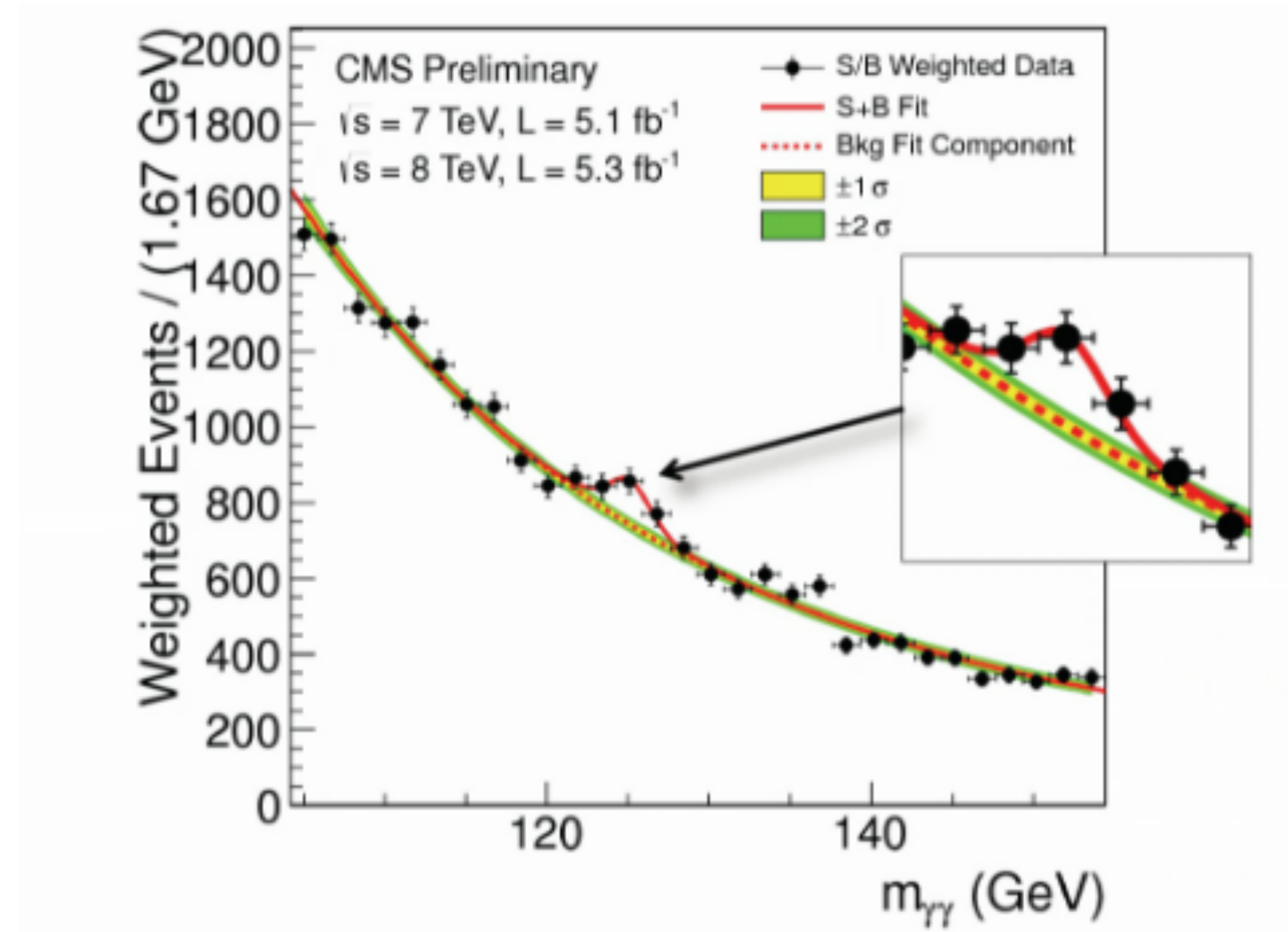
The Space of Quantum Field Theories

Zohar Komargodski
Weizmann Institute of Science

Quantum Field Theory (QFT) arises in the description of Elementary Particles, Statistical Mechanics, Condensed Matter, Stochastic Processes etc.

It is a universal (albeit not rigorously defined) framework.

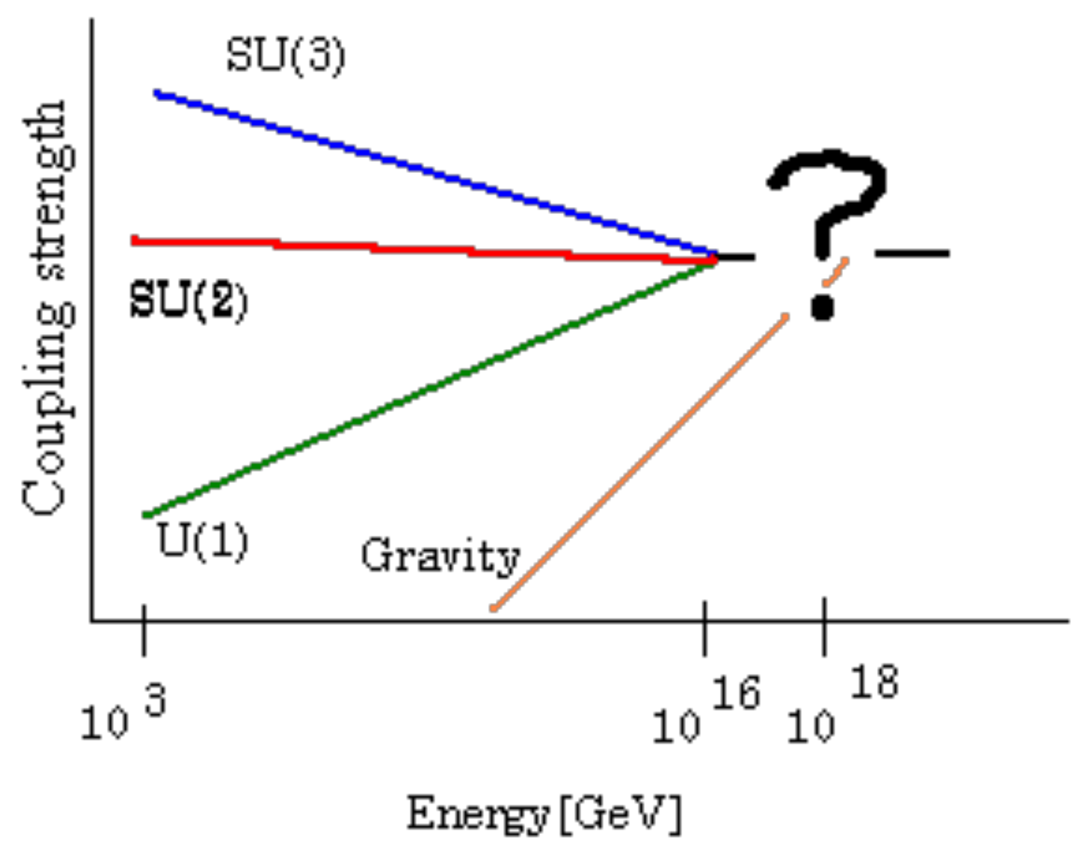


There are many possible QFTs and it is of interest to understand how they are related and what is the space of QFTs.

One can start from a given QFT defined on a d-dimensional space and deform it by some local operators

$$Z[g^i] = \langle e^{\int d^dx \sqrt{g} \sum_i g^i O_i(x)} \rangle$$

The most fundamental and confusing observation is that the g^i are not really well defined. Their actual numerical value depends on the resolution of the experiment.



$$Z[g^i;\mu] = \langle e^{\int d^d x \sqrt{g} \sum_i g^i O_i(x)} \rangle$$

Satisfying an equation that tells us what happens if we change the resolution

$$\frac{\partial}{\partial \log \mu} \log Z[g^i;\mu] + \sum_j \beta_{g^j}(g^i) \frac{\partial}{\partial g^j} \log Z[g^i;\mu] = 0$$

The functions β_{g^j} are in principle computable. They can be viewed as vector fields in the space of theories.

As we decrease our resolution (i.e. decreasing μ), some couplings could be:

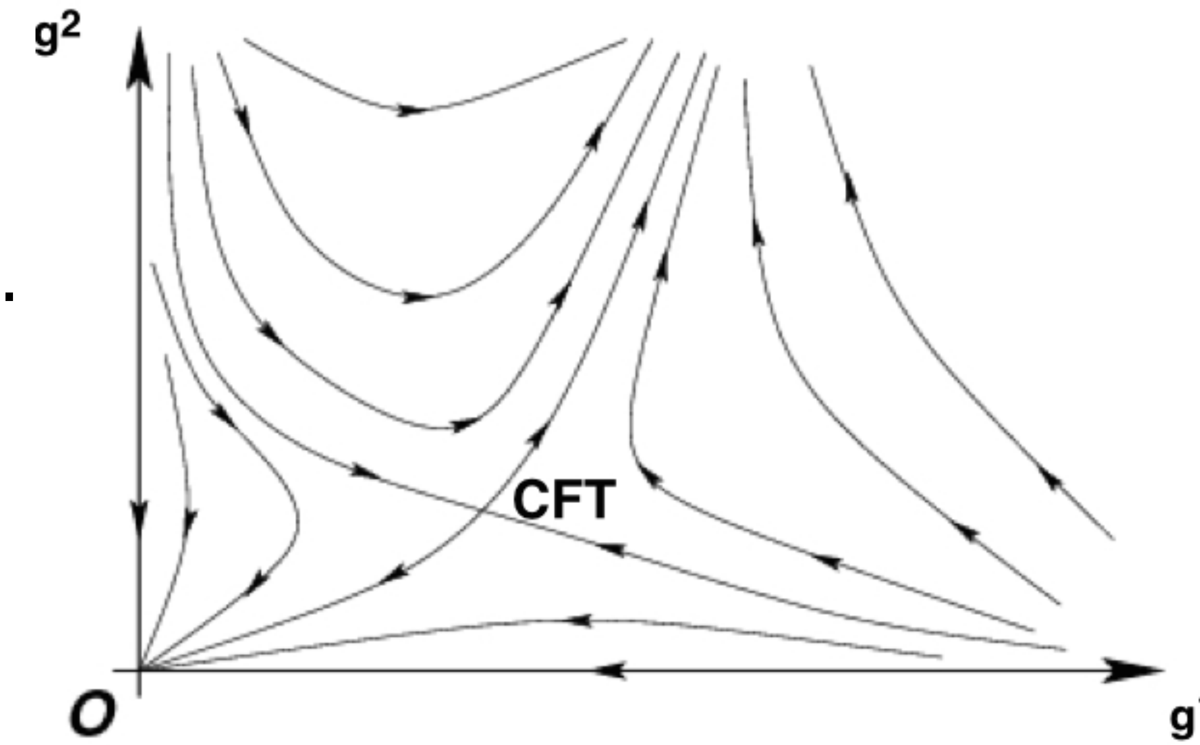
- irrelevant: i.e. they go to zero $\beta > 0$
- relevant: i.e. they increase $\beta < 0$
- exactly marginal: i.e. they don't change $\beta = 0$

If all the couplings are exactly marginal then the theory does not depend on the resolution and the symmetry is enhanced

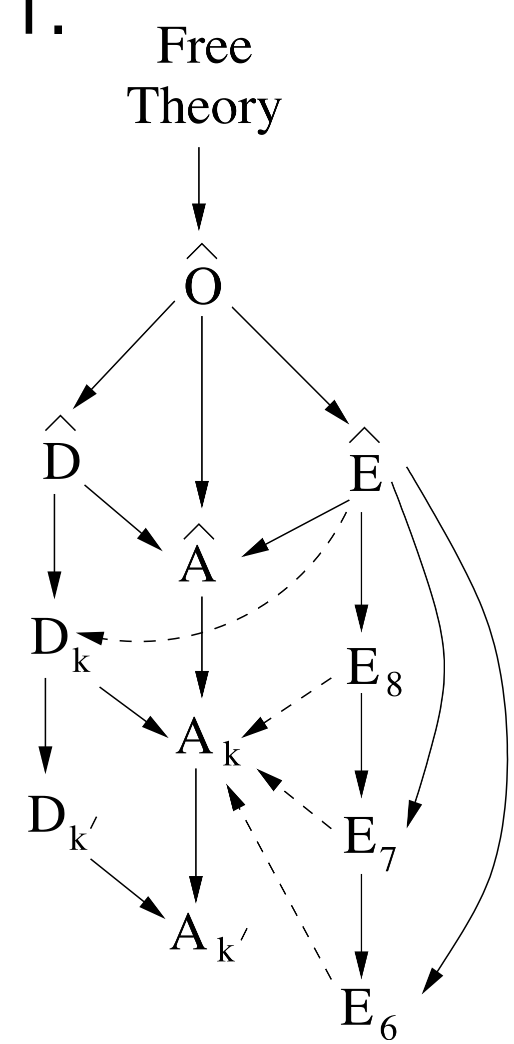
$$ISO(d) \to \mathbb{R} \times ISO(d) \to SO(d+1,1)$$

In this case we get a Conformal Field Theory (CFT)

The general paradigm is that the couplings ‘’flow’’ as we change the scale but for very small (IR) and very large (UV) resolutions we approach some CFTs, where the couplings no longer flow.



So we should imagine a very high-dimensional space with CFTs that are connected by flows that are triggered by relevant couplings. There are finitely many relevant couplings in each CFT.



- Not much is known about this space except for some interesting crude features.

- Many detailed results are however known about deformations of the CFT that do not break $SO(d+1,1)$ i.e. exactly marginal deformations, $\beta_i = 0$.

- It takes some sort of a miracle for $\beta_i = 0$ to happen.

$$\langle O_i(0) O_j(1) O_k(\infty)\rangle = 0$$

$$\int d^d z \langle O_i(0) O_j(z) O_k(1) O_l(\infty)\rangle_c = 0$$

etc.

One can argue that it only happens ‘’naturally’’ in d=2, c=1 models.

E.g. consider the c=1 compact boson (Luttinger Liquid). We have the exactly marginal operator

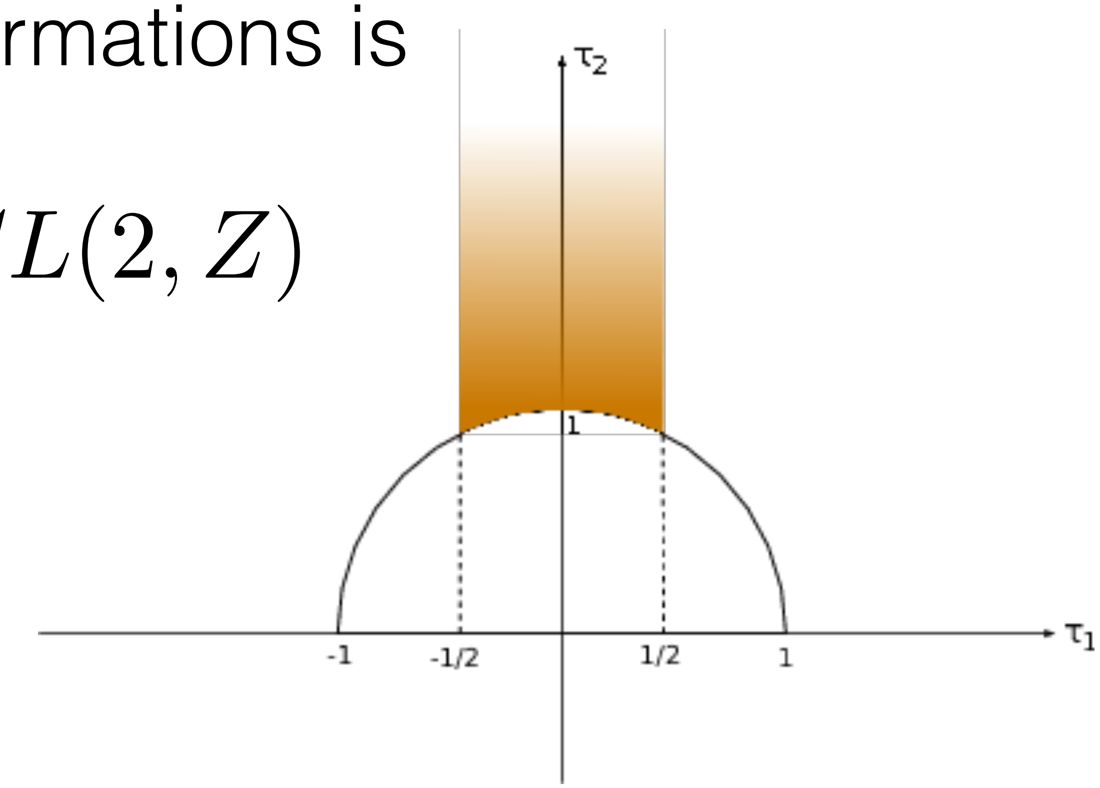
$$\delta R \int d^2x \partial_a X \partial^a X$$

and if the boson is complex we can also add a topological theta-angle term

$$\delta R \int d^2x \partial_a X \partial^a \bar{X} + \delta\theta \int d^2x \epsilon^{ab} \partial_a X \partial_b \bar{X}$$

The space of exactly marginal deformations is

$$H/SL(2,Z) \times H/SL(2,Z)$$



But if $d \neq 2$ or $c \neq 1$ then one has to rely on various miracles in order to have exactly marginal operators.

One “miracle” that can lead to exactly marginal operators is supersymmetry.

Another “miracle” is infinite N.

The space of exactly marginal deformations furnishes a Riemannian manifold, $\mathcal{M}$

The exactly marginal couplings g^I provide coordinates on this space and the metric is given by

$$\langle O_I(x) O_J(0) \rangle_{\{g^I\}} = \frac{\gamma_{IJ}(g^I)}{x^{2d}}$$

The Ricci scalar formed from the metric is completely invariant under redefinitions of coupling constants etc.

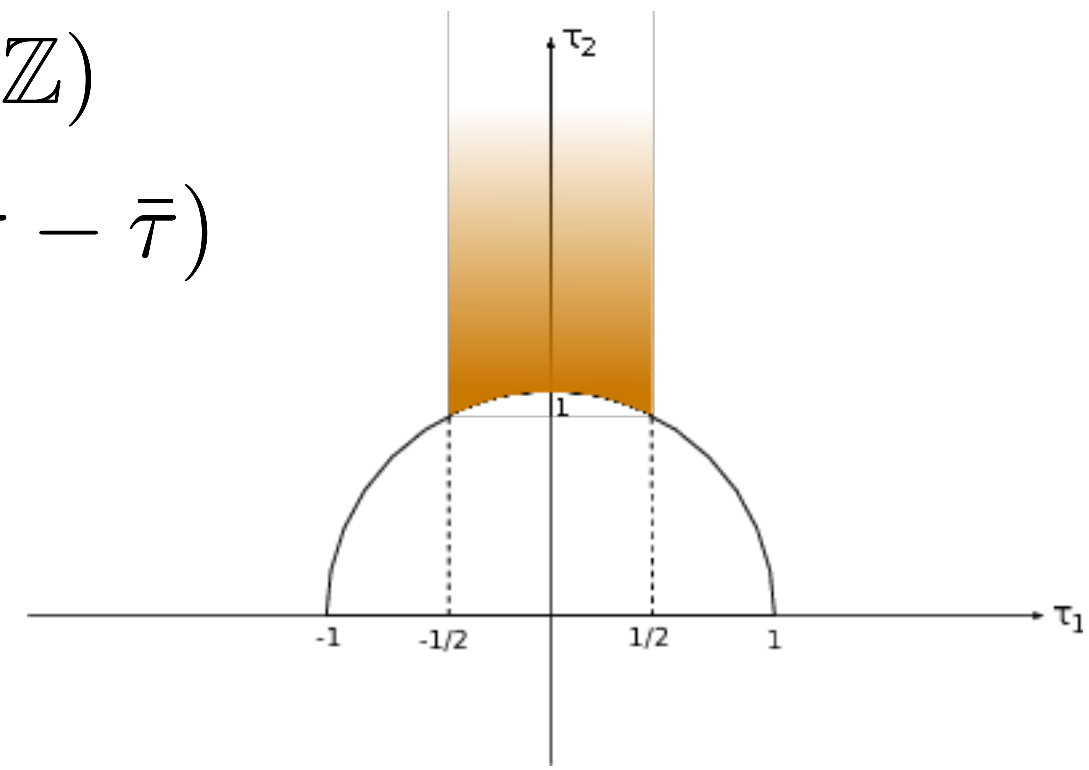
Going back to our complex boson, the metric on $H/SL(2,Z)$ can be calculated exactly. It is given by

$$ds^2 = \frac{1}{R^2}\left(dR^2 + d\theta^2\right)$$

- d=2, (2,2) Superconformal theories. $\mathcal{M}$ is Kahler. If our theory is a sigma model with Calabi-Yau target space, then $\mathcal{M}$ is the moduli space of complex structure and Kahler deformations. In string theory we identify the metric on $\mathcal{M}$ with the metric for the light fields in supergravity.

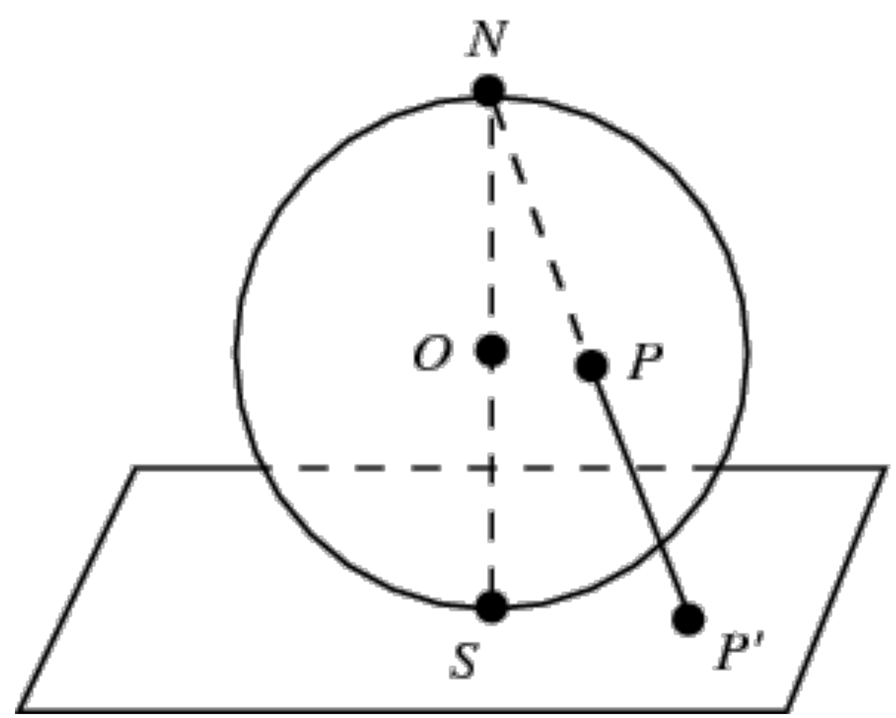
- d=4, $\mathcal{N} = 1$ theories. $\mathcal{M}$ is Kahler. For example, starting from $\mathcal{N} = 4$ we have a $3_{\mathbb{C}}$ dimensional Kahler manifold of which only $1_{\mathbb{C}}$ preserves $\mathcal{N} = 4$. We know the metric in this specific direction $H/SL(2,\mathbb{Z})$

$$K = \log(\tau - \bar{\tau})$$



The information about the metric on these spaces is of interest. It cannot be extracted from standard results on SUSY field theories.

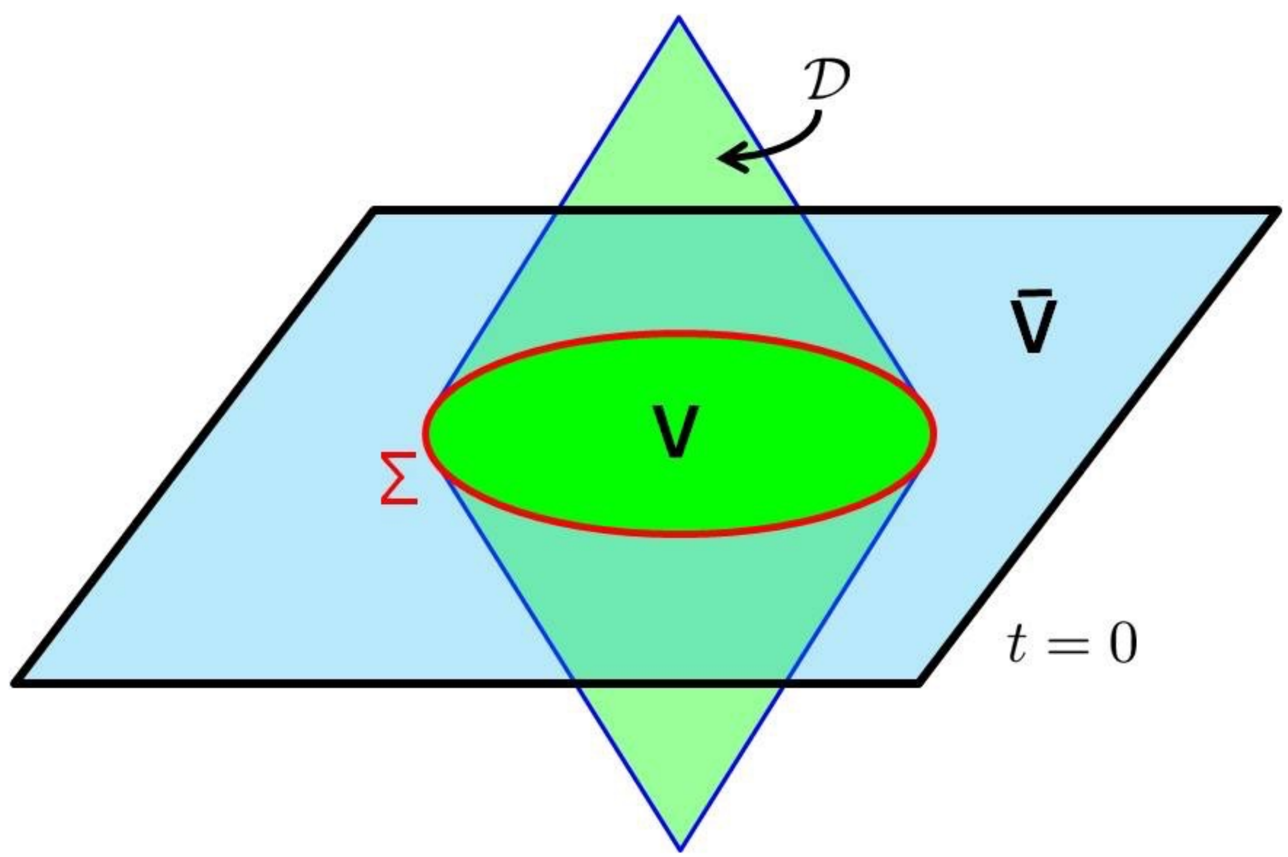
The following idea appears to be very powerful. Given a CFT, one can place it in a canonical fashion on $\mathbb{S}^d$



Physically, these sphere partition functions can be related to the amount of entanglement in the vacuum.

$$Z_{\mathbb{S}^d} = -Tr_V\left(\rho_V \log \rho_V\right)$$

$$\rho_V = Tr_{\bar{V}}\left(|VAC\rangle\langle VAC|\right)$$



Start from d=2n. The partition function $Z_{\mathbb{S}^d}$ in CFTs does not have a preferred normalisation because of the counter-term

$$\int_{\mathbb{S}^d} E_{2n} A(g^I)$$

With E_{2n} the Gauss-Bonnet density. On the other hand, there is a trace anomaly that makes the partition function depend on the radius r as

$$Z_{\mathbb{S}^d} = \left(\frac{r}{r_0}\right)^{a_{2n}} \mathcal{F}(g^I)$$

$\mathcal{F}$ is ambiguous.

We have $a_2 = c/3$, $a_4 = -4a$ in terms of the usual central charges. Other than these interesting coefficients, the partition function does not capture universal information.

In d=2n+1 there is no counter-term. And there is also no trace anomaly. So we have, for instance,

$$Z_{\mathbb{S}^3} = e^{-f}$$

and f is independent of exactly marginal parameters. f is unambiguous. It is conjectured to be always positive. This assertion can be proven for topological QFTs.

A useful observation is that in supersymmetric theories such as d=2, (2,2) theories and $\mathcal{N}=2$, d=4 theories, the normalisation of the partition function becomes physical! One can prove:

$$d=2,(2,2): Z_{\mathbb{S}^2}=\left(\frac{r}{r_0}\right)^{c/3} e^{-K(g^I,\bar{g}^I)}$$

$$d=4,\mathcal{N}=2: Z_{\mathbb{S}^4}=\left(\frac{r}{r_0}\right)^{-4a} e^{\frac{1}{12}K(g^I,\bar{g}^I)}$$

The function $K(g^I, \bar{g}^I)$ is related to the metric on the space of exactly marginal operators by

$$\gamma_{I\bar{J}} = \partial_I \partial_{\bar{J}} K(g^I, \bar{g}^I)$$

Therefore the function K is not intrinsically well defined

$$K(g^I, \bar{g}^I) \to K(g^I, \bar{g}^I) + F(g) + \bar{F}(\bar{g})$$

- In general K is not globally defined. So it would seem like $Z_{\mathbb{S}^d}$ is a section of $L \otimes \bar{L}$ rather than a function. This leads to contradictions.

- One therefore concludes that the Kahler class vanishes $\left[g_{I\bar{J}} dg^I \wedge d\bar{g}^{\bar{J}}\right] = 0$. In particular, $\mathcal{M}$ cannot be compact.

- This means that the moduli space of Calabi-Yau manifolds must have trivial Kahler class!

- A simple check is that for $\mathcal{N} = 4$, the function

$$K = \log(\tau - \bar{\tau})$$

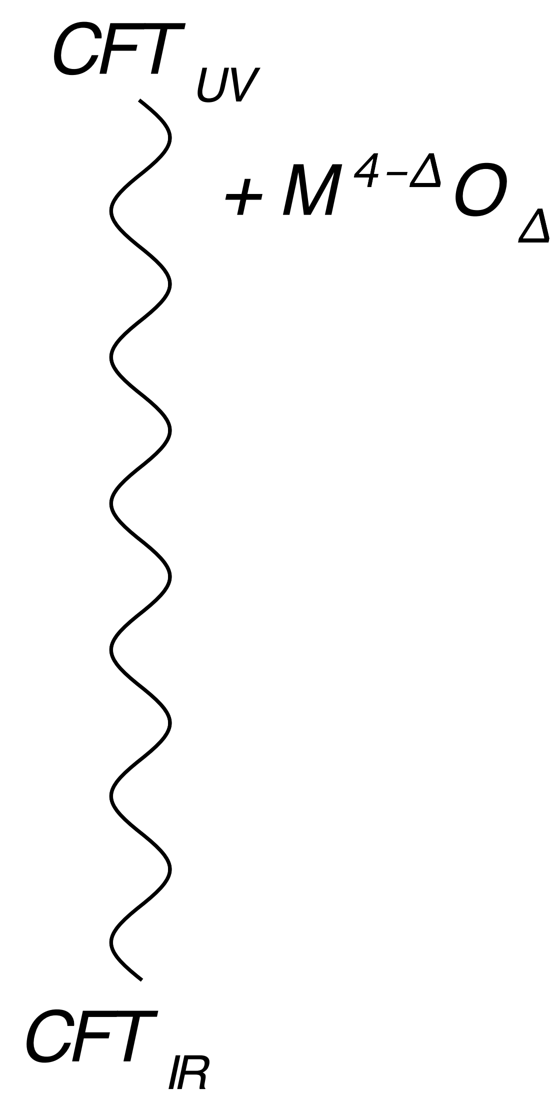
  is well defined in the fundamental domain.

- The $\mathbb{S}^2, \mathbb{S}^4$ partition functions can be computed using localisation. So there are rather explicit expressions for the metric on the moduli space of many interesting CFTs! Weak coupling limits are typically infinitely far away.

- In applications to string theory, it has already been expected that the Kahler class is integer because of considerations in 4d supergravity. We claim that actually it must vanish.

- For the d=4, $\mathcal{N}=2$ case, AdS/CFT relates $\mathcal{M}$ with the vacuum manifold of 5d $\mathcal{N}=4$ supergravity. The known examples are consistent with the claim that the Kahler class vanishes.

Therefore it seems as if computing the sphere partition function in the space of theories produces interesting results. What happens if we allow for relevant perturbations that break the conformal symmetry?



In general, the partition function now takes the form

$$Z_{\mathbb{S}^d} = e^{F(r,M)}$$

Such that for very small and very large radius we asymptote to the respective CFTs and then

$$d = 2 : F(r \to (0,\infty), M) \longrightarrow \frac{c_{(UV,IR)}}{3} \log(r)$$

$$d = 4 : F(r \to (0,\infty), M) \longrightarrow -4a_{(UV,IR)} \log(r)$$

$$d = 3 : F(r \to (0,\infty), M) \longrightarrow -f_{(UV,IR)}$$

- In all the known examples

$$c_{UV} > c_{IR}$$
$$f_{UV} > f_{IR}$$
$$a_{UV} > a_{IR}$$

  So the partition function on the spheres allows to foliate the space of theories — flows are irreversible.

- Other than in d=2, we don't know if the flows are gradient flows or maybe more complicated ones.

- These inequalities appear to be related to a famous inequality in information theory

Consider a tensor product space

$$H = H^1 \otimes H^2 \otimes H^3$$

Define the density matrix (trace=1, semi-positive definite) to be ρ^{123}. Then one can define ρ^{12} as a partial trace $\rho^{12} = Tr_{H^3}\rho^{123}$.
One finds a nontrivial inequality

$$S(\rho^{123}) + S(\rho^2) \leq S(\rho^{12}) + S(\rho^{23})$$

$$S(\rho) \equiv -Tr(\rho \log \rho)$$

- The space of theories has natural geometry on it. It is particularly rich in supersymmetric theories.
- The space of QFTs has a natural foliation. Maybe even a gradient flow structure (remains to be proven/disproven).
- The sphere partition function allows to probe these fascinating structures.
- There seems to be some intriguing relation to information theory.