Models of visual perceptual Space

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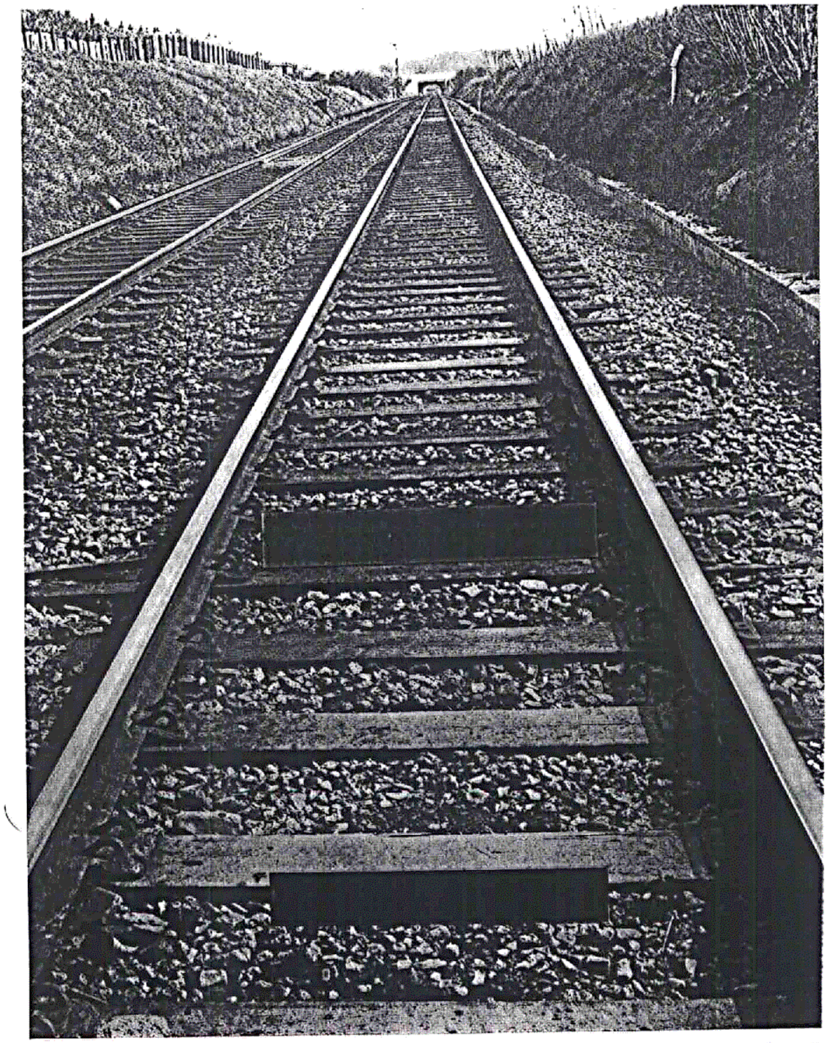

ILLUSION INVOLVING PERSPECTIVE is remarkably constant for all human observers. The two rectangles superposed on this photograph of railroad tracks are precisely the same size, yet the top rectangle looks distinctly larger. The author regards this illusion as the prototype of visual distortions in which the perceptual mechanism, involving the brain, attempts to maintain a rough size constancy for similar objects placed at different di tances. Since we know that the distant railroad ties are as larg as the nearest ones, any object lying between the rails in the mi dle distance (the upper rectangle) is unconsciously enlarged. I deed, if the rectangles were real objects lying between the rail we would know immediately that the more distant was large

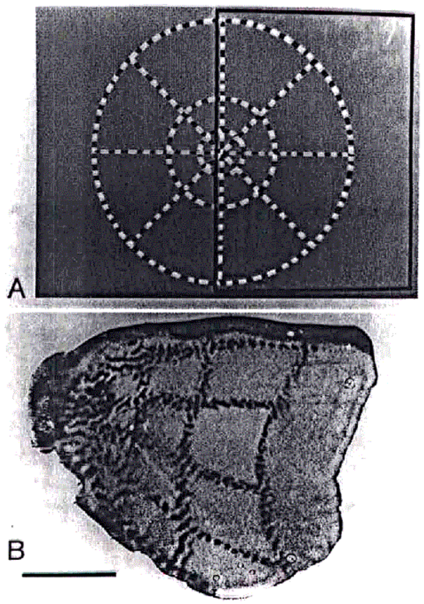

FIG. 4.1 (A) Stimulus used for mapping the retinotopic projection onto striate cortex. (B) pattern of 2-DG uptake in macaque striate cortex produced by this stimulus pattern. See text for details (from Tootell, Silverman, Switkes & R.L. De Valois, 1982a, *Science, 218*, 902-904. Copyright 1982, AAAS. Reprinted with permission).

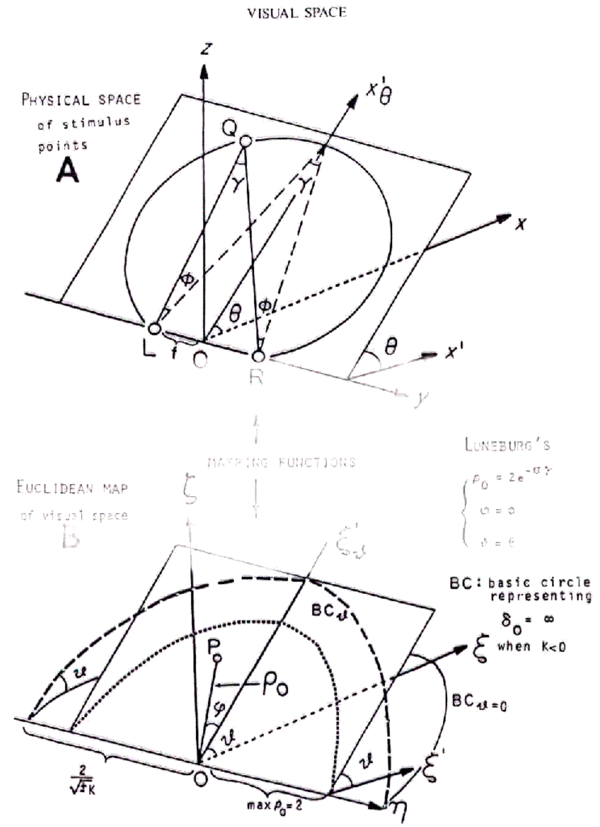

Figure 2. A: A stimulus point Q in the physical space X; Cartesian and bipolar coordinates. B: A point P in the Euclidean map EM; Cartesian and polar coordinates. (The whole visual space VS is represented within the sphere of radius max ρ_0)

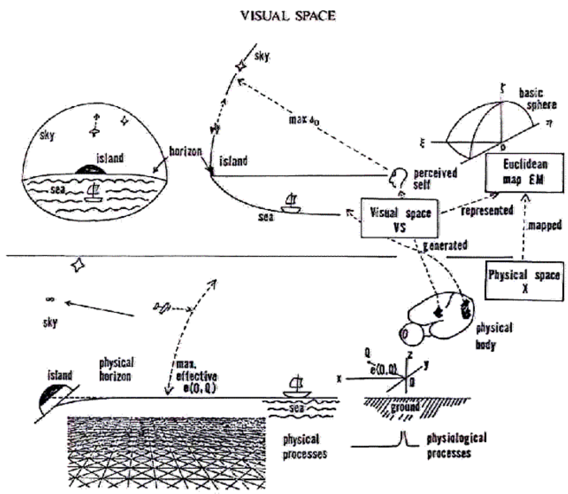

Figure 1. Physical space X and visual space VS in the large.

Properties of Visual Space

- VS has three major directions, which correspond to the three dimensions in X, the physical world.
- Percepts in VS are hierarchically related to each other. Each percept is localized with respect to other percepts which act as its framework.
- We can perceive geometrical properties in VS.
- How VS is structured depends on stimulus conditions in X.

  VS is frameless in the sense that stimuli are presented in a homogenized ganzfeld

- In orthogonal sensory co-ordinates, the line element ds in VS is given in terms of sensory co-ordinates α,β and γ by

$$ds^2 = \frac{d\alpha^2 + d\beta^2 + d\gamma^2}{\left[1 + \frac{1}{4}\kappa\left(\alpha^2 + \beta^2 + \gamma^2\right)\right]^2}$$

where κ refers to the curvature of the space

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A Visual space formalism

Experimental Foundation

***Class I*:** Discrimination of simple patterns with simple size scaling. Here comparisons involved discriminations of one and two dimensional patterns with relative sizes scaled to unity or in a 2:1 ratio in either one or two dimensions

***Class II*:** Discriminations among similar patterns with rotations. Here, similar patterns were either reflected about a vertical axis, inverted about a horizontal axis or rotated about the line of sight.

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- ***Class III***: Comparisons of position defined by dissimilar features. In this class, the component features themselves differed between reference and comparison patterns. Only their relative positions remained comparable
- ***Class IV***: Discriminations among similar patterns with transformations. Here, the test patterns were transformed under a set of transformation conditions.

Fig. 1. Stimulus patterns used in Class 1 experiments (see text for details). (a) Experiment 1 ("length"). (b) Experiment 2 ("equal"). (c) Experiment 3 ("profile"). (d) Experiment 4 ("odd").

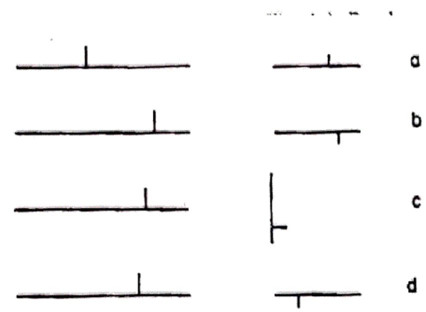


Fig. 4. Stimulus patterns used in Class 2 experiments. (a) Experiment 5 ("left–right"). (b) Experiment 6 ("up–down"). (c) Experiment 7 ("upright"). (d) Experiment 8 ("upside").

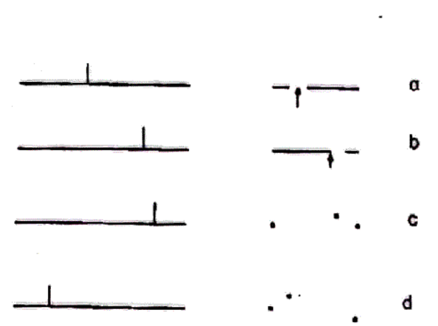
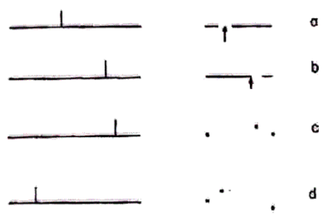


Fig. 7. Stimulus patterns used in Class 3 experiments. (a) Experiment 9 ("symmetric gap"). (b) Experiment 10 ("asymmetric gap"). (c) Experiment 11 ("aligned dots"). (d) Experiment 12 ("nonaligned dots"). The arrows in the figure are for purposes of illustration only. There were no arrows present in the patterns used in actual experiments.



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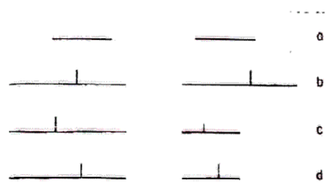

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Discrimination of relative spatial position 1655

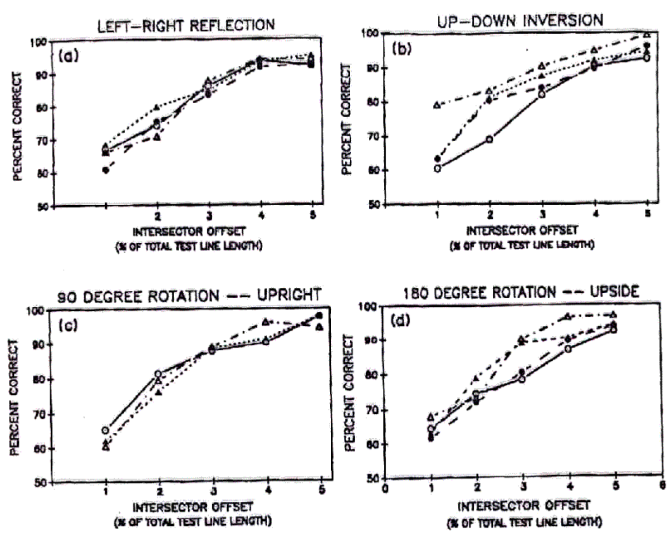

Fig. 5. Psychometric functions obtained from Class 2 experiments: 1, "left–right" (a); 2, "up–down" (b); 3, "upright" (c); and 4, "upside" (d). Subjects: ○, RN; ●, SS; △, KDeV; ▲, JS.

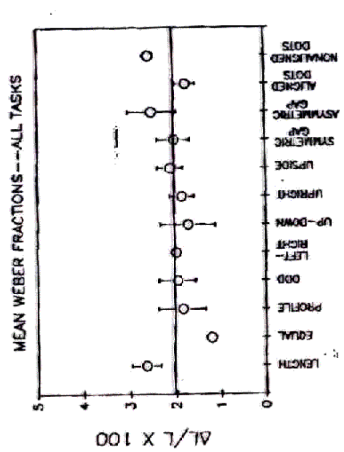

Fig. 6. Mean Weber fractions obtained from all tasks for all observers. The solid line represents the overall average of Weber fractions from all observers on tasks 1–12. Error bars illustrate one standard deviation. In the case of a small number of points (as here), it effectively shows the range of the data.

- Magnification, translation, inversion, reflection and rotation of objects in the fronto-parallel plane do not adversely affect performance.
- Discriminations along one axis do not depend upon position in the perpendicular axis.
- Positions to be compared can be equivalently defined by any of several features
- Performance does not depend on length ratios.
- Certain classes of axis transformations have little effect on performance (ex., Cartesian to polar) but other classes of axis transformations (ex., linear to log) have significant adverse effect on performance.

The Model

- There is an orthogonal, labeled co-ordinate system;
- Simple linear transformation of scale and orientation are allowable;
- Some complex transformations are allowed, while others are not.

Affine Connection

$$V_\alpha = f_{\alpha\beta} v^\beta$$

$$\Gamma^\lambda_{\alpha\beta} = \frac{\partial f_{\alpha\gamma}}{\partial x^\beta} f^{\gamma\lambda}$$

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$$\Gamma^\lambda_{\lambda\beta} = \frac{1}{2}\frac{\partial \log |G|}{\partial x^\beta}$$

$$\frac{\partial \log |F|}{\partial x^\beta} = \frac{1}{2}\frac{\partial \log |G|}{\partial x^\beta}$$

which results in

$$G = (kF)^2$$

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Riemannian Curvature

$$R^{\alpha}_{\eta\beta\gamma} = \left(\Gamma^{\alpha}_{\beta\eta}\right)_{,\gamma} - \left(\Gamma^{\alpha}_{\eta\gamma}\right)_{,\beta} + \Gamma^{\alpha}_{\tau\gamma}\Gamma^{\tau}_{\beta\eta} - \Gamma^{\alpha}_{\tau\beta}\Gamma^{\tau}_{\gamma\eta}$$

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Orientation of Vectors in VS

$$\partial W = \oint dx^{\alpha}(S) \int_0^S \Gamma^{\beta}_{\alpha\gamma}(S')V_{\beta}(S')\frac{dx^{\gamma}}{dS'}dS'$$

$$\frac{\partial f_{\alpha\rho}}{\partial x^{\gamma}} f^{\rho\beta} V_{\beta}(S')\frac{dx^{\gamma}}{dS'}dS$$

$$\partial W = v^{\rho} \oint dx^{\alpha}(S) \int_0^S \frac{\partial f_{\alpha\rho}}{\partial x^{\gamma}} \frac{dx^{\gamma}}{dS'} dS'$$

$$= v^{\rho} \oint dx^{\alpha}(S)\left[\delta f_{\alpha\beta}(S)\right]$$

$$= 0$$

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- Multiplying by $f_{\sigma\lambda}$ we can show that the above is an exact differential given by

$$\frac{d}{dS}\left(f_{\sigma\alpha} \frac{dX^{\alpha}}{dS} \right) = \frac{d^{2} f_{\sigma}}{dS^{2}} = 0$$

which has the solution

$$f_{\sigma} = a_{\sigma} S + b_{\sigma}$$

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Length of a vector

$$V^{\alpha} = f^{\alpha\beta} v_{\beta}$$

$$V^{\alpha} V_{\alpha} = f^{\alpha\beta} v_{\beta} f_{\alpha\gamma} v^{\gamma}$$

$$= v_{\beta} v^{\beta}$$

Thus, all length orthogonal transformations preserve the vector lengths and vice versa.

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The metric of VS

$$f^{\rho\lambda} = \frac{1}{F}\frac{\partial F}{\partial f_{\lambda\rho}}$$

$$\frac{\partial f_{\lambda\rho}}{\partial x^{\beta}} f^{\rho\lambda} = \frac{\partial f_{\lambda\rho}}{\partial x^{\beta}}\left(\frac{1}{F}\frac{\partial F}{\partial f_{\lambda\rho}}\right) = \frac{1}{F}\frac{\partial F}{\partial x^{\beta}} = \frac{\partial \log |F|}{\partial x^{\beta}}$$

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Geodesics in VS

- The geodesic of an affine space is given by

$$\frac{d^2 X^{\lambda}}{dS^2} + \Gamma^{\lambda}_{\alpha\beta}\frac{dX^{\alpha}}{dS}\frac{dX^{\beta}}{dS} = 0$$

where S is some parameter (which can be arc length). Substituting for $\Gamma^{\lambda}_{\alpha\beta}$, we get

$$\frac{d^2 X^{\lambda}}{dS^2} + \frac{df_{\alpha\rho}}{dx^{\beta}} f^{\rho\lambda}\frac{dX^{\alpha}}{dS}\frac{dX^{\beta}}{dS} = 0$$

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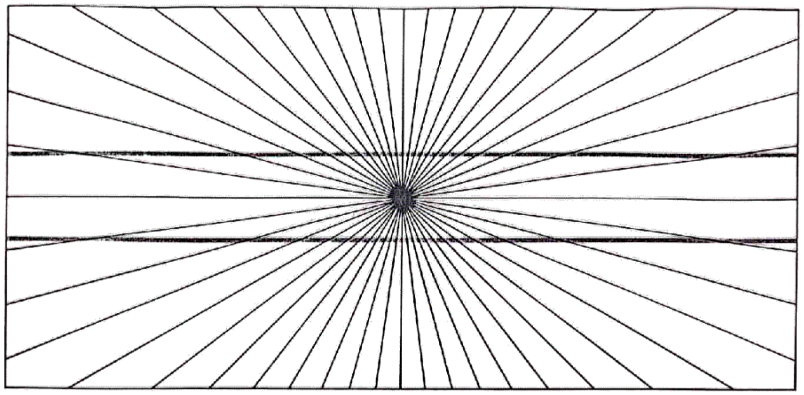
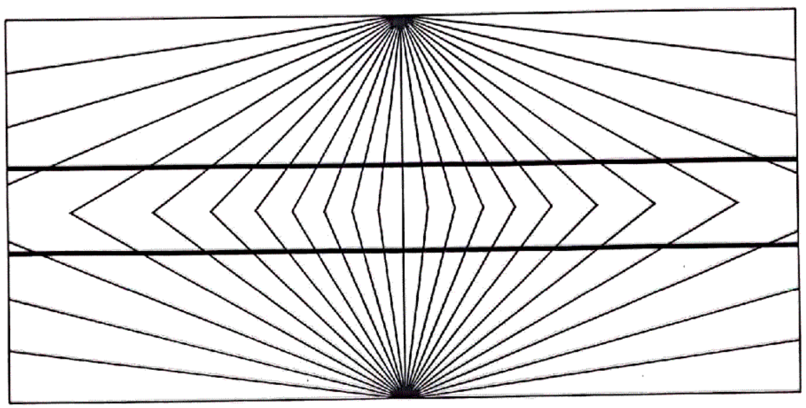


HERING ILLUSION was published in 1861 by Ewald Hering. The horizontal lines are of course straight. Physicists and astronomers of that period took a lively interest in illusions, being concerned that visual observations might sometimes prove unreliable.



CONVERSE OF HERING ILLUSION was conceived in 1896 by Wilhelm Wundt, who introduced experimentation into psychology. Wundt earlier described the simplest of the visual illusions: that a vertical line looks longer than a horizontal line of equal length.

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Allowed and non-allowed transformations

$$f_{xy}^{\,2} = f_{xx} f_{yy}$$

$$\left(qq_{xy} - q_x q_x\right)^2 = \left(qq_{xx} - q_x^{\,2}\right)\left(qq_{yy} - q_y^{\,2}\right)$$

Discussion and Conclusion

- No single simple manifold can be defined which will adequately describe every spatial visual pattern.
- It should be possible, at least in principle, to specify the rule by which a given pattern gives rise to a particular perceptual representation.

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- If the rules are known, the unique spatial percepts that result from any given geometric pattern should be predictable. Thus, geometric visual illusions should follow from the application of the spatial transformation rules to the particular stimulus patterns which give rise to them.
- The prediction of these unique percepts should not depend upon any assumptions about the ecological significance of or meaning attached to a particular feature or stimulus.

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- Optics Application
- Vision / perception
- Spatial Vision

- Hoffman LTG approach – Lie Germ
- Caell Pattern or Contour extraction
- Foster Apparent movement – permissible paths

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- Optical retinal image – continuous transformations
- non-uniform projections – Constant change extract image properties which are invariant and stable and contiguous image → affine properties – constancies – color, shape, distance (length), etc.

3

- Experimental set up for apparent motion :

Parameters : time, distance

1. Timing should be right. Stimulus Onset Asynchrony SOA time between onset of one view and onset of the other is not too short.
2. Interstimulus Interval (ISI) – time between offset of one view and onset of the other is not too long.

➡ Experience a path between alternate positions

Perceptual Illusion of an actual rigid motion of the body along a well-defined trajectory

SOA critical value ∝ disparity between the two positions

Korte's laws

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- **Mental Rotation Experiments**
- Prospective line drawings
- 3D structures
- When two structures are different (cannot be rotated into congruence) they differ only by a reflection

- Subjects: Determine if 2 stoboscopically presented objects are related by a rotation.
- - Same object rotates or reflected.

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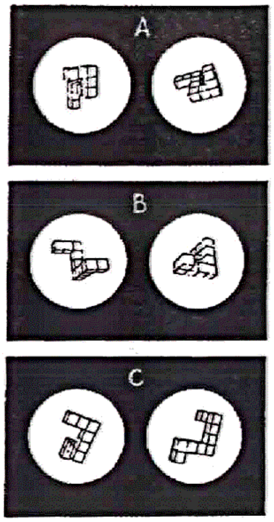

Figure 3.1 Illustrative pairs of perspective views, in-
uding a pair differing by an 80 degree rotation in the
icture plane (A), a pair differing by an 80 degree rotation
depth (B), and a pair differing by a reflection as well as a
otation (C).

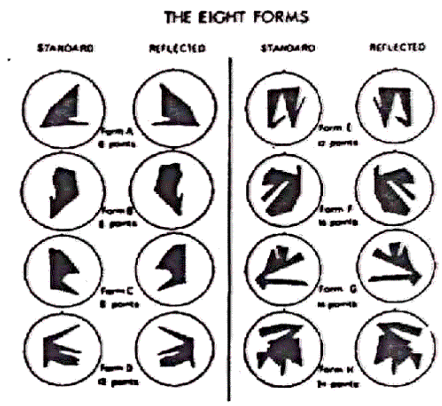

Figure 6.1 The eight random forms used in Experiment I, displayed in both standard and reflected versions.

Shepard and Metzler, Mental rotation of three dimensional objects, Science 171: 701-703, 1971

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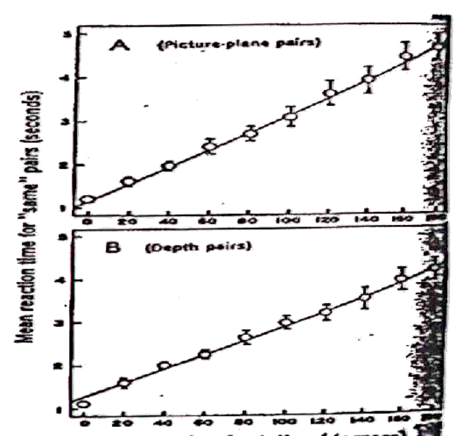

Fig. 2. Mean reaction times to two perspective line drawings portraying objects of the same three-dimensional shape. Times are plotted as a function of angular difference in portrayed orientation: (A) for pairs differing by a rotation in the picture plane only; and (B) for pairs differing by a rotation in depth. (The centers of the circles indicate the means and, when they extend far enough to show outside these circles, the vertical bars around each circle indicate a conservative estimate of the standard error of that mean based on the distribution of the eight component means contributed by the individual subjects.)

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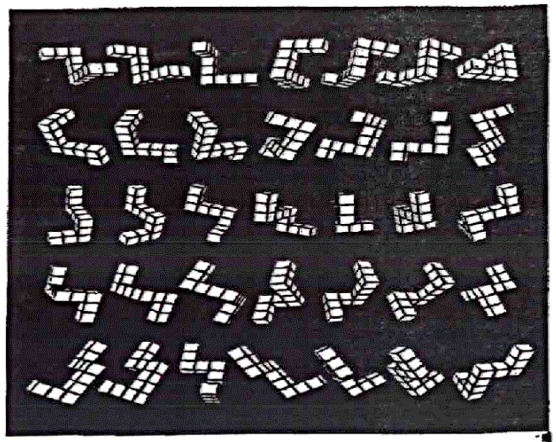

Figure 2.2 Half of the set of computer-generated perspective views used in the subsequent experiments. (Reproduced from the cover of the February 19, 1971 issue of *Science*.)

Psych Phenomenon of apparent motion and mental rotation Of rigid bodies in 3-D Euclidean space – there is a continuous Path of motion - an observed path in A.M. imagined path in M.R.

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- " Paths of transformation" experienced in real, apparent and imagined motion of an objects between 2 positions can be represented as paths in the Euclidean motion group.

$$E^{+} = R^{3} s\ SO(3)$$

- E^+ - group of transformations of 3-D space R^3 preserves Euclidean distance left/right orientation of orthonormal frames (Mackey, 1968)
- R^3 – Translation sub-group
- SO(3) – Rotation sub group

- Path in $E^+ : \gamma : [a, b] \rightarrow E^+$

Mapping from [a,b] set of real numbers $a \le t \le b$

To any given rigid motion of the object from

$$Q_1 = \gamma(a) \ to \ Q_2 = \gamma(b)$$

- there corresponds a unique path

$$\gamma : [a,b] \rightarrow E^+ \qquad \gamma(a) = Q_1, \gamma(b) = Q_2$$

"Analog" character of internal process [illegible] internal representation of rigid transformation is by such a path.In terms of path, $\gamma : [a,b] \rightarrow E^+$ by an intermediate value of the path parameter t, t_o such that $a \le t_0 \le b$; intermediate point $= \gamma(t_o)$ along curve $\gamma(t)$ in E^+

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"Continuous mapping"

→ Internal representation of each intermediate position change continuously into the internal representation of the succeeding position.

Nature of the path:

→ Psychologically favored paths/motions are not arbitrary but determined by internal constraints that reflect the natural geometry of the rigid transformation formalized by E^+

→ Geodesics of an appropriate connection on E^+ that reflects the relative dominance in the perceptual/representational system of algebraic or metric principles.

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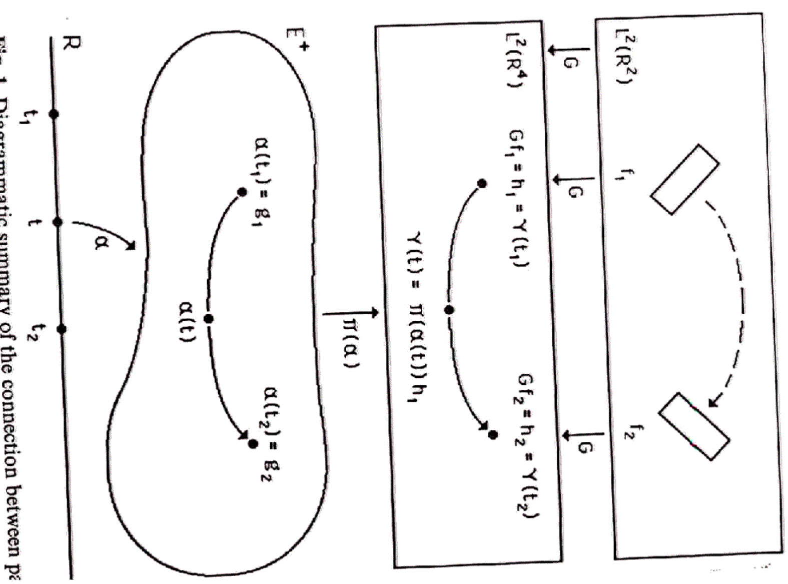

Fig. 1. Diagrammatic summary of the connection between pa
$\alpha \in E^+$ and path $\gamma \in L^2(R^4)$

- Participants

  "by imagining one of the 2 objects rotated into the same orientation as the other and then assessing whether the resulting shapes did or did not match"

- Time Required

  "Increased linearly as a function of the angular difference between the orientations in which the two objects were portrayed"

Direct proportionality of R.T to orientation

→ Consistent with continuity of imagined rotation

Random Polygon Shapes

Analog process → " intermediate stages of the internal process represent intermediate positions of the transformed object"

9

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- Other experiments: Apparent motion is also an analog process in the same sense of representing intermediate positions at intermediate times.

- " Although the experienced motion is perceptual and involuntary, the paths of experienced transformations evidently are the same as those traversed in voluntarily imagined transformations suggesting that the same representational system is operative in both cases"

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- Q: How are rigid transformations internally represented as path in E^+ connected to cortical patterns?
- A: A unitary Representation of E^+ on L^2 (R^4)

Why?

How?

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- Description of responses of simple cortical cells.
- Simple cells in visual cortex
- a. spatially localized receptive fields
- b. Distinct elongated excitatory and inhibitory zones.
- c. Cells respond strongly to specifically oriented lines or edges.
- d. Can be studied in spatial frequency domain – cells tuned to specific spatial frequencies bandwidth ~ 1 octave.

$\psi\phi$ = Evidence that visual scene is analyzed in terms of independent spatial frequency channels

→ Visual cortex acts as a spatial frequency analyzer or at least does a "piecewise spatial frequency analysis".

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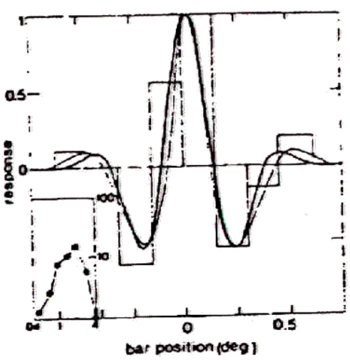

Fig 1: Comparison of the experimentally measured responses of a simple cell in the visual cortex

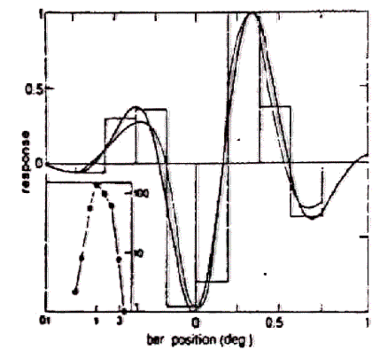

Fig 2: Comparison of a response of a cell with an antisymmetric receptive field.

JOSA, 70(11) 1980

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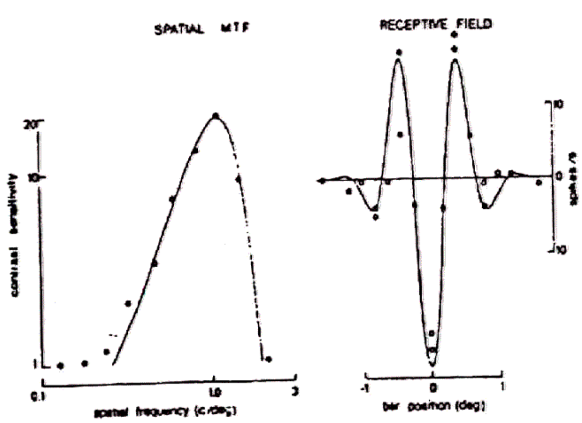

Fig 3: Comparison of the response of a simple X-cell from A monkey visual cortex and the functional form of Gabor elementary signals.

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- Representation of image in the visual cortex involves both spatial and spatial frequency variables in its description.
- Does not include: preferred direction of stimulus movement, coding of color, neural connections.
- Marceelja, JOSA 70: 1297-1300, 1980

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Receptive Field:

Gabor elementary functions

$$g(p,u)(x) = e^{\left(-(x-p)^2\right)} e^{\left(-iu(x+p)\right)}$$

Fourier transform → spatial frequency

Pribram and Carlton Acta Psychol. 63: 175-210, 1986

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- Model: Neural activity resulting from visual cortical response to a given retinal image is represented as a function of space and spatial frequency variables – as a function on a 4 –D retinal by spatial frequency manifold.
- $g(p,u)(x)$ = square integrable function if 2 real variables

$x=(x_1,x_2)$

$p=(p_1,p_2)$ retinal point where receptive fields is centered .

$u=(u_1,u_2)$ perceptual spatial frequency

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- What is the interaction of f (retinal image) with receptive field profile $g(p,u)$ for retinal point $x=(x_1,x_2)$,
- $f(x)$ = complex electromagnetic fields at x
- $f:R^2 \rightarrow c$ vanishes outside a compact set contribution to the response of a single receptive field $g(p,u)$ from retinal area
- $d^2x = dx_1dx_2$ about the point x is $f(x)g(p,u)(x)d^2x$

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Assume linearity,

response of $g(p,u)$ to retinal image is f

$$\int_{R^3} f(x)g(p,u)(x)d^2x = Gf(p,u)$$

p ranges over retina, u over all spatial frequencies G

By appropriate choice of constants, the mapping is an isometry of $L^2(R^2) \rightarrow L^2(R^4)$

Integration of delta function shows that for appropriate f $in\ L^2(R^2)$ the norm Gf in $L^2(R^4)$ is propotional to the norm of f in $L^2(R^2)$

- The Hilbert space $L^2(R^4)$ pf the four variable transforms Gf of the 2 variable retinal images gives the link between cortical activity patterns and paths experienced in real apparent and imagined rigid transformation of objects in 3-D space

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- **Unitary Representation of the Euclidean Group on Gabor Transform Space.**
- Bargmann → unitary representation of SU(2) pf unitary modular transformations of 2 –D complex vector space C^2 on Hilbert space F_2 of analytic functions of 2 complex variables.
- → irreducible unitary representation of SU(2) method of harmonic Analysis: linear representations of a group on a space of functions from a known action of the group on the domain of the functions.

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- Obtain unitary representation of on

Q = Quarternions

$x = x^0 e_o + x^1 e_1 + x^2 e_2 + x^3 e_3$ $\quad e_i = \textit{Quarternion basis vectors}$

$e_0 = 1; e_i^2 = -e_0; i = 1,2,3$

$X = \left(x^0, x^1, x^2, x^3\right); x^i = real$

$e_1 e_2 = -e_2 e_1 = e_3;\ e_3 e_1 = -e_1 e_3 = e_2;$

$e_2 e_3 = -e_3 e_2 = e_1$

3-D real vector space R^3 = subspace of Q spanned by e_1, e_2, e_3

Norm $X \in Q$

$$|X|^2 = \sum_{i=0}^{3} (x^i)^2$$

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- Lie groups S^3 and SU(2) are isomorphic and form a 2 fold covering of the rotation group SO(3).
- The subspace of Q spanned by e_1, e_2, e_3 is mapped on to itself by Quarternion conjugation.
- The set S^3 of unit Quarternions is a group under Quarternion multiplication.
- $a \in S^3$ define a rotation R(a) of R^3 by Quarternion conjugation C(a)

$$R(a)(x) = axa^{-1} = C(a)(x)$$

- $C(-a) = C(a)$ antipodal unit Quarternians define same rotation

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$X = (x^1 x^2 x^3) \in R^3$

$S^3 \square SO(3)$

- S^3 also acts by conjugation on R^4 identified with Q. This action of S^3 on R^4 gives a natural representation Π, of S^3 as a group of linear operators on any space of functions defined on R^4.

$$(\Pi(a)f)(x) = f(c(a^{-1})(x)) = f(a^{-1}xa)$$

- Π is a homomorphism of S^3 since the composition of 2 linear operators

$$(\Pi(a)\Pi(b)f)(x) = (\Pi(b)f)(a^{-1}xa) = f(b^{-1}(a^{-1}xa)b)$$

$$= f((ab)^{-1} \times (ab)) = (\Pi(a,b)f)(x)$$

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$$(\Pi(e_0)f)(x) = f(x)$$

$$(\Pi(a))^{-1} = \Pi(a^{-1})$$

- Therefore, the desired representation of E^+ on $L^2(R^4)$ comes from extending the domain of Π to include the translation subgroup R^3 by simply embedding R^3 in Q. The resulting representation of Π of the covering group $\overline{E^+} = R^3 s S^3$ of carries E^+ both +1 and –1 to the identity operator on $L^2(R^4)$
- So defines a representation E^+ of $L^2(R^4)$.
- A few more properties:

$\overline{E^+} = R^3 s S^3 \qquad (a,b) \quad a \in S^3, b \in R^3$

$b - (b^1 b^2 b^3) \qquad b = b^1 e_1 + b^2 e_2 + b^3 e_3$

$$(a_1 \ b_1)(a_2 \ b_2) = (a_1 a_2 C(a_1)(b_2) + b_1)$$

$$= (a_1 a_3, a_1 b_2 a_1^{-1} + b_1)$$

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- $\overline{E^+}$ acts on R^4 identified with Q by the composition of Quarternion conjugation with vector translation

$$(a,b)(x) = C((a)(x)) + b = a \times a^{-1} + b$$
$$a \in S^3Q, b \in R^3Q, x \in Q$$

This action preserves the semi direct group product on $\overline{E^+}$

$$(a_1b_1)((a_2b_2)(x)) = ((a_1b_1)(a_2b_2))(x)$$

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- Identity preservation

$$(e_0,0)(x) = e_0 x e_0^{-1} + 0 = x$$
$$(a,b)^{-1}(x) = (a^{-1}, -a^{-1}ba)(x) = a^{-1}xa - a^{-1}ba$$
$$\therefore \quad \Pi(a,b)(f)(x) = f\left((a,b)^{-1}x\right)$$
$$= f\left(a^{-1}xa - a^{-1}ba\right)$$

Just as for S^3, preservation of E^+group product under composition of operators:

$$\Pi(a_1b_1)\Pi(a_2b_2) = \Pi((a_1b_1)(a_2b_2))$$

Identity and inverses are preserved.

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- $(a,b) \in \overline{E^+}, \Pi(a,b)$ preserves inner product on $L^2(R^4)$

$$(\Pi(a,b) f \Pi(a,b) h)$$

$$= \int_{R^4} (\Pi(a,b) f)(x) \overline{(\Pi(ab) h)(x)} d^4x \quad \text{Lebesque Measure}$$

$$= \int_{R^4} f(a^{-1}xa - a^{-1}ba) \overline{h(a^{-1}xa - a^{-1}ba)} d^4x$$

$$(x \to axa^{-1})$$

$$= |a|^4 |a|^{-4} \int_{R^4} f(x - a^{-1}ba) \overline{h(x - a^{-1}ba)} d^4x$$

using tranlation invarinace $x \to x + a^{-1}ba$

$$= \int f(x) \overline{h(x)} d^4x$$

$$= (fh)$$

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- Connection between paths in Euclidean group $\overline{E^+}$ and paths in Gabor transform space $L^2(R^4)$

→ Cortical activity element $L^2(R^4)$

Connected to a particular intermediate rigid transformation of an object (element of $\overline{E^+}$) is given by the action of the Euclidean group element on the cortical activity representation function.

Retinal image = function f in $L^2(R^4)$ corresponding primary visual cortical activity pattern is represented by Gf in $L^2(R^4)$ function space $L^2(R^4)$ contains potential neural microstructure activity patterns which are individually designated by different functions in a Hilbert Space

a path in $L^2(R^4)$ represents a succession of cortical activity pattern and a continuous path $\gamma : R \to L^2(R^4)$ represents a continuous transformation of one cortical activity pattern into the succeeding one.

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- Intermediate positions are represented by

$$t \to \gamma(t) = \Pi(\alpha(t))h_1 \qquad t_1 \le t \le t_2$$

$\therefore$ cortical activity pattern represented at parameter value t by $\gamma(t) \in L^2(R^4)$ connected to a particular rigid transform of an object represented by is given by $\alpha(t) \in E^+$ the action of the Euclidean group element $\Pi(\alpha(t))$ on the cortical activity representation function h_1

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- **Problems:**

Key step is

$$(a,b) = C(a)x + b = axa^{-1} + b \qquad a \in S^3$$

$$b \in R^3$$

$$x = x^0e_0 + x^1e_1 + x^2e_2 + x^3e_3 = x^0e_0 + x^1$$

Let b be a quarternion with

$$b = b^1e_1 + b^2e_2 + b^3e_3$$

$$(a,b)(x) = (a,b)(x^0e_0 + x^1)$$

$$= x^0e_0 + ax^1a^{-1} + b$$

$$= x^0e_0\left[C(a)x^1 + b\right]$$

4D space has 1-D subspace unchanged!

Constant: x^0e_0 is kept fixed biologically implausible

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- $\therefore \Pi(a,b)$ is a unitary operator

Map $x \rightarrow axa^{-1} + b$ of Q into itself define a unitary operator $\Pi(a,b)$ on $L^2(R^4)$ for any non-zero quarternion a and any quarternion b

Maps $a \rightarrow a^{-1}x + b$ and $x \rightarrow xa + b$ define unitary operators for $a \in S^3$ $\therefore \Pi : \overline{E^+} = R^3 s S^3 \rightarrow U(L^2(R^4))$ in a unitary representations of $\overline{E^+}$ on $L^2(R^4)$

This unitary action of the Euclidean group on the Gabor transform space $L^2(R^4)$ gives a natural way of connecting the experienced (in apparent and real motion) or imagined (in mental rotational) paths with successive transformations of the underlying cortical activity patterns.

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→ one of the 4 parameters $(p_1\ p_2\ u_1\ u_2)$ is left fixed if p_1 is fixed → brain pattern can move along one of the cortical axis only.

- Restricted to images which would follow along a circle or along a radius in the visual field [follows from the fact that retino-cortical projection follows an eccentric complex log map].
- If you fix one of the spatial frequencies → exclude any receptive field which deviates in one of it's frequency components from some same set tuning.
- $\therefore$ biologically restrictive
- $\therefore$ The formalism can yield a reducible representation leaving x_0 unchanged.
- The group E^+ is non-compact (due to translation R^3) and any finite dimensional representation of E^+ has to be necessarily reducible on simple lifts from 3 –D

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- Group Extensions:

"embed a given space in a larger space"

use Lorentz group O(3,1) leaves a 4-D length unchanged under a rotation in 4-D space.

Has been used to discuss motion invariance

Extend $E^{+} = SO(3)sR^{3}$ to Lorentz group O(3,1)

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- Write the Lie algebra commutators as

$[J_i, J_j] = i \in_{ijk} J_k$ $\in_{ijk} = 0$ for any 2 indices equal

$[J_i, P_j] = i \in_{i,j;k} P_k$ $\pm$ depending on permutation

$[P_i\, P_j] = 0$

J = generators whose exponents give the elements of SO(3)

P = 3 generators of the translation.

P_i form an Abelian invariant subgroup

J_i close by themselves → form a subgroup

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- New generators:

$$K = \left(\frac{1}{2\mu}\right)(P \times J - J \times P)$$

μ^2 = eigen value of P.P (casimir operator)

Casimir operator: function of the generators which commutes with all the generators, In the space on which the generators act.

The Casmir operator has to be a multiple of identity.

In the case of the rotation group SO(3) Casmir Operator

$$= J_1^2 + J_2^2 + J_3^2$$

- eigen value is not changed by the generators and labels a representation.

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- For Lorentz group, there are 2 Casmir operators whose eigen values label a representation.
- J,K constructed to form the Lie algebra of Lorentz group O(3,1) satisfy

$$J \times J = iJ$$
$$K \times K = -iJ$$
$$J \times K + K \times J = 2iK$$

- K is representation dependent (since it involves the Casimir operator specifying a definite basis)
- J,K → generate an algebra isomorphic to O(3,1)

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→Infinite dimensional Majorana representation

Representation of the Lorentz group is usually specified by J,M which are the eigen values of J^2 and J_3 (expressed in terms of subgroups SO(3))

K will take a representation labeled by J to (J+1, J-1)

- In Majorana representation → eigen values (J +1, J-1)

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For these, the second Casimir operator of

we have

- 1. Extended to Lorentz by using a Lie group Germ.
- 2. Results maintain homomorphism properties postulated by Carlton.
- 3. Representation infinite dimensional, but it is still unitary and irreducible → no subspace remains unchanged by the action.
- 4. Need to derive a finite dimensional representation O(3,2) ?
- 5. Extend analysis to study if on higher functional dimension spaces.

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- Jones and Palmer :

6 free parameter to fit data:

1 Amplitude coefficient for entire 2D Gabor

2 Coefficients for Gaussian spreads in both dimensions

2 Frequencies for oscillation terms

1 Phase term

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$\therefore$ adjusted Gabor $\rightarrow L^2(R^6)$

$\therefore L^2(R^4) \rightarrow$ sufficient to take into account full details of even just simple cells.

$\therefore$ higher dimensions

- Other approaches besides Bargmann's →a Hamiltonian approach as in QM?
- Trajectories are one parameter families from initial states ϕ_0 *to* ϕ_1 via an operator

$$\phi_t = U_t(\phi_0)$$

- assume Markovia property

$$U_s\left(U_t(\phi_0)\right) = U_{s+t}(\phi_0)$$ assoc. set of Diff. eq.

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- Suggestion

$U_t\phi = \Pi_{\alpha_t}\phi \rightarrow$ representation of E^+

Where d_t = 1- parameter subgroup.

Role of Lorentz group in the context of theoretical physics well known.

In the perceptual context ?

→ **what instruction are introduced by the group for presentation of visual stimuli or for neurophysiologal measurements (SQUID,PET, Metabolic scans etc.)**

→ **prediction of new measurable effects?**

Form bridge connecting different domains of neuroscience ?

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