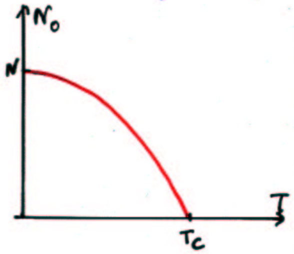
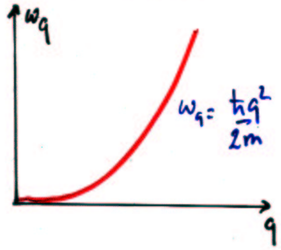
Bose Einstein Condensation and Superfluidity

Ideal gas

- Condensate at $T=0$ if $d>2$



- Quasiparticle excitations



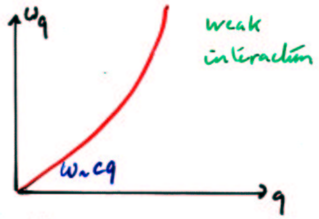
Repulsive gas

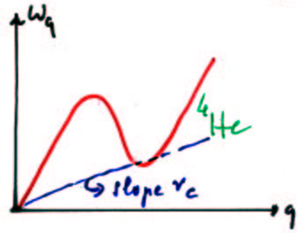
- Cross over to phonons at long wave length

$\Downarrow$

$q_c = 1/\xi$

coherence length





- Quantum fluctuations at $T=0$

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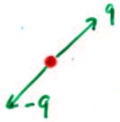
depletion of condensate



- Critical velocity for superfluid flow

$\Downarrow$

Landau criterion

$$\tilde{\omega}_q = \omega_q - q v_s > 0$$

Superfluidity is controlled by particle repulsion !

Hartree approximation at $T=0$: Gross – Pitaevski

- Condensation in a state with wave function $\varphi(r)$

  ↳ order parameter $\Phi(r) = \sqrt{N}\,\varphi(r)$

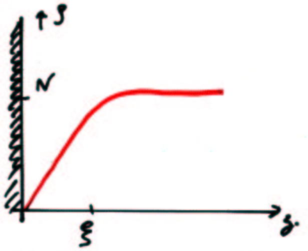
  energy $E - \mu N = \int d^3r \left\{ \frac{\hbar^2}{2m} |\nabla\Phi|^2 + \frac{V}{2} |\Phi|^4 - \mu |\Phi|^2 \right\}$

- Ground state : $\varphi = 1$ (volume $V = 1$)

  ↳ $\Phi = \sqrt{N}$ $\boxed{\mu_0 = NV}$

- Coherence length

  $\frac{\hbar^2}{m\xi^2} \sim NV \rightarrow \xi = \frac{\hbar}{\sqrt{mNV}}$



Profile near a wall

- Dynamics

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m} \Phi + \left\{ V|\Phi|^2 - \mu \right\} \Phi$$

– linearization → coupling of $\delta\Phi$ and $\delta\Phi^*$

↳ Bogoliubov spectrum $\omega_q = \sqrt{\frac{NVq^2}{m} + \frac{\hbar^2 q^4}{4m^2}}$

– Sound velocity $c = \sqrt{\frac{NV}{m}}$ (thermodynamics)

Fragmentation of Condensate

Why all particles in same state 0? $|0\rangle = b_0^{*N} |vac\rangle$

$\Downarrow$

"fragmented" state $|0\rangle = b_1^{*N_1} b_2^{*N_2} |vac\rangle$, $N_1 + N_2 = N$

↳ same energy if $\varepsilon_1 \approx \varepsilon_2 \approx \varepsilon_0$ (in thermodynamic limit)

- Each state has a wave function $\varphi_1 \sim e^{iS_1}$, $\varphi_2 \sim e^{iS_2}$.

The crucial issue is the phase coherence of S_1 and S_2

PHASE LOCKING

Coherent superposition of φ_1 and φ_2

$\Downarrow$

single PURE STATE

Density matrix $(\theta = S_2 - S_1)$ $\rho = \begin{pmatrix} N_1 & \sqrt{N_1 N_2}\, e^{i\theta} \\ \sqrt{N_1 N_2}\, e^{-i\theta} & N_2 \end{pmatrix}$

PHASE INCOHERENCE

STATISTICAL MIXTURE

$$\rho = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

- Superfluidity, macroscopic quantization are all consequences of phase coherence (with two condensates, the circulation $C = L[N_1 v_1 + N_2 v_2]$ is not quantized!)

$\Downarrow$

crucial issue!

Exchange interactions

Particle repulsive interaction

$$V = \frac{1}{2} \sum_{kk'q} V_q\, b_k^* b_{k'}^* b_{k'-q} b_{k+q}$$

Ordinary condensate

$$|\Psi_0\rangle = \frac{1}{\sqrt{N!}}\, b_0^{*N} |vac\rangle$$

$\Downarrow$

Interaction energy

$$E_0 = \frac{1}{2} V_0 N(N-1) \approx \frac{1}{2} V_0 N^2$$

HARTREE

Fragmented condensate

$$|\Psi_0\rangle = \frac{1}{\sqrt{N_1! N_2!}}\, b_1^{*N_1} b_2^{*N_2} |vac\rangle$$

HARTREE $b^* b^* b\, b$

FOCK $b^* b^* b\, b$

$\Downarrow$

$$E = \frac{1}{2} V_0 \left[N_1(N_1 - 1) + N_2(N_2 - 1) + 2N_1 N_2 \right]$$

$+ V_{1-2} N_1 N_2$: FOCK EXCHANGE

Conclusion: fragmenting the condensate costs an extensive macroscopic energy

Exchange is holding the condensate together!

⑤

- Comments

1) Argument unaffected by Hartree approximation
(interaction with uncondensed particles is the same for single and fragmented condensates)

2) Argument reversed in attractive case $\underline{\underline{?}}$

– Unphysical in extended systems : DENSITY COLLAPSE

$E_0 < \frac{1}{2} N^2 V < 0$ (No cost in kinetic energy if N, V large)

– But a finite attractive gas inside a trap may be metastable : collapse = divergent kinetic energy

$\boxed{{}^7Li}$

$$\frac{E}{N} \sim \underset{\text{trap}}{m\omega_0^2 r^2} + \underset{\text{kinetic}}{\frac{\hbar^2}{mr^2}} - \underset{\text{attraction}}{\frac{N|V|}{r^3}}$$

⇓

relevant parameter

$$\eta = \frac{N\,|a|}{a_0}$$

$|a|$ → scattering length of V

a_0 → radius of harmonic trap $\sqrt{\hbar/m\omega_0}$

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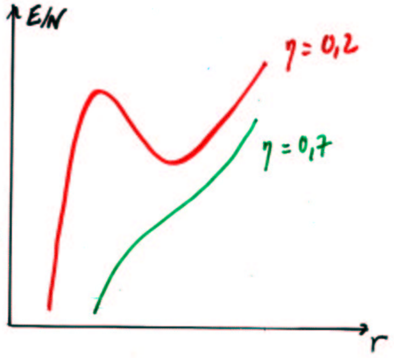
Metastability if $\eta < 0{,}5$



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Ground state of a rotating fluid

GUNN, WILKIN, SMITH, PRL 80, 2265 (1998)

- 2 d gas in an harmonic, isotropic trap

$$H = -\frac{1}{2}\nabla_i^2 + \frac{1}{2} r_i^2 + \eta \sum_{i<j} \delta(r_i - r_j)$$

Dilute gas $|\eta| \ll 1$ $\begin{cases} \text{repulsion if } \eta > 0 \\ \text{attraction if } \eta < 0 \end{cases}$

Angular momentum is a good quantum number

$L = \sum_m m N_m$ ⇒ ground state at fixed $L > 0$ $\underline{\underline{?}}$

- Eigenstates of a single atom

$E_\lambda = |m| + 1 + 2\nu$ → radial quantum number

Energy is lowest if $\nu = 0$, $m > 0$ → $E_0 = \sum_m (m+1) N_m = L + N$

- First order perturbation theory among degenerate manifold (arbitrary choice of $m > 0$ in building L)
↳ exact calculation

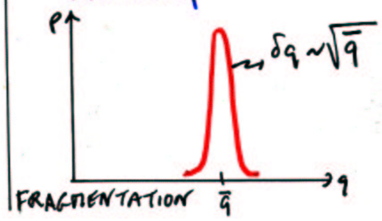
$\eta > 0$

All atoms have the same angular momentum

$$\bar{q} = \frac{L}{N}$$

$\eta < 0$

Distribution of q around $\bar{q}$



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Internal degrees of freedom

Internal state α

↳ Nuclear spin for polarized atomic $H_\uparrow$
Electric dipole moment for optically active excitons
L and S for triplet pairs of ^3He atoms

QUESTION : is the condensate a pure state or a mixture?

Density matrix $\rho_{\alpha\beta} = \langle b^*_{0\alpha} b_{0\beta} \rangle$

↳ one or several non zero eigenvalues?

Examples.

- Superfluid $H_\uparrow$ in pure state ⇒ nuclear ferromagnetism
- " excitons " ⇒ Superradiance

The coherence is lost if the condensate is a mixture.

↳ spectacular macroscopic consequences

(A8)

A forgotten example

- 1960 : Superfluidity of ^3He

P.W. ANDERSON + P. MOREL
"Two atoms bound in a L=1, S=1 state"
↳ coherent, anisotropic condensate

- 1961 L.P. GOR'KOV + V.M. GALITSKII
(Sov. Phys. JETP 13, 792 (1961))
↳ isotropic ground state !!!

Equivalent to an incoherent mixture of all rotational states

↳ unit density matrix
isotropic anomalous self energies

Not realized in nature - but is it a matter of numbers

QUESTION Is an incoherent condensate physically sensible?

An example: "spin 1/2" bosons

local *pair* interaction $\begin{cases} \text{Scalar } U \\ \text{exchange } J \end{cases}$

1) Hartree approximation, coherent state

$$|\Psi_0\rangle = \left[\alpha\, b_{0\uparrow}^{*} + \beta\, b_{0\downarrow}^{*}\right]^{N} |vac\rangle \qquad \begin{cases} N_\uparrow = M|\alpha|^2 \\ N_\downarrow = M|\beta|^2 \end{cases}$$

Density matrix
$$\rho = \begin{bmatrix} N_\uparrow & \sqrt{N_\uparrow N_\downarrow}\, e^{i\theta} \\ \sqrt{N_\uparrow N_\downarrow}\, e^{-i\theta} & N_\downarrow \end{bmatrix}$$

Ferromagnetic alignment in a intermediate direction
$$E_0 = \frac{1}{2}\left[U + \frac{J}{4}\right] N^2$$

- Incoherent state

$$|\Psi_0\rangle = b_0^{*N_\uparrow}\, b_{0\downarrow}^{*N_\downarrow} |vac\rangle$$

No transverse magnetization

Because of symmetry, *one* pair in the same orbital state is necessarily in *triplet* state

↳ same energy $E_0 = \frac{1}{2}\left[U + \frac{J}{4}\right] N^2$

Fragmentation does not matter ??

2) Beyond mean field approximation

Different fluctuation spectra in coherent and incoherent case.

$N_\uparrow = N_\downarrow = \frac{N}{2}$

↗ quadratic spin waves in coherent ferromagnet $\omega \sim q^2$

↘ linear spin waves in incoherent paramagnet $\omega \sim q$

↓

Different ground state energies

⇓

GUESS

(i) The coherent state always wins for ferromagnetic exchange.

(ii) For antiferromagnetic J ?

(Tendancy to form singlet pairs

↳ pair condensation instead of particle condensation)

3) Return to mean field: anisotropic exchange.

$J_z \neq J_\perp$: splitting of $m = 0$ and $m = \pm 1$ triplet states

effective "anisotropy" field

$N_\uparrow = N_\downarrow$

Coherent state	Incoherent state
$E_0^{(c)} = \left(U + \frac{J_\perp}{4}\right)\frac{N^2}{2}$	$E_0^{(i)} = U\frac{N^2}{2} + \frac{J_z}{8}(N_\uparrow - N_\downarrow)^2 + \frac{J_\perp}{2} N_\uparrow N_\downarrow$

$$E_0^{(c)} - E_0^{(i)} = \frac{(J_z - J_\perp)}{4} N_\uparrow N_\downarrow \qquad \begin{cases} \text{pure state if } J_\perp > J_z & \text{(OK)} \\ \text{incoherent state if } J_\perp < J_z & \text{(?)} \end{cases}$$

Superfluidity vs localization

^{4}He becomes a solid above 26 bars !!

- The liquid optimizes the kinetic energy, but increases the potential energy → cost ~ nV
- The solid optimizes the potential energy (atoms avoid each other), but localization increases the kinetic energy → cost $\frac{\hbar^2}{md^2}$.

Transition when both terms are comparable

Cristallization if $nV > \frac{\hbar^2}{md^2}$ ⇒ $\boxed{\xi < d}$

Strong coupling destroys superfluidity !

⇓

- QUESTIONS :
  - 1st order transition ?
  - Intermediate range with a fully depleted condensate ? $N_0 = 0$
- Can a superconductor turn directly into an insulator ?

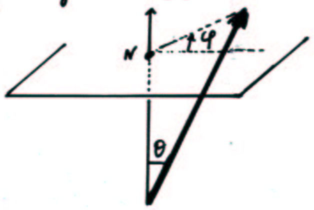
The lattice hard core Bose gas

$$H = -t\, b_i^* b_j + \frac{V}{2} n_i\, n_{i+\delta} \qquad \left[+U n_i(n_i - 1)\right] \quad U \to \infty$$

Isomorphic to spins 1/2 (MATSUBARA, 1969)

$$\begin{cases} n_i = 0 : & \downarrow \\ n_i = 1 : & \uparrow \end{cases} \qquad \begin{cases} b_i = S_i^- \\ n_i = \frac{1}{2} + S_{iz} \end{cases} \qquad \begin{cases} t : & J_\perp \text{ ferro} \\ V : & J_z \text{ antiferro} \end{cases}$$

The number of bosons is the magnetization $N_L/2$



- $V = 0$

Transverse ferromagnetism = superfluidity

$$\begin{cases} N/N_L = \cos^2\theta/2 \\ \phi = \sqrt{N_L} < b_i > = \sqrt{\frac{N(N_L - N)}{N_L}} \end{cases}$$

$\hookrightarrow \frac{\sin\theta}{2}$

(The holes are condensed if N close to N_L !)

- $V > t$, $N = N_L$

Longitudinal antiferromagnetism = crystal

↳ good description of ^{4}He phase diagram

"Supersolid" if $N \neq N_L$? LIU + FISHER J. Low. Temp. Phys. 10, 655 (1973)

Bose Hubbard model
Mott localization

$$H = -t\, b_i^* b_j + \frac{U}{2}\, n_i\,(n_i - 1)$$

Two parameters $\begin{cases} U/t \\ \text{filling } n = N/N_L \end{cases}$

- Mean field approximation

$$b_i \rightarrow \bar{b}_i + \delta b_i$$

(i) Linearization of Interaction

↳ Bogolubov : no information on commensurability N/N_L

(ii) linearization of Kinetic energy

$$b_i^* b_j \rightarrow b_i^* \phi + \phi^* b_j - \phi^* \phi$$

single site problem, equivalent to a Gutzwiller approximation

Mean field

SHESHADRI, KRISHNAMURTHY, PANDIT, RAMAKRISHNAN
Europhys. Lett. 22, 257 (1993)

KRAUTH, CAFFAREL, BOUCHAUD
Phys Rev B45, 3137 (1992) ≠ Gutzwiller

Set $z\,t = 1$: $H = \frac{U}{2}\, n_i\,(n_i - 1) - \mu\, n_i - \phi(a_i + a_i^*) + \phi^2$

(i) Numerical diagonalization with $n_i < n_{max}$ $(= 2^{4}$ when $U \to \infty)$
(ii) Minimization with respect to ϕ
(iii) Adjustment of $\mu \rightarrow N$

Superfluid $\phi \neq 0$
$\rho_s = |\phi|^2$

localized solid $\phi = 0$
↓
Gap Δ in elementary excitations

$N = N_L$ localization if $U > U_c$

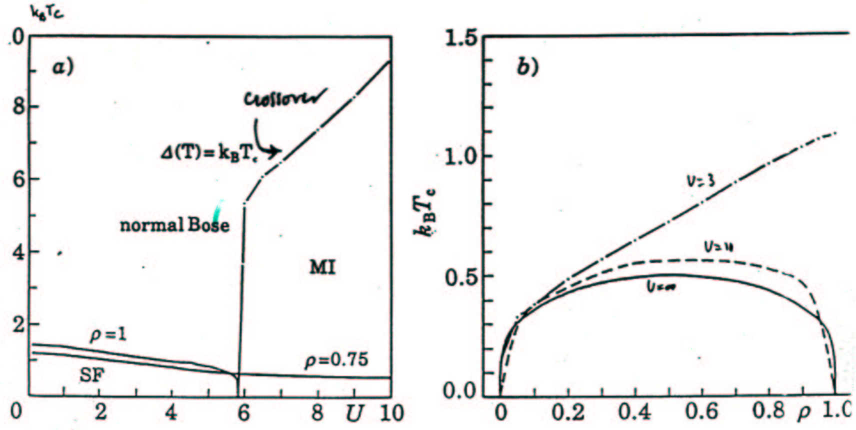
"Bose" Mott transition

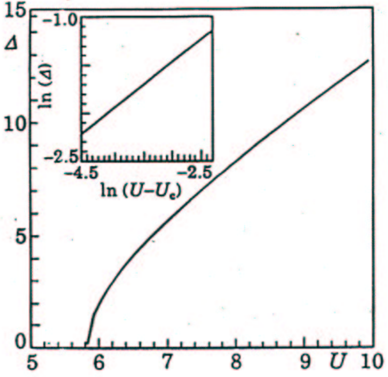
Superfluid T_c if $U < U_c$

Crossover if $U > U_c$

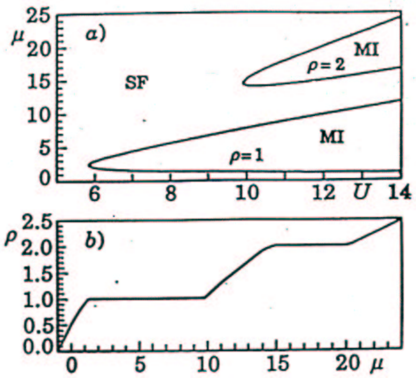
$N < N_L$ The ground state is always superfluid

(cf. metal-insulator transition)





$\Delta \sim \sqrt{U-U_c}$



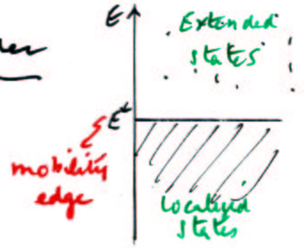
"Pockets" of insulator in the (μ, U) plane

(cf liquid-gas transition)

Anderson localization via disorder

$H = -t\, b_i^* b_j + W_i\, b_i^* b_i$

random

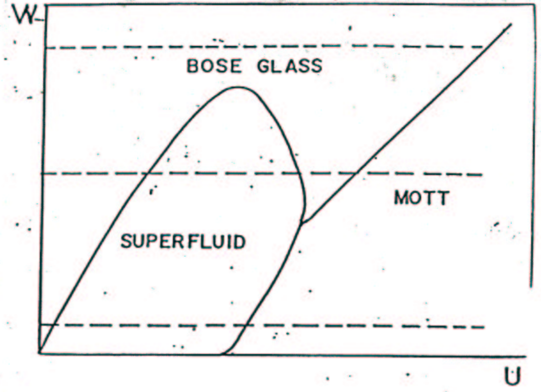


- localized state if $\mu < E^*$: Bose glass
- But for an ideal gas μ is at the lowest state at $T = 0$

  ↳ need of a repulsion U in order to push μ above E^*

Two parameters

$$\begin{cases} U \\ W = \sqrt{\overline{W_i^2}} \end{cases}$$



Renormalization FISHER et.al. Phys.Rev B40, 546 (1989)

Numerical TRIVEDI, J. Low Temp. Phys. 89 67 (1992)

(Competition with Mott localization when $N = N_L$)

- Applications
  - ^{4}He in porous media (disorder)
  - Granular superconductors (Mott)

    $t =$ Josephson coupling between grains

    $U =$ capacitive energy

    ↳ U/t is controlled by grain radius R

Another enemy of superfluidity : DISSOCIATION of composite bosons

Bose condensate ⇒ normal Fermi liquid

- Singlet bound pair of fermions = "boson"

$$b_q^* = \sum_k \psi_k \, C^*_{k+q\uparrow} C^*_{k\downarrow}$$

Total momentum

Internal wave function (even)

↓

Bose Einstein condensation

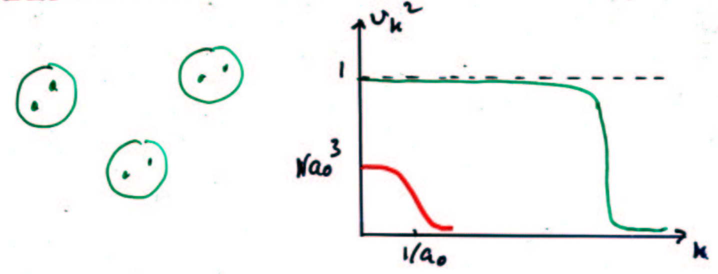
$$\exp[\phi b_0^*]|vac\rangle = \exp\Big[\sum_k \phi\psi_k C^*_{k\uparrow} C^*_{-k\downarrow}\Big]|vac\rangle$$

$$= \prod_k \big[u_k + v_k C^*_{k\uparrow} C^*_{-k\downarrow}\big]|vac\rangle$$

BCS wave function with $\frac{v_k}{u_k} = \phi\psi_k$, $\sum_k v_k^2 = N$

Superconductivity = Bose Einstein condensation of fermion pairs !

- Dilute limit $Na_0^3 \ll 1$ ⇒ "atome" regime



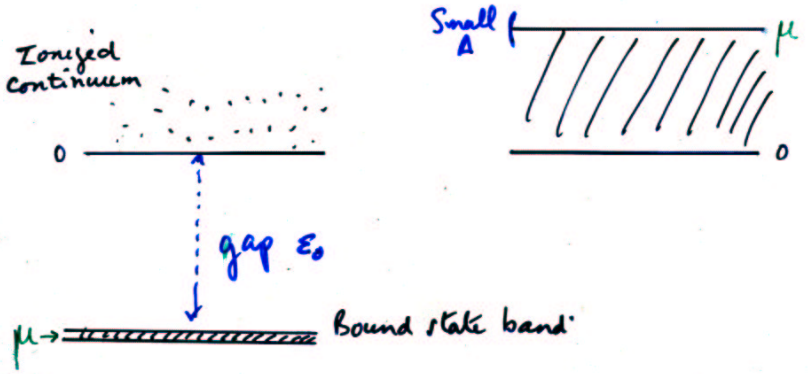
- Dense limit

Exclusion principle : $v_k^2 < 1$

n_k extends to $k \gg 1/a_0$

↓

Degenerate Fermi liquid with small BCS gap



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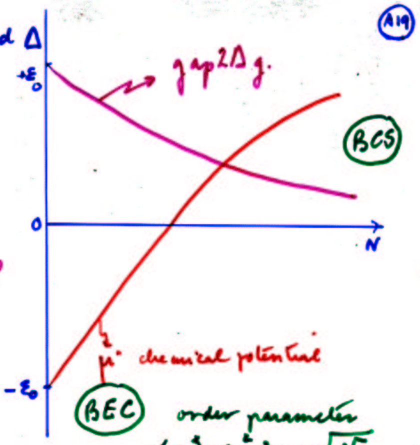
- Continuous evolution of μ and Δ at zero temperature



Excitation gap Δ_g $=\begin{cases}\Delta \text{ if } \mu>0 \\ \quad\text{(in the band)} \\ \sqrt{\mu^2+\Delta^2} \text{ if } \mu<0\end{cases}$

order parameter $\langle C^+_{k\uparrow} C^+_{-k\downarrow}\rangle \sim \sqrt{N}$

- But the physics of the critical temperature is different in the two limits

Dense case	Dilute case
$\mu>0,\ \Delta \ll E_F$	$\mu<0,\ E_F \ll \Delta_g$.
Pair breaking, well described by BCS	Center of mass motion of bound pairs
$\Downarrow$	$\Downarrow$
$T_c \sim \Delta = \Delta_g$.	$T_c \sim \dfrac{\hbar^2}{md^2} \ll \Delta_g$.
One energy scale	Two energy scales

QUESTION : where is the crossover?

$\mu = 0$? $\qquad T_c = T_c^{BCS}$?

Ⓐ20

- Phase fluctuations : BCS vs Kosterlitz - Thouless

Center of mass motion : $\Delta e^{i\vec{q}.\vec{r}}$ → phase fluctuations

$\Downarrow$

macroscopic picture : $\Delta e^{i\phi(r)}$ → $E = \frac{1}{2}\int d\tau\, \underset{\text{phase stiffness}}{\Lambda} |\text{grad}\,\phi|^2$.

Thermal fluctuations of ϕ → T_c when $\overline{|\delta\phi|} \sim 2\pi$

- "Exact" calculation in 2 dimensions by Kosterlitz - Thouless } → universal result $\boxed{T_c^{KT} = \frac{\pi}{2}\Lambda}$

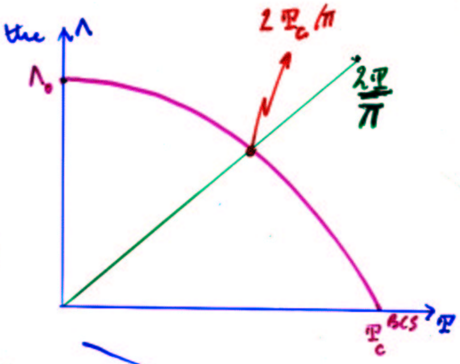
Superconductivity is destroyed when (+,-) vortex pairs unbind → phase slippage

- Actually, Λ depends on the temperature because of pair breaking (BCS)

$\Downarrow$

Implicit equation for T_c

$T_c = \frac{\pi}{2}\Lambda(T_c)$



Small Λ_0	Large Λ_0
$T_c = T_c^{KT}$: fluctuation dominated $\ll T_c^{BCS}$	$T_c \approx T_c^{BCS}$ (narrow fluctuation dominated critical regime)

(A2i)

- The crossover corresponds to $T_{c0}^{KT} \sim T_c^{BCS}$

↳ calculation of Λ_0 ?

Straightforward for free electrons : Δe^{iqr} means a shift of the Fermi surface by $q/2$

$$\Delta E = \underset{\substack{\wr \\ 2 \cdot \frac{\pi k_F^2}{4\pi^2}}}{N} \frac{\hbar^2}{2m} \frac{q^2}{4} = \frac{\Lambda q^2}{2} \Rightarrow \boxed{\Lambda = \frac{\varepsilon_F}{4\pi}}$$

⇓

crossover

of $\Delta \sim 0{,}2\ \varepsilon_F$

The chemical potential has hardly moved,

and the idea of preformed bosons makes no sense!

Conclusion : there is a broad range in which

$T_c \ll T_c^{BCS}$ is dominated by fluctuations.

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NO RELIABLE THEORY

How can one have bound pairs inside the continuum as an explanation of the pseudo gap?

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③

Another possibility

- {A semiconducting gap opens due to some other mechanism
  Superconductivity develops on top of that gap

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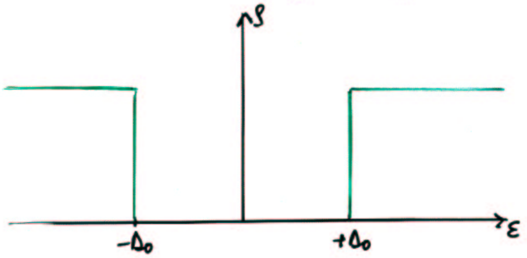
{Simple "2d" model for the density of states



What is the effect of an effective attraction between 2 electrons (or 2 holes) ?

- The origin of the gap Δ_0 is left open
  - Magnetic SDW (FRIEDEL) ?
  - Lattice distortion ?

OUR AIM : Understand the qualitative physics of that model!

Characteristic energies

- Band width ω_m : $\gg$ everything else
- Semi conducting gap. $2\Delta_0$
- Interaction strength. $-U n_\uparrow n_\downarrow$

  ↳ would give a superconducting gap Δ_m if Δ_0 were zero

$$1 = \frac{\rho U}{2} \int_{-\omega_m}^{+\omega_m} \frac{d\xi}{\sqrt{\xi^2+\Delta_m^2}}$$

Dimensionless parameter Δ_0/Δ_m : in an undoped insulator, can an attraction produce carriers across the gap → superconductivity

- Doping : $\delta N = \rho\,\varepsilon$ excess carriers in the conduction band: at $T=0$ and with no interaction μ jumps from 0 to $(\Delta_0+\varepsilon)$.

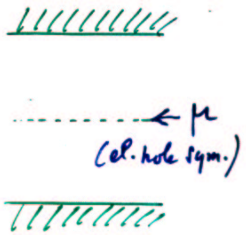
  Superconductivity is always present: how does it evolve?

  ↳ Second dimensionless parameter $\frac{\varepsilon}{\Delta_m}$.

Undoped system $(\varepsilon = 0)$

- If the ground state is superconducting, its order parameter Δ obeys

$$\Delta = \rho U \int_{\Delta_0}^{\omega_m} \frac{\Delta}{\sqrt{\Delta^2+\xi^2}}\, d\xi$$



$$\Downarrow$$

$$\Delta_m = \Delta_0 + \sqrt{\Delta_0^2+\Delta^2}$$

$$\downarrow$$

$$\Delta = \sqrt{\Delta_m(\Delta_m - 2\Delta_0)}$$

Direct transition from superconductor to insulator at $\Delta_0 = \Delta_0^* = \Delta_m/2$

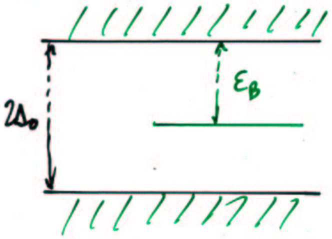
When $\Delta_m > 2\Delta_0$, gain in superc. condens. energy. > cost in producing carriers

- Quasiparticle gap : $\Delta_g = \sqrt{\Delta_0^2+\Delta^2} = \Delta_m - \Delta_0$

  At the transition, Δ_g joins with the semiconducting gap Δ_0 (measured from μ)

(57)

- Analogy with Kohn's "excitonic insulators"



If $\varepsilon_B > 2\Delta_0$, excitons appear spontaneously

⇓

Bose condensation at $T=0$

"Free" carriers are triggered by interactions

Here, bound electron-hole pairs (U repulsive)

↓

bound electron-electron pairs (U attractive)

Excitonic insulator ($\overline{a^* b} \neq 0$)

⇓

Superconductor ($\overline{a^* a^*}$, $\overline{b^* b^*}$) finite

- Question

Should one expect "inter band pairing" $\overline{a^* b^*} \neq 0$

(Single pairs do not respect time reversal symmetry)

Y. MORITA, Y. HATSUGAI, K. KOHMOTO, Phys. Rev. B53, 8561 (1996)

↳ (?)

(A26)

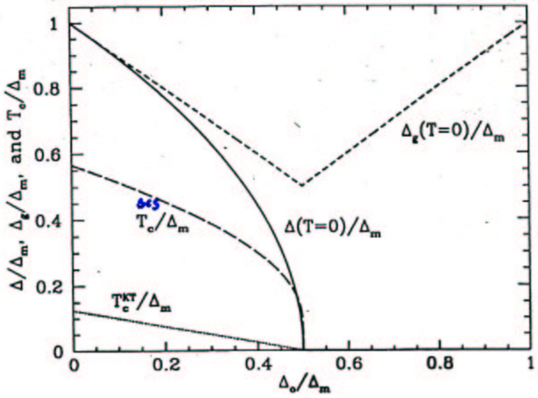
Critical temperature

(i) BCS

$T_c \ll \Delta_g$ near the transition!

(ii) Kosterlitz-Thouless

Always $< T_c^{BCS}$



Phase fluctuations are always the dominant mechanism!!

WHY: The phase stiffness would be $\equiv 0$ without superconductivity!

Translating a filled band by $\frac{q}{2}$ is doing nothing!

Consequence: only superconducting fluctuations are sensitive to superflow ↳ $\Lambda \to 0$ as $\Delta \to 0$

Exact calculation

$$\Lambda = \frac{1}{4\pi}\int d\xi \left[1-\frac{\xi}{\sqrt{\xi^2+\Delta^2}}\right] \rightarrow T_c^{KT} = \frac{\Delta_0^* - \Delta_0}{4}$$

QUESTION: is it a general feature that a system close to being an insulator has a small phase stiffness (?)