Anyons in an exactly solved model and beyond.

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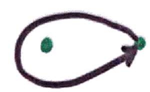
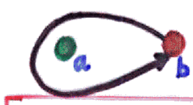
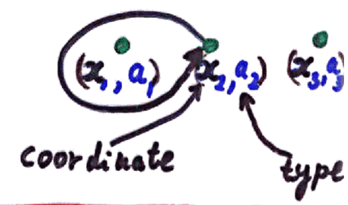
- Exact solution of a spin model on the honeycomb lattice
- Classification of possible phases with $\mathbb{Z}_2$ gauge field coupled to fermions (Vortex statistics depends on $\nu \bmod 16$)
- Chern number for quasidiagonal matrices (semi - original)
- Review of algebraic theory of anyons
- Some ramifications

Anyons : particles with unusual statistics (only occur in 2D)

	Bosons:	**Fermions**
	$\lvert\Psi\rangle \mapsto \lvert\Psi\rangle$	$\lvert\Psi\rangle \mapsto -\lvert\Psi\rangle$
	$\lvert\Psi\rangle \mapsto \lvert\Psi\rangle$	$\lvert\Psi\rangle \mapsto \lvert\Psi\rangle$

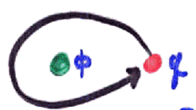
Anyons:

Abelian	**Non-Abelian**
	
$\boxed{\lvert\Psi\rangle \mapsto e^{i\varphi_{ab}}\lvert\Psi\rangle}$	$\boxed{\begin{array}{l}\lvert\Psi\rangle \mapsto U\lvert\Psi\rangle \\ \lvert\Psi\rangle \in \mathcal{L}(x_1,a_1;x_2,a_2;\ldots)\end{array}}$
Laughlin quasiparticles in FQHE systems	(FQHE at $\nu=5/2$)
$\varphi_{++}=\varphi_{--}=\frac{2\pi}{3}$, $\varphi_{+-}=-\frac{2\pi}{3}$ $\left(q=\pm\frac{e}{3}\right)$	

General idea of anyons
(F. Wilczek 1982)

Let us consider a $U(1)$ gauge theory in 2D.

Magnetic vortex: $\phi \in \mathbb{R}/\mathbb{Z}$ ($= U(1)$)
Electric charge: $q \in \mathbb{Z}$
} Bosons if considered separately

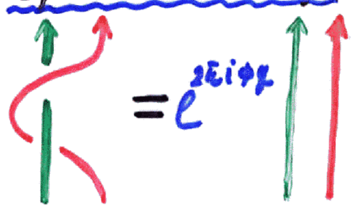


$$|\psi\rangle \mapsto e^{2\pi i \phi q}|\psi\rangle$$

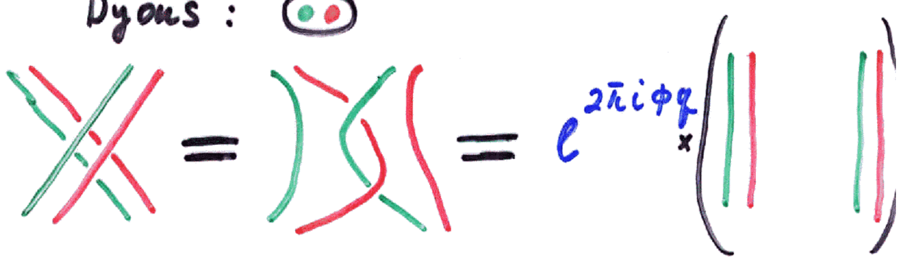
(Aharonov-Bohm phase)

Nontrivial mutual statistics

Space-time diagram



Dyons :



$\mathbb{Z}_2$ - gauge anyons

Particle types (superselection sectors)

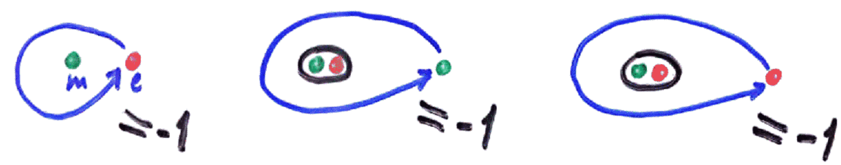
- m (magnetic vortex)
- e (electric charge)

} ... Bosons

ε (dyon) .... Fermion

1 (vacuum) .... Boson

Mutual statistics:



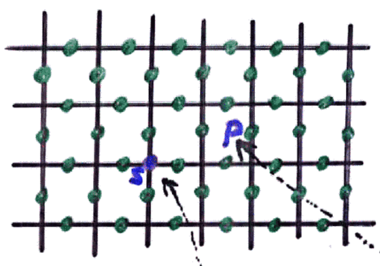
Fusion rules:

$e \times e = 1$

$m \times m = 1$

$e \times m = \varepsilon, \quad \varepsilon \times \varepsilon = 1, \quad e \times \varepsilon = m, \quad m \times \varepsilon = e$

A simple realization:
spins on the links of a 2D-lattice



$S_j = \pm 1$ indicates the $\mathbb{Z}_2$ vector potential on link j

$$H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p$$

$$\underbrace{A_s = \prod_{\text{star}(s)} \sigma_j^x}_{\text{electric charge on vertex } s} \qquad \underbrace{B_p = \prod_{\text{boundary}(p)} \sigma_j^z}_{\text{magnetic flux through plaquette } p}$$

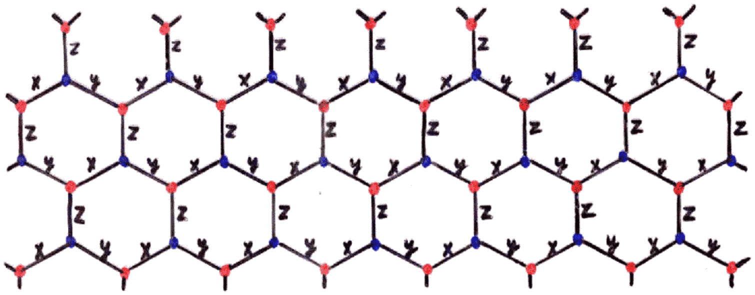
Rules of the game

- Spins on a lattice
- Local Hamiltonian (the simpler the better)
  (not local Lagrangian)
- Interested mostly in gapped phases
  (Gapless phases are also interesting but the particle statistics may not be well-defined)
- No exact symmetry assumed
  (Some symmetry may be helpful for an exact solution but we are interested in properties that are stable to a generic perturbation)
  (Gauge symmetry may be introduced by adding fictitious degrees of freedom)

Anyons in a hexagonal lattice model

L1

1. The model



Spin $\frac{1}{2}$ on each site.

Two sublattices :  and 

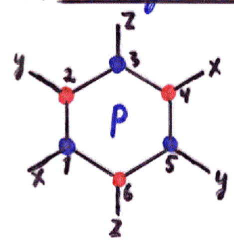
Three types of links: 

Spin - spin interactions :
$$K_{jk} = \begin{cases} \sigma_j^x \sigma_k^x & \text{for } x\text{-links} \\ \sigma_j^y \sigma_k^y & \text{for } y\text{-links} \\ \sigma_j^z \sigma_k^z & \text{for } z\text{-links} \end{cases}$$

$$\boxed{H_0 = -J_x \sum_{x\text{-links}} K_{jk} - J_y \sum_{y\text{-links}} K_{jk} - J_z \sum_{z\text{-links}} K_{jk}}$$

2. Integrals of motion

L2



$$\boxed{B = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z}$$

A separate operator B_p for each face p

$[B_p, K_{jk}] = 0$, hence $\boxed{[B_p, H_0] = 0}$

individual term in the Hamiltonian

the Hamiltonian

$[B_p, B_s] = 0$

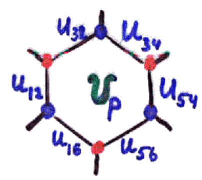
$B_p^2 = 1$, $B_p^\dagger = B_p$ $\Rightarrow$ the eigenvalues $= \pm 1$

Hilbert space decomposition :
$$\boxed{\mathcal{H} = \bigoplus_{v_1,\ldots,v_M} \mathcal{H}_{v_1,\ldots,v_M} \quad (v_p = \pm 1)}$$

"gauge sectors"

$$\boxed{|\psi\rangle \in \mathcal{H}_{v_1,\ldots,v_M} \iff B_p|\psi\rangle = v_p|\psi\rangle}$$

$v_p = +1$ – non-vortex $v_p = -1$ – vortex



$v_p = \prod_{\text{(p)}} u_{jk}$ $u_{jk} = \pm 1$ ($\mathbb{Z}_2$ gauge field)

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Counting degrees of freedom.

N spins $\longleftrightarrow$ N/2 hexagons

$$\frac{2^N}{2^{N/2}} = 2^{N/2} \text{ states}$$

$\leftarrow$ fixing v_p

($\sqrt{2}$ states per vertex)

(1 real Majorana) fermions.

In fact, each sector can be described by free (real) fermions

$$\boxed{c_j^+ = c_j} \quad \boxed{c_j c_k + c_k c_j = 2\,\delta_{jk}}$$

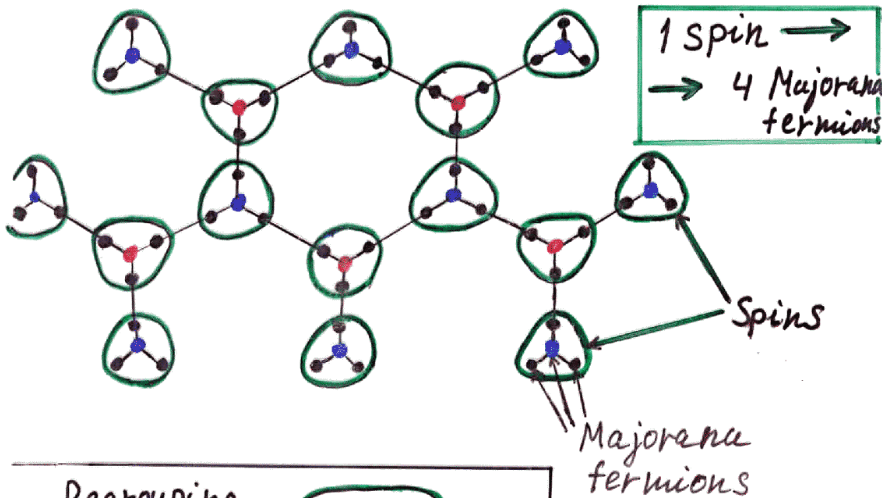
Ordinary fermions:

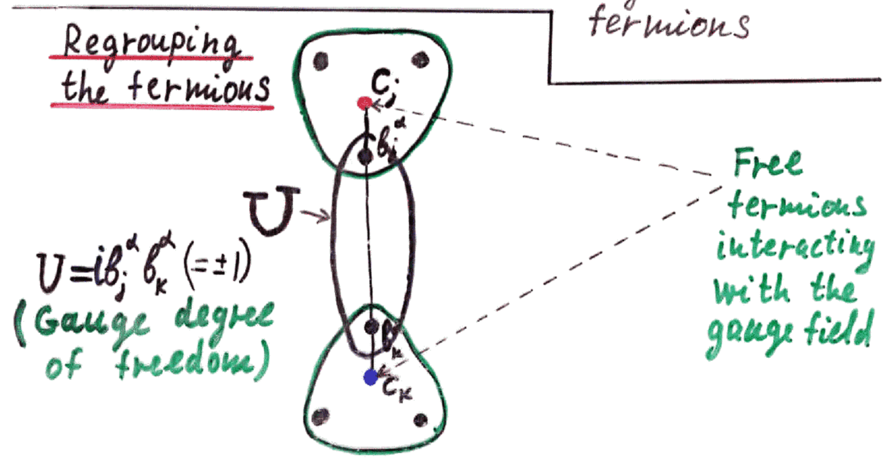
$$a_m = c_{2m-1} + i\,c_{2m}$$
$$a_m^+ = c_{2m-1} - i\,c_{2m}$$

$$a_m a_m^+ + a_m^+ a_m = \delta_{mm}$$
$$a_m a_n + a_n a_m = 0$$

Fermionization

13a





L19

For each "vortex configuration" $(v_1, \ldots, v_{N/2})$ one can compute the energy of fermionic vacuum

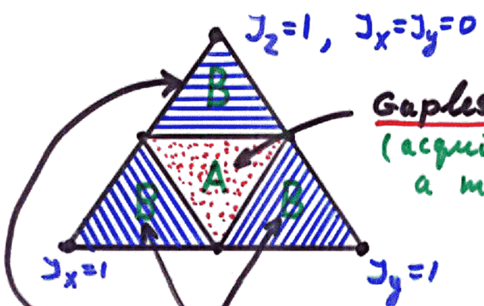
$$E(v_1, \ldots, v_{N/2})$$

$\boxed{H_{v_1,\ldots,v_{N/2}} \text{ can be described in terms of free fermions}}$

Remarkable fact : $E(v_1, \ldots, v_{N/2})$ is minimal if $v_j = +1$ (no vortices) for all $J_x, J_y, J_z > 0$.

Follows from Lieb's work (1994)

Phase diagram



Gapless phase (acquires a gap in a magnetic field)

Gapped phases Abelian anyons (the same univers. class as the toric code)

L2

Each sector can be described by free real fermions in the gauge field

3. Representing spins by fermions (a general procedure)

Two fermionic modes : $\square \quad \boxdot$

Fock space : $\mathcal{F}(2) \cong \mathbb{C}^4$: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

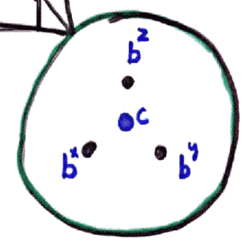
$a_1, a_1^\dagger, a_2, a_2^\dagger$ (annihilation and creation operators) or

$$c_0 = a_1 + a_1^\dagger = c \qquad c_2 = a_2 + a_2^\dagger = b^y$$

$$c_1 = \frac{a_1 - a_1^\dagger}{i} = b^x \qquad c_3 = \frac{a_2 - a_2^\dagger}{i} = b^z$$

(Majorana operators)

$$\boxed{c_j^\dagger = c_j \qquad c_j c_k + c_k c_j = 2\delta_{jk}}$$



The even subspace

$\mathcal{L} \subseteq \mathcal{F}(2)$: $|00\rangle$, $|11\rangle$

$$|\psi\rangle \in \mathcal{L} \iff \boxed{b^x b^y b^z c\, |\psi\rangle = |\psi\rangle}$$

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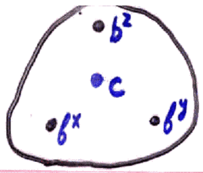
1 spin = 

(subspace $\mathcal{L}$)

$$\boxed{\begin{array}{ll} D = b^x b^y b^z c & \text{(a stabilizer operator)} \\ \sigma^\alpha \equiv i\, b^\alpha c & (\alpha = x, y, z) \end{array}}$$

"equal on $\mathcal{L}$"

$[\sigma^\alpha, D] = 0 \quad \Rightarrow \quad \sigma^\alpha$ preserves the subspace $\mathcal{L}$

$$\sigma^x \sigma^y \sigma^z = i^3 b^x c\, b^y c\, b^z c = i\, \underbrace{b^x b^y b^z c}_{D} \equiv i$$

$b_j^x,\ b_j^y,\ b_j^z,\ c_j,\quad D_j = b_j^x b_j^y b_j^z c_j$

for each site j

Any spin Hamiltonian $= H\{i b_j^\alpha c_j\}$

$[H, D_j] = 0 \qquad [D_j, D_k] = 0$

$D_j |\Psi\rangle = |\Psi\rangle$ (gauge invariance)

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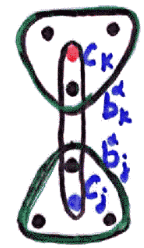
4. Special properties of the Hamiltonian H_0.

$$\hat{H}_0 = -\sum_{(j,k)} J_{\alpha(j,k)} \hat{K}_{jk}$$

$\alpha(j,k) = x, y, z$



$$\hat{K}_{jk} = (i \hat{b}_j^\alpha \hat{c}_j)(i \hat{b}_k^\alpha \hat{c}_k) = -i\, \hat{U}_{jk} \hat{c}_j \hat{c}_k$$

where $\boxed{\hat{U}_{jk} = i \hat{b}_j^\alpha \hat{b}_k^\alpha}$ — $\mathbb{Z}_2$ gauge field

$\hat{U}_{kj} = -\hat{U}_{jk}$ (careful about signs)

$$\boxed{\begin{array}{l} \hat{H}_0 = \frac{i}{2} \sum_{(j,k)} \hat{A}_{jk} \hat{c}_j \hat{c}_k \\ \hat{A}_{jk} = 2 J_{\alpha(j,k)} \hat{U}_{jk} \end{array}} = \frac{i}{4} \sum_{j,k} \hat{A}_{jk} \hat{c}_j \hat{c}_k$$

(each pair (j,k) appears twice)

New integrals of motion: $\hat{U}_{jk}$ $\quad [\hat{U}_{jk}, \hat{H}_0] = 0$

Did not exist in the spin model because $[\hat{D}_j, \hat{U}_{jk}] \neq 0$

Old integrals of motion: L6

$$\boxed{\hat{B}_p = \prod_{\partial p} \hat{U}_{jk}}$$

$$[\hat{D}_j, \hat{B}_p] = 0$$

("field tensor" is gauge-invariant)

5. A special "fermionization" procedure for H_0

0) Choose a vortex configuration $\{v_p\}$. ($v_p = \pm 1$)

1) Fix a gauge: $\hat{U}_{jk}|\psi\rangle = u_{jk}|\psi\rangle$

$$\boxed{\prod_{\partial p} u_{jk} = v_p} \qquad u_{jk} = \pm 1$$

2) Solve for the ground state of

$$\boxed{\begin{aligned} \hat{H}_{\{u\}} &= \frac{i}{4}\sum_{j,k} A_{jk}\hat{c}_j\hat{c}_k \\ A_{jk} &= 2J_{d(j,k)}u_{jk} \end{aligned}}$$

$E = E(v_1, \dots, v_{N/2})$

vortex configuration

A is a real skew-symmetric matrix

3) Symmetrize over "gauge transformations" $\hat{D}_j$ while keeping $v_p = \prod_{\partial p} u_{jk}$ fixed.

(projecting onto the physical subspace)

Quadratic fermionic Hamiltonians

$$\boxed{H(A) = \frac{i}{4}\sum_{j,k} A_{jk}c_jc_k}$$

$j,k \in \{1, \dots, 2N\}$

c_j are Majorana operators (Clifford algebra generators)

$$c_jc_k + c_kc_j = 2\delta_{jk}$$

c_j act in the Fock space $\mathcal{F}$, whose dimension is 2^N

A is a real, skew-symmetric matrix. $A \in so(2N)$

$$\boxed{[iH(A), iH(B)] = iH([A,B])}$$

$A \mapsto iH(A)$ is a representation of the Lie algebra $so(2N)$

$so(2N) \to \mathbb{L}(\mathbb{C}^{2^N})$

6. The one-particle spectrum

$$H_{\{u\}} = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k$$

Canonical form: $\frac{i}{2} \sum_m \varepsilon_m b'_m b''_m \left(= \sum_m \varepsilon_m \tilde{a}^\dagger_m \tilde{a}_m + \text{const} \right)$

normal modes

$$\begin{pmatrix} b'_1 \\ b''_1 \\ \vdots \\ b'_N \\ b''_N \end{pmatrix} = W \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2N-1} \\ c_{2N} \end{pmatrix} \qquad W A W^T = \begin{pmatrix} 0 & \varepsilon_1 & & & & \\ -\varepsilon_1 & 0 & & & 0 & \\ & & \ddots & & & \\ & 0 & & & 0 & \varepsilon_N \\ & & & & -\varepsilon_N & 0 \end{pmatrix}$$

$$W \in O(2N)$$

$\varepsilon_m \geq 0$ Ground state: $-i b'_m b''_m |\psi\rangle = |\psi\rangle$

$$E = -\frac{1}{2} \sum_m \varepsilon_m$$

(Translational invariance allows further simpification: $\varepsilon(q)$ $(\varepsilon(-q) = -\varepsilon(q))$)

Observation:

$$A_{\text{canonical}} = \begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \varepsilon_N \\ & & & -\varepsilon_N & 0 \end{pmatrix} \quad \text{and}$$

$$B_{\text{canonical}} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

define the same fermionic ground state (though the energies and spectra are different)

Same is true for:

$$A = W^T A_{\text{canonical}} W, \qquad B = W^T B_{\text{canonical}} W$$

$\boxed{B = -i\, \operatorname{sgn}(iA)}$ – the structural matrix

(we assume that $\det A \neq 0$)

Properties:

- $B_{jk} \in \mathbb{R}$
- $B_{kj} = -B_{jk}$
- $B^2 = -I$
- B is quasidiagonal $|B_{jk}| < c\, f(d(j,k))$; $f(r) \sim e^{-r/r_0}$ if A has a spectral gap

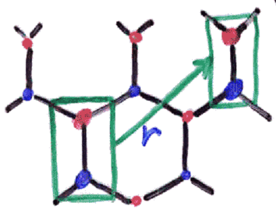
Spectral Chern number

Let P be the projector onto negative-energy states

$$P = \frac{1}{2}(I - iB)$$

$P^{\dagger} = P$, $P^2 = P$, P is quasidiagonal

Traditional approach: applies if A_{jk} (and so P_{jk}) has translational symmetry: $A_{j+r,\,k+r} = A_{jk}$



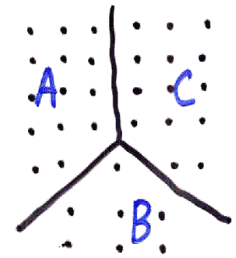
2 sites per unit cell

Fourier transform: $P_{\alpha\beta}(q)$ is a 2×2 matrix, $q \in \mathbb{T}^2$

Consider the corresponding subbundle in the trivial 2-d bundle

Alternatives:

- Noncommutative geometry
- Elementary expression



$$\nu(P) = \sum_{j\in A,\, k\in B,\, l\in C} 12\pi i \left(P_{jk}P_{kl}P_{lj} - P_{jl}P_{lk}P_{kj}\right)$$

Spin of the vortex:

 $= \theta_\sigma$

$$\theta_\sigma = e^{2\pi i \frac{\nu}{16}}$$

[11]

10 Superselection sectors for Abelian anyons

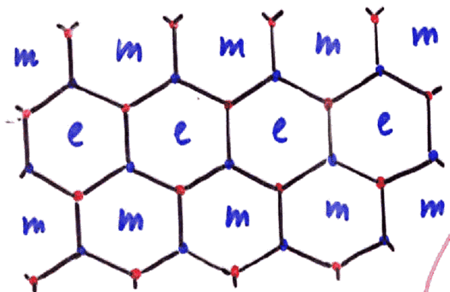
$\nu = 0$

Without vortices:

- 1 (the vacuum)
- ε (the fermion)

Vortices (assuming that the fermions form the lowest energy state)

- m ("magnetic vortex")
- e ("electric charge")



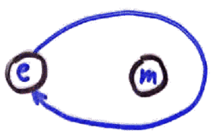
$e \times e = 1$

$m \times m = 1$

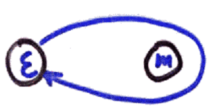
$e \times m = \varepsilon$

$\varepsilon \times \varepsilon = 1 \qquad e \times \varepsilon = m \qquad m \times \varepsilon = e$

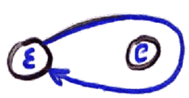
fusion rules



$= -1$



$= -1$



$= -1$

[12]

The gapless phase acquires a gap in a magnetic field.

$$H' = H + \sum_j \left(\vec{h}, \vec{\sigma}_j\right)$$

3-rd order of the perturbation theory

$$V^{(3)} \sim h_x h_y h_z \, \sigma_1^x \sigma_2^y \sigma_3^z$$

Quadratic fermionic Hamiltonian

Now excitations are non-Abelian anyons!

$\nu = \pm 1$

Fusion rules

$(1, \varepsilon, \sigma)$ — ε: fermion, σ: vortex

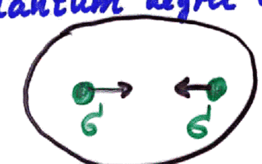
$\varepsilon \times \varepsilon = 1$

$\sigma \times \varepsilon = \sigma$

$\sigma \times \sigma = 1 + \varepsilon$

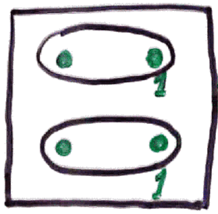
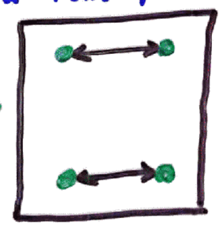
The signature of Non-Abelian anyons: more than one possibility

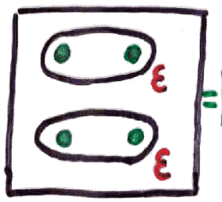
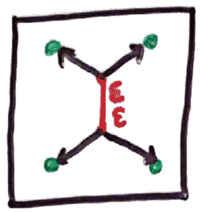
Hidden (protected) quantum degree of freedom



The vortices either annihilate or fuse into an ε particle

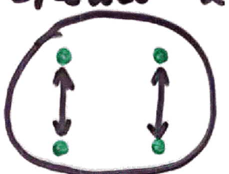
Encoding a qubit into 4 vortices with total "charge" 1. [Here we mean a real qubit] 22

$|0\rangle_L \longmapsto$  $= |1\,1\rangle$ 

$|1\rangle_L \longmapsto$  $= |\varepsilon\,\varepsilon\rangle$ 

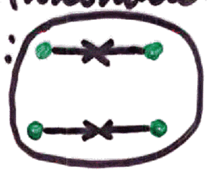
Thought experiment

1) Create 2 pairs

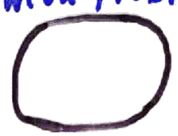


$$\psi = \frac{1}{\sqrt{2}}|0\rangle_L + \frac{1}{\sqrt{2}}|1\rangle_L$$

2) Annihilate them in a different way:



with prob. $\frac{1}{2}$



with prob. $\frac{1}{2}$

Basic computational operations 23

1) Initializing a "qubit"	
2) Measuring a "qubit"	 (ε or 1 ?)
3) Unitary gates	 $\boxed{\begin{array}{l} |1\rangle \mapsto e^{i\frac{\pi}{8}}|1\rangle \\ |\varepsilon\rangle \mapsto e^{-i\frac{3\pi}{8}}|\varepsilon\rangle \end{array}}$

- Unfortunately, the braiding gate is not universal (for this type of anyons)
- The anyons are still good for the realization of quantum memory and may relax the threshold requirement