_{xy}^{-1} = \frac{e^2}{h} t$$

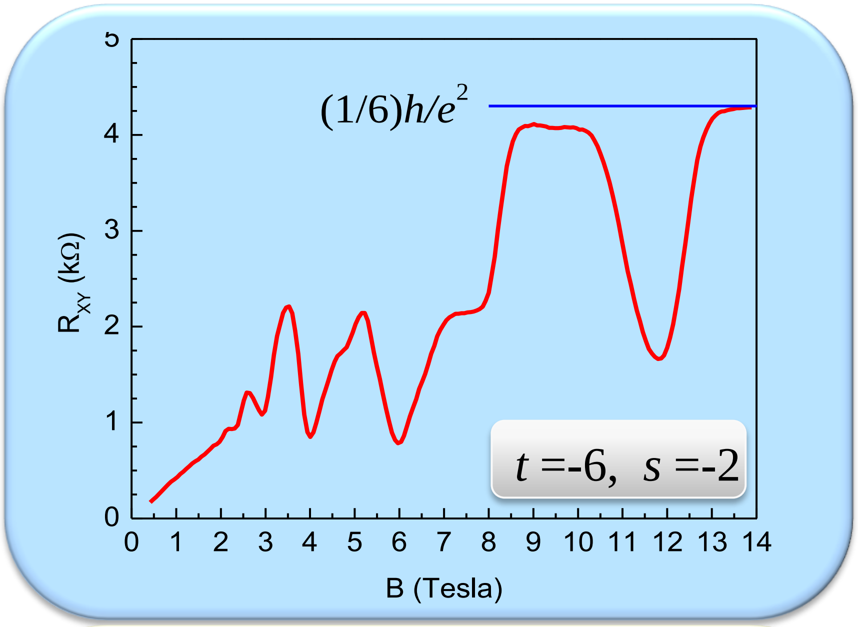
Quantum Hall Effect with Two Integer Numbers

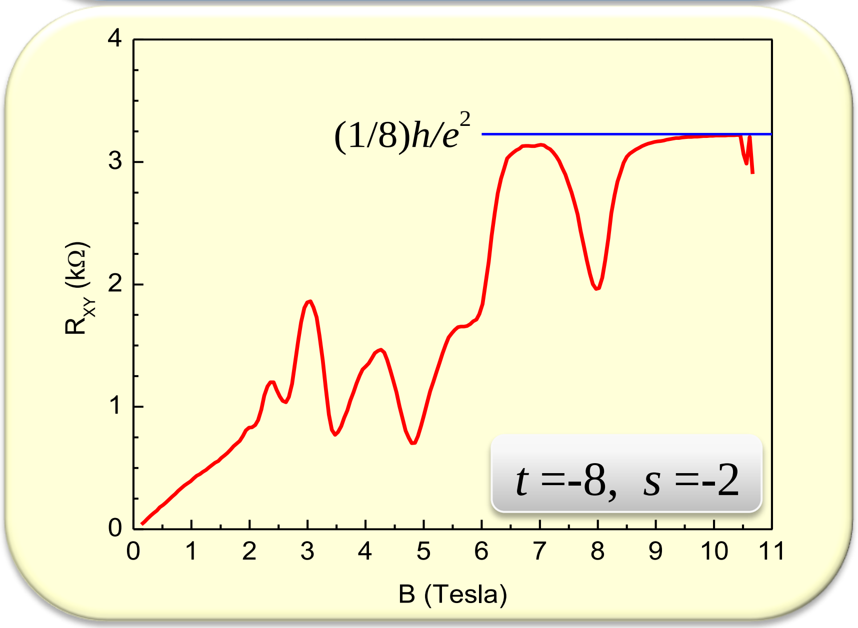


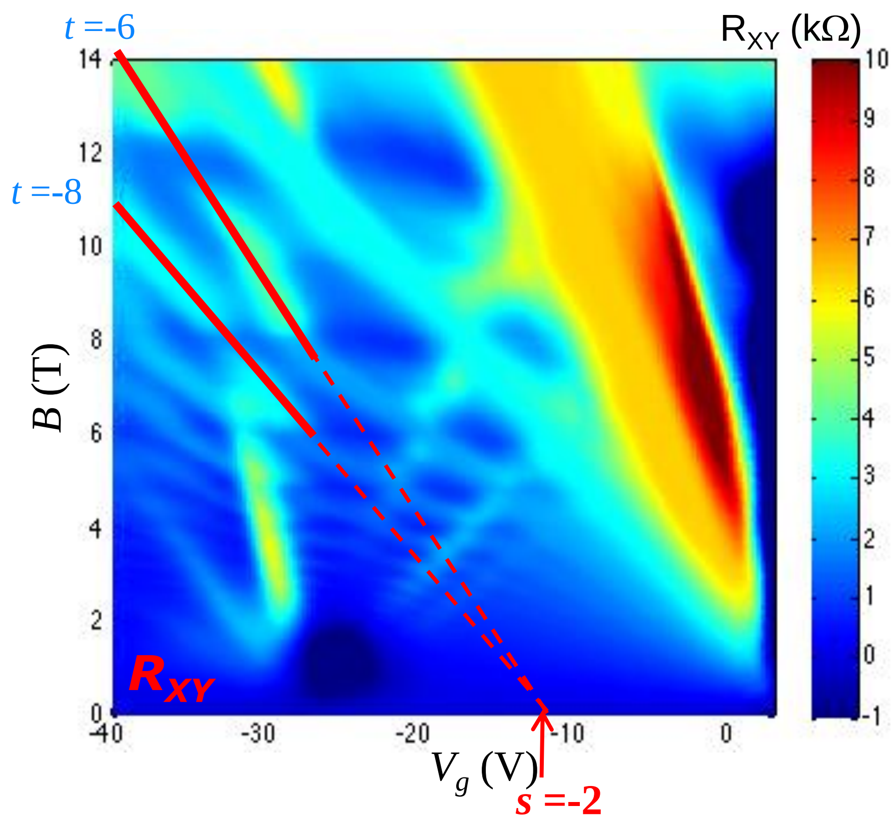
$$(n/n_0) = t(\phi/\phi_0) + s$$

$$R_{xy}^{-1} = \frac{e^2}{h} t$$

Quantum Hall Effect with Two Integer Numbers





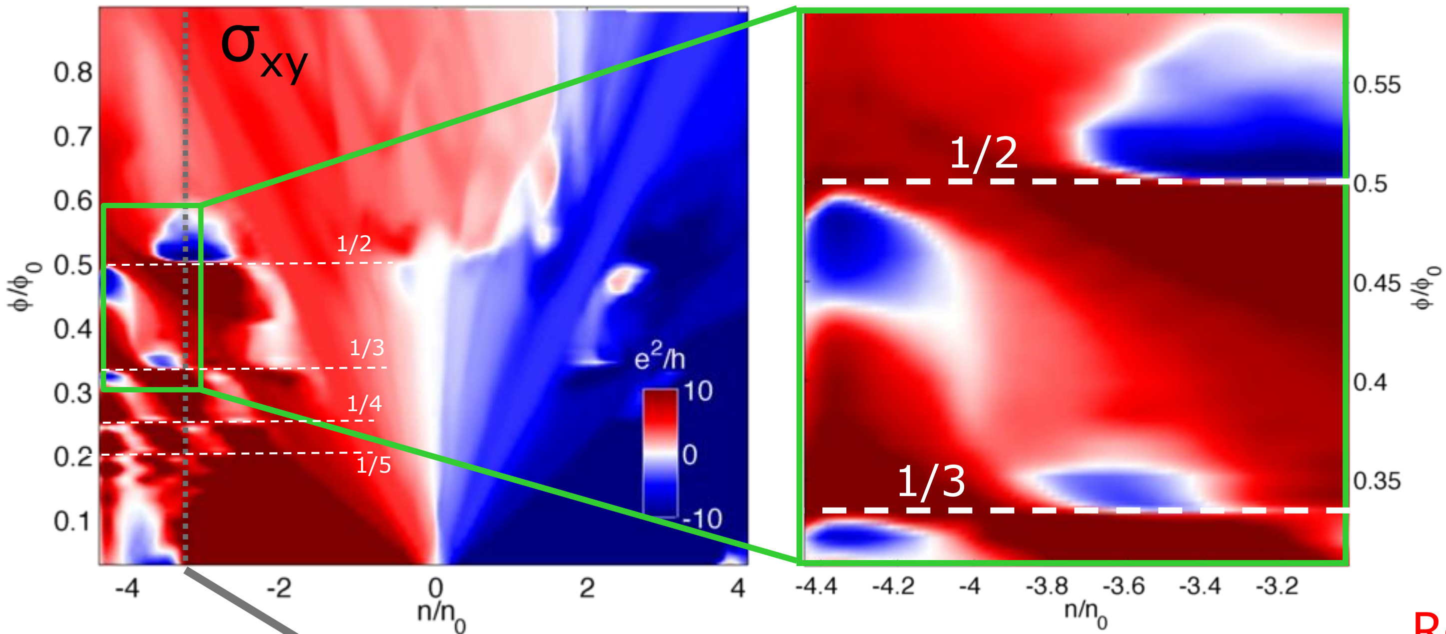


$$(n/n_0) = t(\phi/\phi_0) + s$$

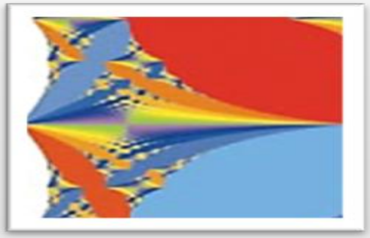
$$R_{xy}^{-1} = \frac{e^2}{h} t$$

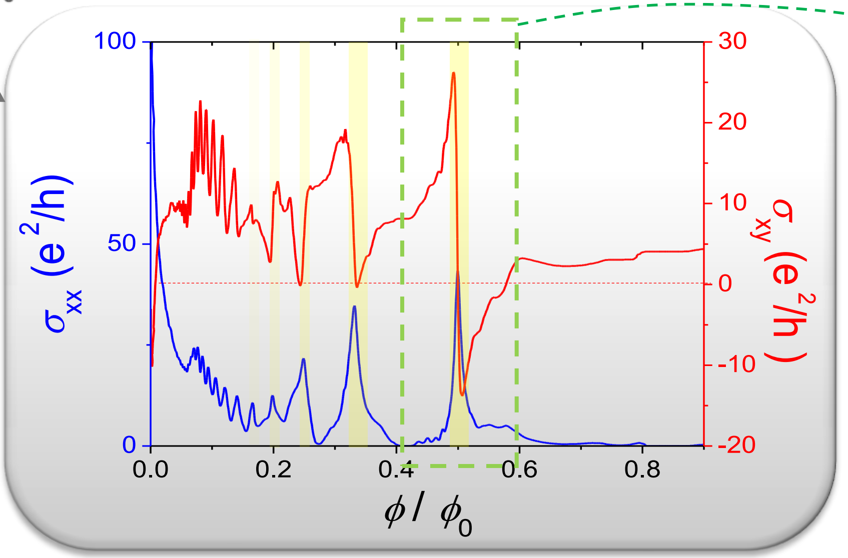
Recursive QHE near the Fractal Bands

Higher quality sample with lower disorder

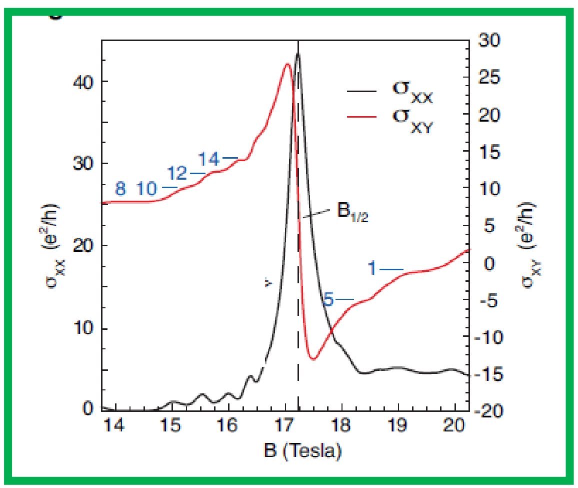


Hall conductivity across Fractal Band





Recursive QHE!



At the Fractal Bands

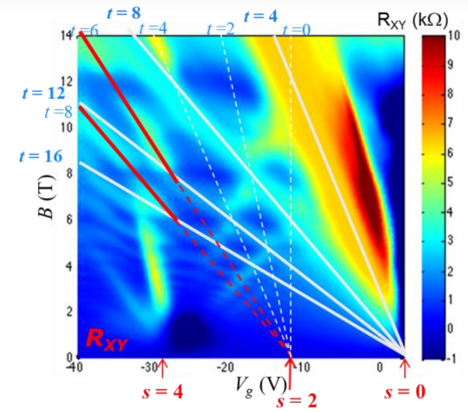
Sign reversal of σ_{xy}

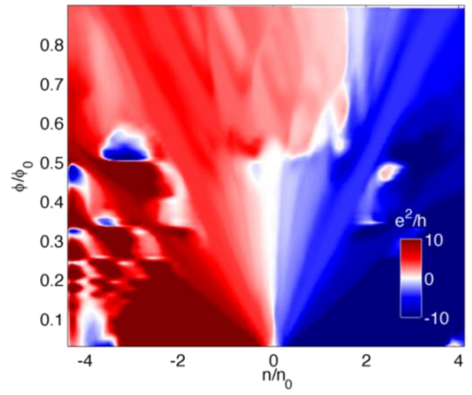
Large enhancement of σ_{xx}

Summary and Outlook

$$(n/n_0) = t(\phi/\phi_0) + s$$

- Graphene on hBN with high quality interface created Moire pattern with supper lattice modulation
- Quantum Hall conductance are determined by two TKNN integers.
- Anomalous Hall conductance at the fractal bands

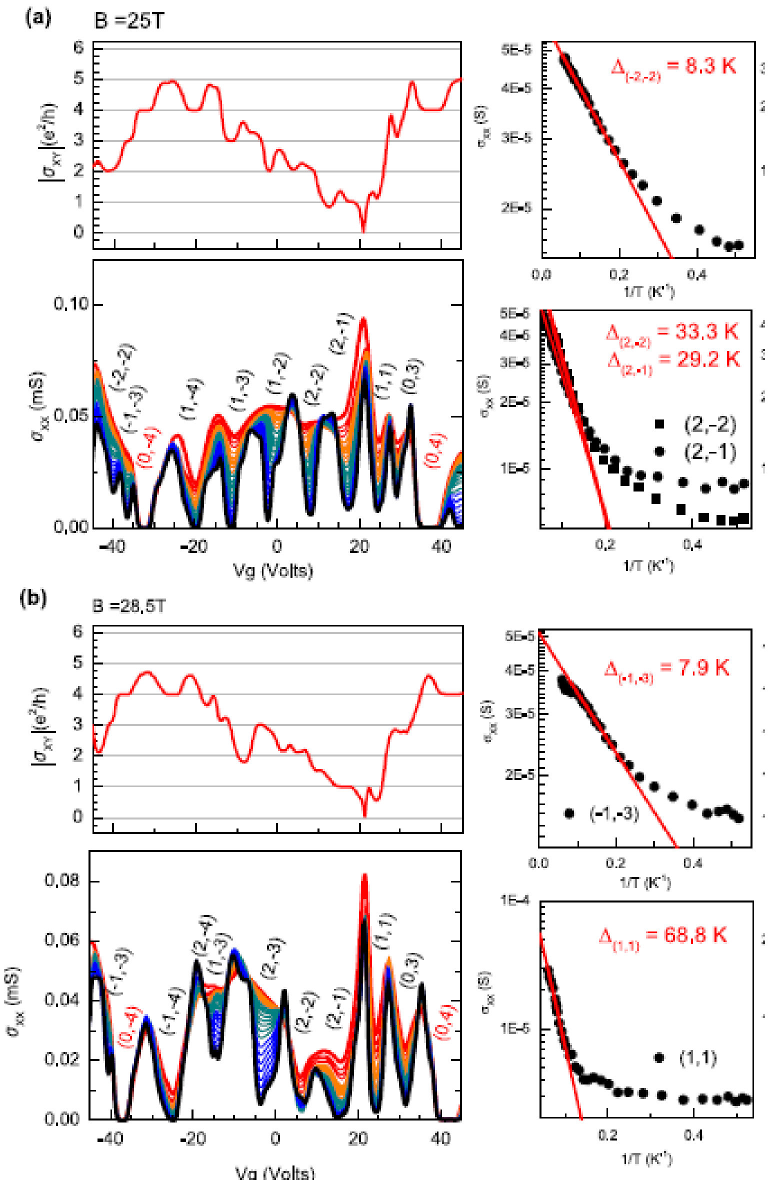




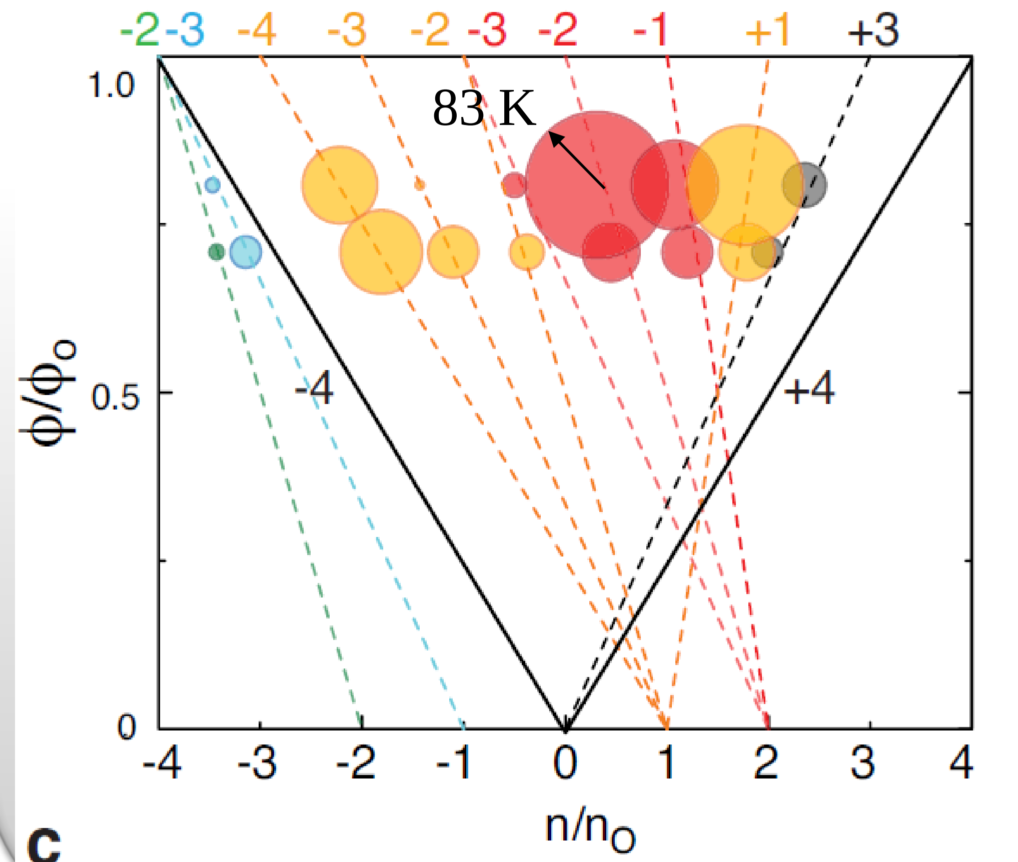
Open Questions:
- Elementary excitation of the fractal gaps?
- Role of interactions, Hofstadter Butterfly in FQHE?

Energy spectroscopy is needed for the next step

Fractal Gaps: Energy Scales

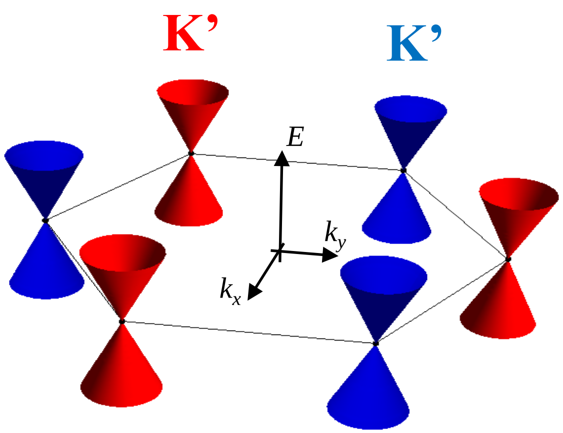


Fractal Gap Size



Large odd integer gap indicates (fractal) quantum Hall ferromagentism!!

SU(4) Quantum Hall Ferromagnet in Graphene



Magnetic Wave Function

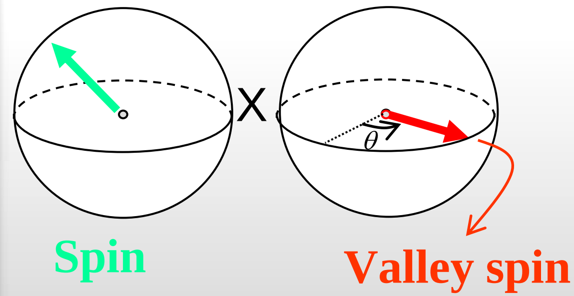
$$\vec{\Phi}_n^K \propto \begin{pmatrix} \phi_n \\ D_n\phi_{n-1} \end{pmatrix}, \qquad n \geq 1$$

$$\vec{\Phi}_n^{K'} \propto \begin{pmatrix} D_n\phi_{n-1} \\ \phi_n \end{pmatrix}, \qquad n \geq 1$$

$$\vec{\Phi}_0^K \propto \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix}, \qquad n = 0$$

$$\vec{\Phi}_0^{L'} \propto \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}, \qquad n = 0$$

$$SU(2) \quad SU(2) \quad < \quad SU(4)$$



$$\begin{bmatrix} \psi_{K\uparrow} \\ \psi_{K\downarrow} \\ \psi_{K'\uparrow} \\ \psi_{K'\downarrow} \end{bmatrix}$$

Degree of freedom:

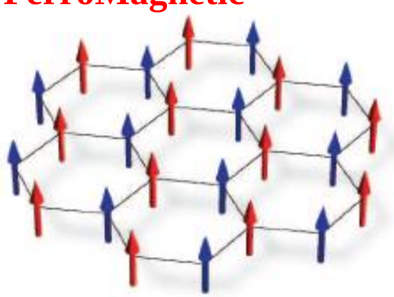
Spin (1/2), Valleys

Under magnetic fields:
pseudospin = **valley spin**

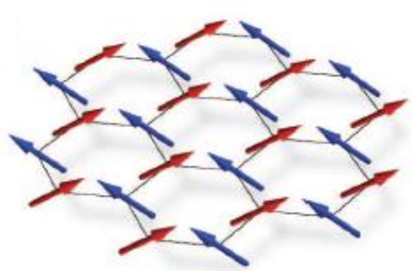
Yang, Das Sarma and MacDonal, PRB (2006);

Possible SU(4) Quantum Hall Ferromagnetism at the Neutrality

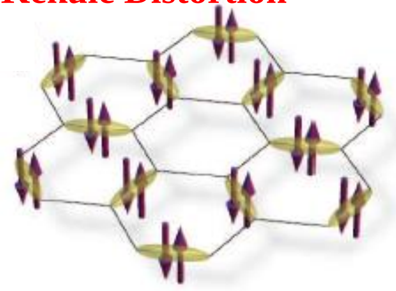
FerroMagnetic



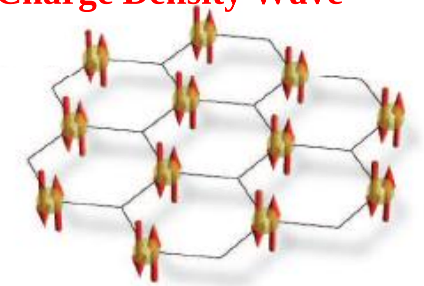
Anti FerroMagnetic



Kekule Distortion



Charge Density Wave



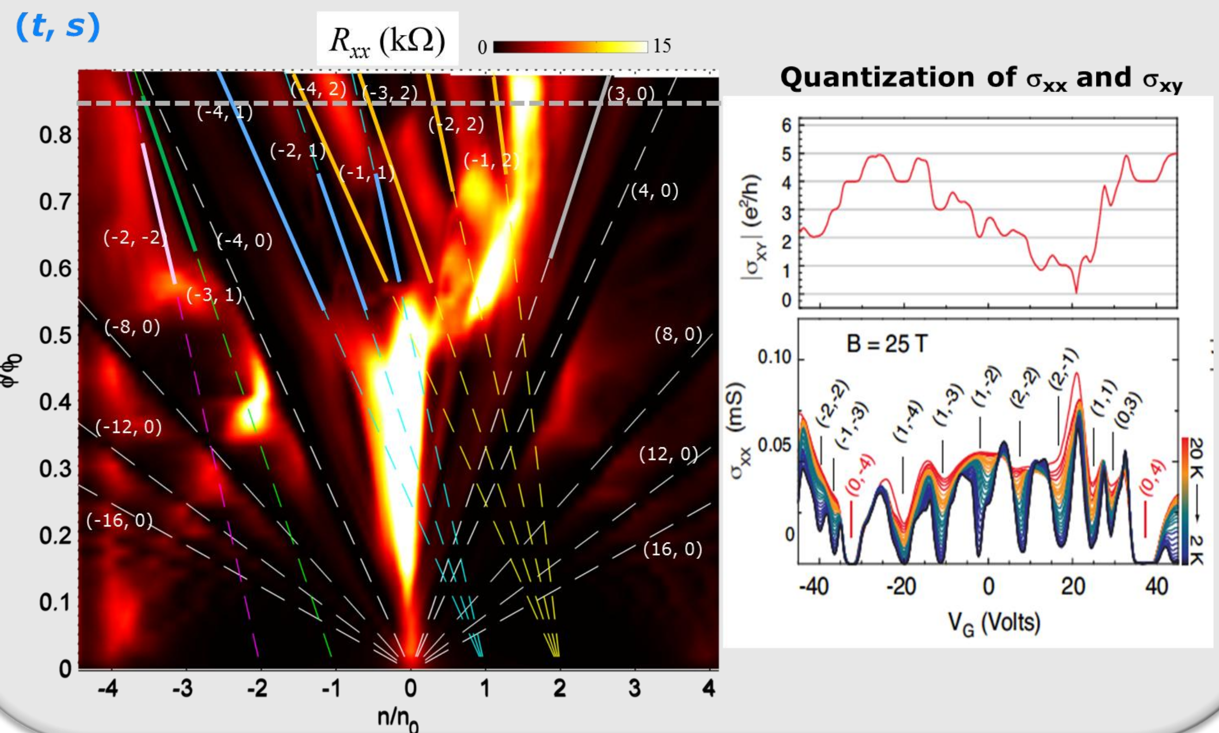
Nature of Quantum Hall Ferromagnetism in Graphene

Partial list of references

Abanin, D. A., Lee, P. A. & Levitov, L. S. Spin-filtered edge states and quantum Hall effect in graphene. *Phys. Rev. Lett.* **96,** 176803 (2006).
Fertig, H. A. & Brey, L. Luttinger liquid at the edge of undoped graphene in a strong magnetic field. *Phys. Rev. Lett.* **97,** 116805 (2006).
Kharitonov, M. Edge excitations of the canted antiferromagnetic phase of the $\nu = 0$ quantum Hall state in graphene: A simplified analysis. *Phys. Rev. B* **86,** 075450 (2012).
Kharitonov, M. Canted antiferromagnetic phase of the $\nu = 0$ quantum Hall state in bilayer graphene. *Phys. Rev. Lett.* **109,** 04680
Feldman, B. E., Martin, J. & Yacoby, A. Broken
divergent resistance in suspended bilayer gr
889–893 (2009).
Zhao, Y., Cadden-Zimansky, P., Jiang, Z. &
in the zero-energy Landau level in bilayer g
066801 (2010).
Weitz, R. T., Allen, M. T., Feldman, B. E.,
Broken-symmetry states in doubly gated su
Science **330,** 812–816 (2010).
Kim, S., Lee, K. & Tutuc, E. Spin-polarized
in graphene bilayers at $\nu = 0$ in high magn
016803 (2011).
Velasco, J. Jr *et al.* Transport spectroscopy
states in bilayer graphene. *Nature Nanotech.*
Veligura, A. *et al.* Transport gap in suspende
field. *Phys. Rev. B* **85,** 155412 (2012).
Freitag, F., Trbovic, J., Weiss, M. & Schenb
ground state in suspended bilayer graphene.
Gorbar, E. V., Gusynin, V. P. & Miransky,
diagram of the $\nu = 0$ quantum Hall state in
155451 (2010).
Lemonik, Y., Aleiner, I. L. & Falko, V. I. Com
and spin-flux orders in the ground state of b
245451 (2012).

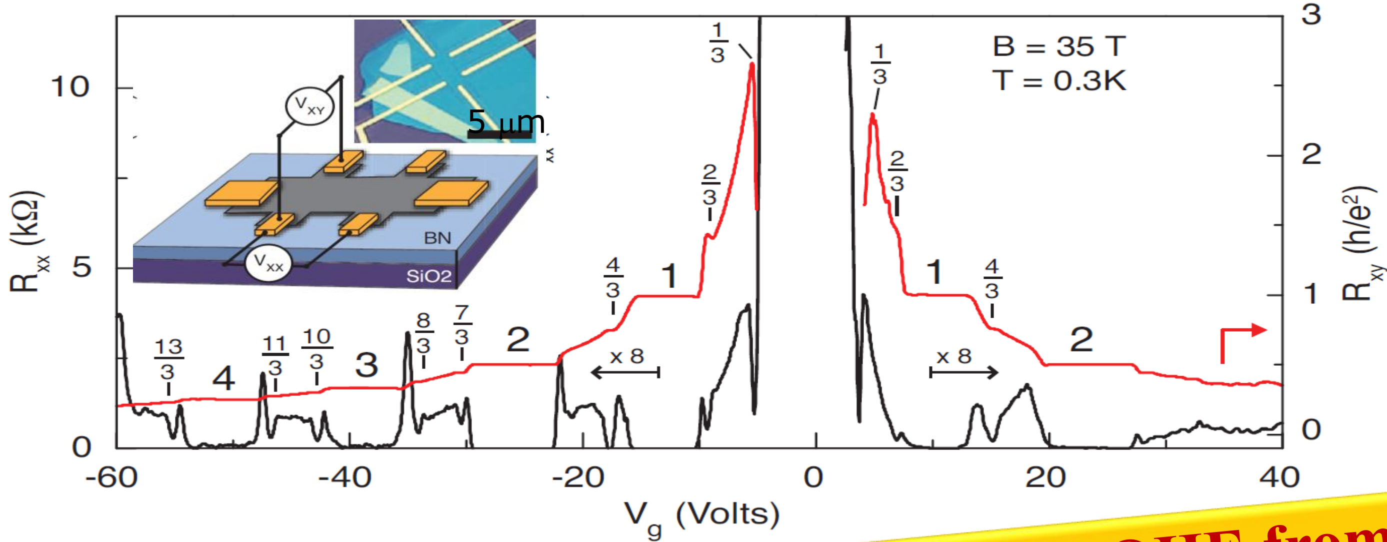
Kharitonov, M. Antiferromagnetic state in bilayer graphene. *Phys. Rev. B* **86,** 195435 (2012).
Zhang, F. & MacDonald, A. H. Distinguishing spontaneous quantum Hall states in bilayer graphene. *Phys. Rev. Lett.* **108,** 186804 (2012).
Lemonik, Y., Aleiner, I. L., Toke, C. & Falko, V. I. Spontaneous symmetry breaking and Lifshitz transition in bilayer graphene. *Phys. Rev. B* **82,** 201408 (2010).
Vafek, O. & Yang, K. Many-body instability of Coulomb interacting bilayer
010).

Electron Interaction Fractal Spectrum

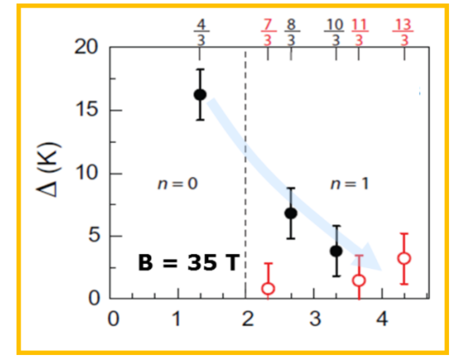


Graphene Franctional Quantum Hall Effect

Dean *et al*. Nature Physics (2011)



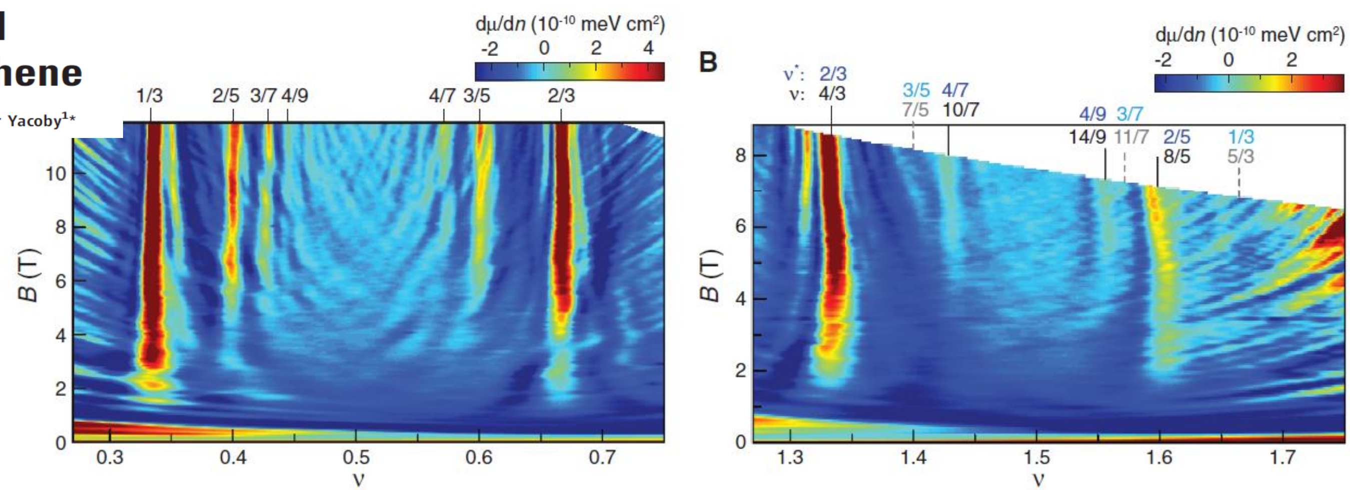
Fractional Quantum Hall Gaps



Hierarchical Fractional QHE from SU(4)

Unconventional Sequence of Fractional Quantum Hall States in Suspended Graphene

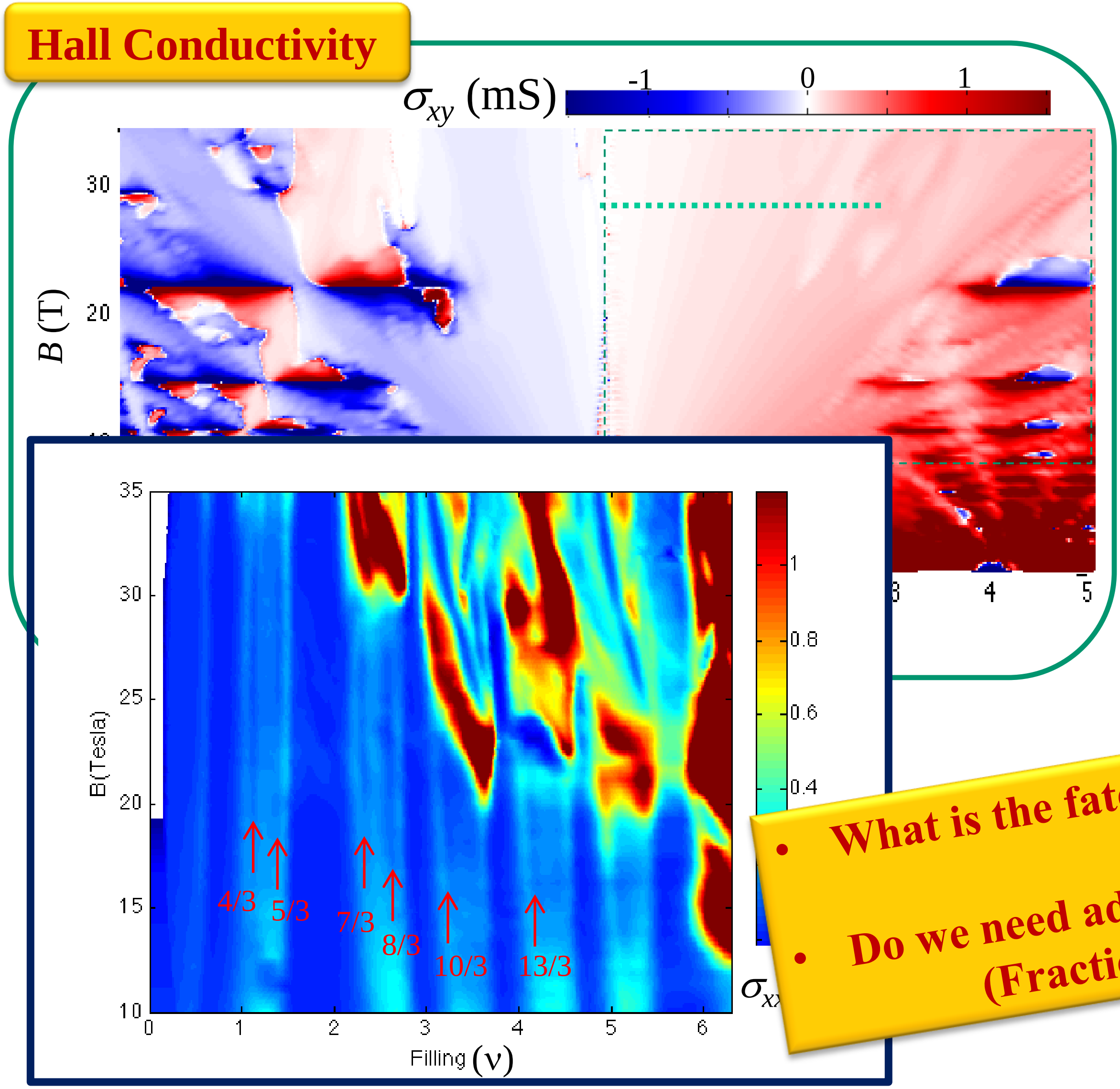
Benjamin E. Feldman,[1] Benjamin Krauss,[2] Jurgen H. Smet,[2] Amir Yacoby[1*]

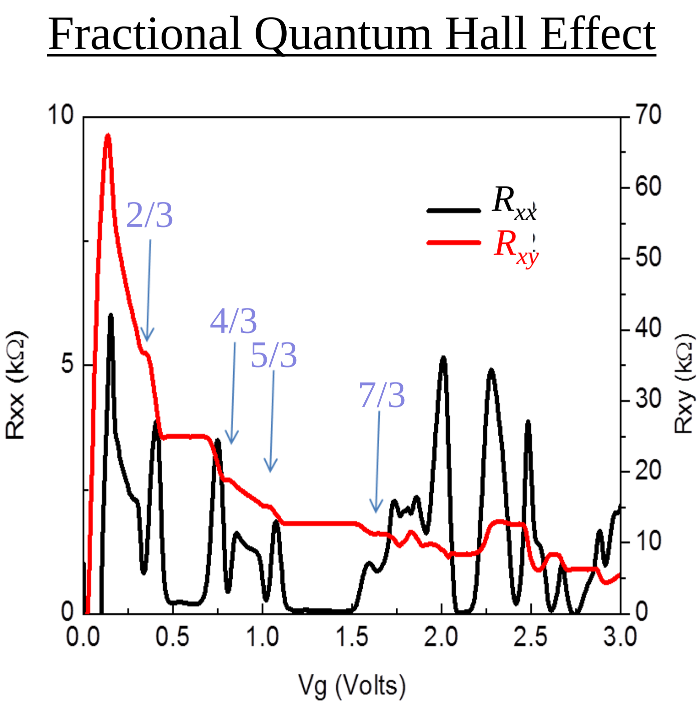


7 SEPTEMBER 2012 VOL 337 SCIENCE

Fractional Quantum Hall Effect in Moire Superlattice

Single layer graphene on hBN @ 20 mK up to 35 T



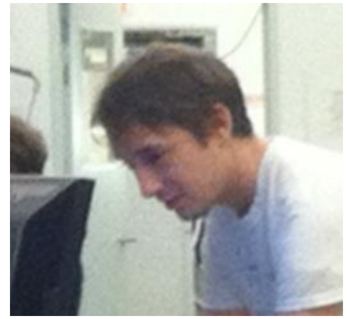


- **What is the fate of FQHE in fractal spectrum?**
- **Do we need additional quantum numbers? (Fractional Fractal Quantum Hall Effect)**

Acknowledgment

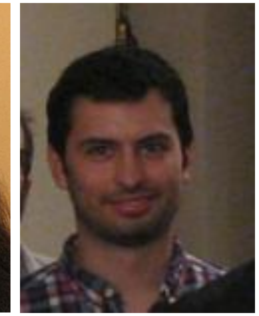
Prof. Cory Dean
(now at CUNY)

Lei Wang



Patrick Maher

Fereshte Ghahari



Carlos Forsythe

Prof. Jim Hone

Prof. Ken Shepard

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hBN samples: T. Taniguchi & K. Watanabe (NIMS)

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