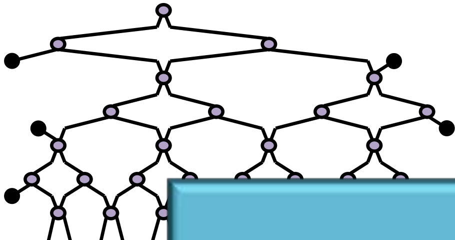
KITP, December 7th 2010

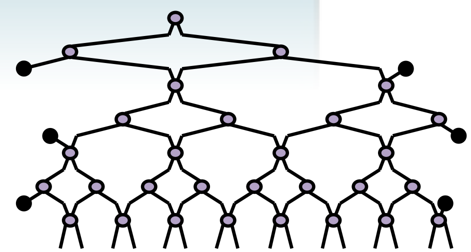


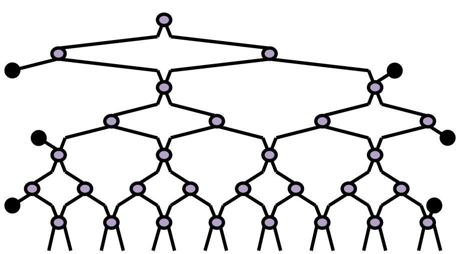
Holographic Branching and Entanglement Renormalization

Glen Evenbly

Guifre Vidal

The University of Queensland
Australia





Tensor Network Methods

(DMRG, PEPS, TERG, MERA)

Potentially offer **general formalism** to efficiently describe many-body wave functions

- ground states of large systems in D=1,2 (maybe 3) spatial dimensions
- strong or weak interactions, frustrated interactions etc
- different particle statistics (e.g. spins/bosons, fermions, or even anyons),

Only limited by the amount of entanglement in the state!

As Numerical Methods:

- Given Hamiltonian H what are properties x,y,z of the ground state?

Conceptual Aspects:

- Framework for describing many-body systems - entanglement structure!

boundary law for entanglement entropy scaling:

$$\boxed{S_L \propto L^{D-1}}$$

branching MERA

(Evenbly, Vidal, in preparation)

- generalization of the MERA to more complex **holographic geometries** in order to produce **violations of the boundary law** for entanglement entropy scaling

  - e.g. 2D branching MERA with entanglement: $\boxed{S_L \propto L \log L}$

  an ansatz for 2D systems with a 1D Fermi surface?

- we propose to use **holographic geometry** to **classify entanglement** in many-body ground states

  - geometric understanding of violations to the boundary law

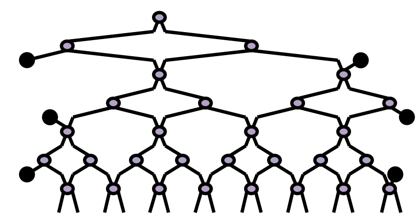
- new notions of **RG flow** / **RG fixed points**

  - 2D fermions (with 1D fermi surface) as RG fixed points

Outline

- **Entanglement and tensor network methods**
  - Scaling of entanglement entropy in ground states
  - Scaling of entanglement entropy in tensor network ansatz
    - physical geometry vs holographic geometry
  - Comparison of entropy scaling:
    - ground states vs tensor network ansatz

- The branching MERA
  - Decoupling a many-body theory
  - Holographic trees
  - Scaling of entropy in the branching MERA
  - Example: $S_L = L \log L$ entropy scaling in 2D fermions
  - Example: $S_L = (\log L)^2$ entropy scaling in 1D fermions



- Work in progress...
  - **Classification** of many-body entanglement via **holographic geometry**
  - Generalised **RG flow / RG fixed points**

Scaling of entanglement entropy for free fermions

1D

	Gap.	Crit.
S_L	const.	$\log(L)$

2D

	Gap.	Crit.I	Crit.II
S_L	L	L	$L\log(L)$

1D

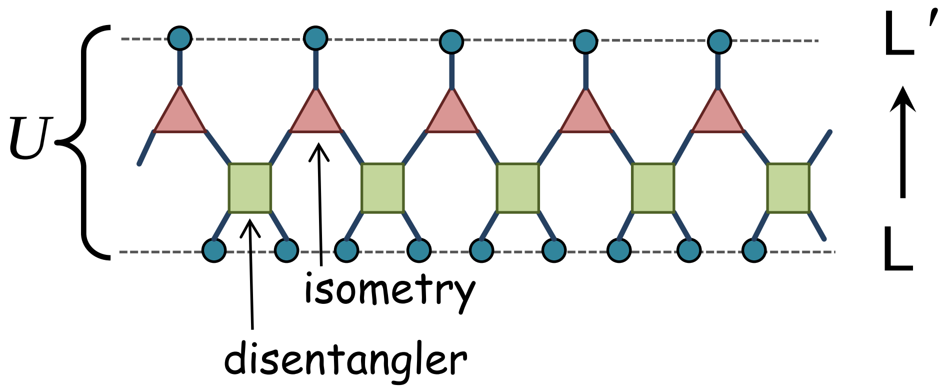
Vidal, Latorre, Rico, Kitaev, PRL 2003
Srednicki, PRL 1993
Callan, Wilczek, Phys Lett B 1994.
Fiola, Preskill, Strominger,Trivedi, PRD 1994.
Holzhey, Larsen,Wilczek, Nucl.Phys.B 1994.
Jin, Korepin, J. Stat. Phys. 2004
Calabrese, Cardy, J. Stat. Mech. 2004

2D

Wolf, PRL 2006.
Gioev, Klich, PRL 2006.
Barthel, Chung, Schollwock, PRA 2006.
Li, Ding, Yu, Haas, PRB 2006.
Ding, Bray-Ali, Yu, Haas, PRL 2008.
Helling, Leschke, Spitzer 2009, arXiv:0906.4946.
Swingle 2009, arXiv:0908.1724.

- Can Tensor Network methods reproduce the appropriate entanglement entropy?

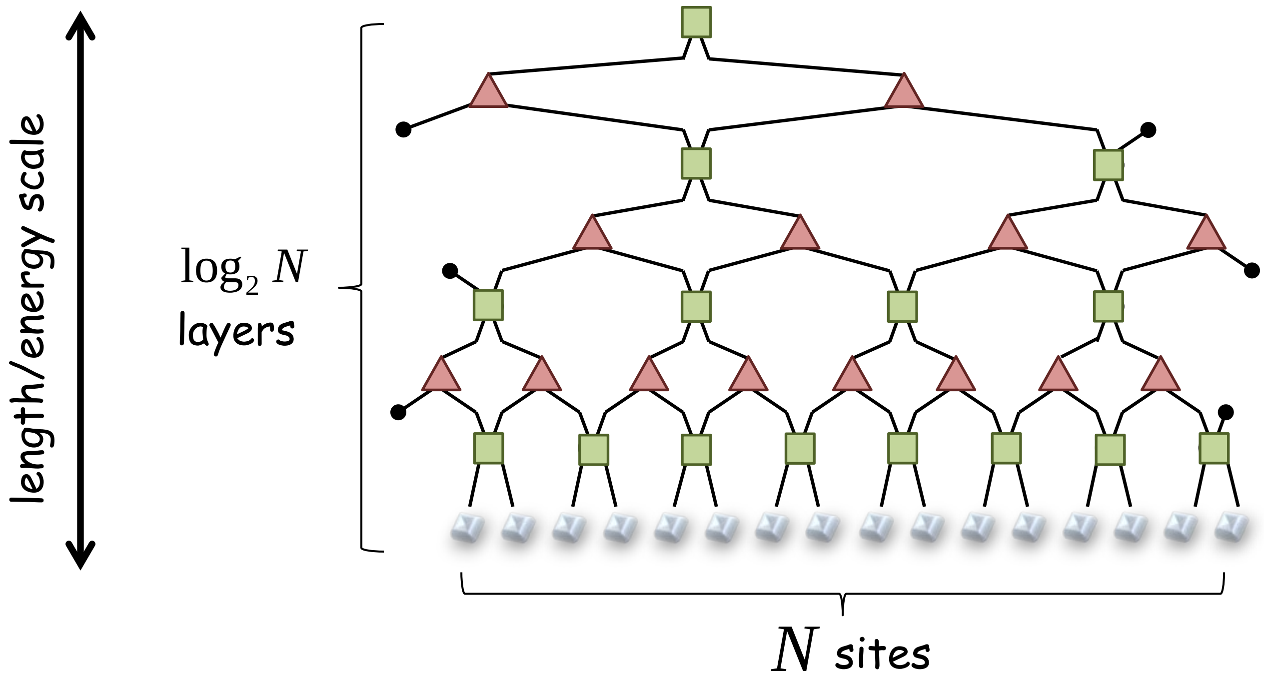
Entanglement Renormalization and the MERA



Coarse-graining transformation:

Entanglement Renormalization

MERA (multi-scale entanglement renormalization ansatz)



MERA ↔ Holography

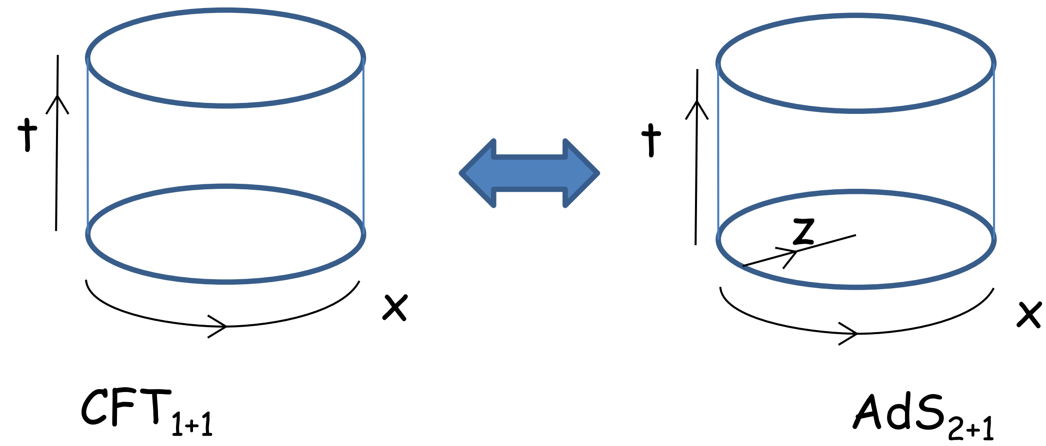
Brian Swingle

arXiv:0905.1317

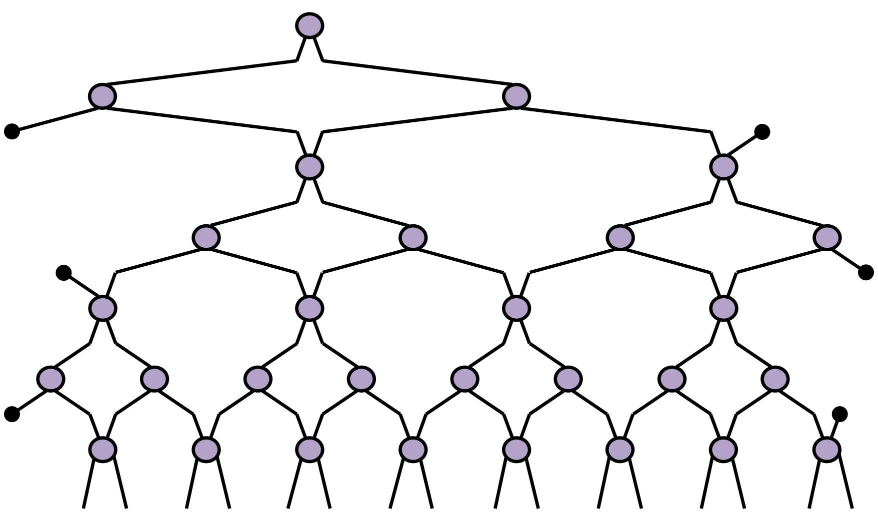
Holographic geometry

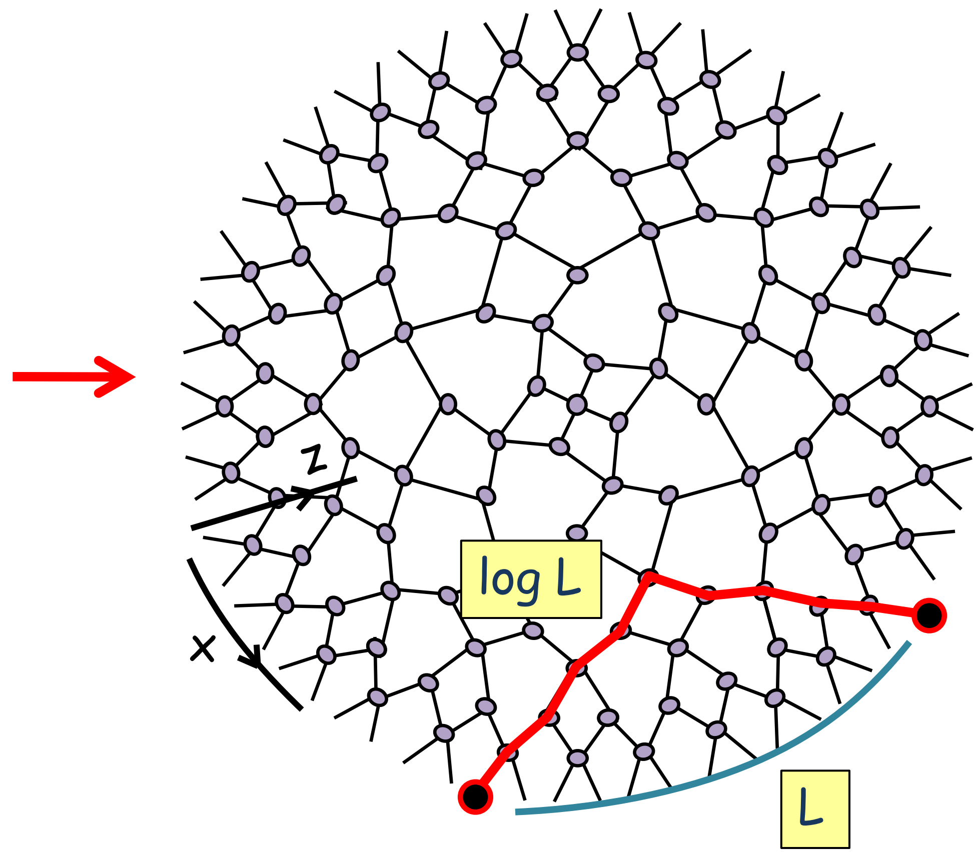
- Reproduce the pattern of entanglement in the ground state

AdS/CFT Correspondance



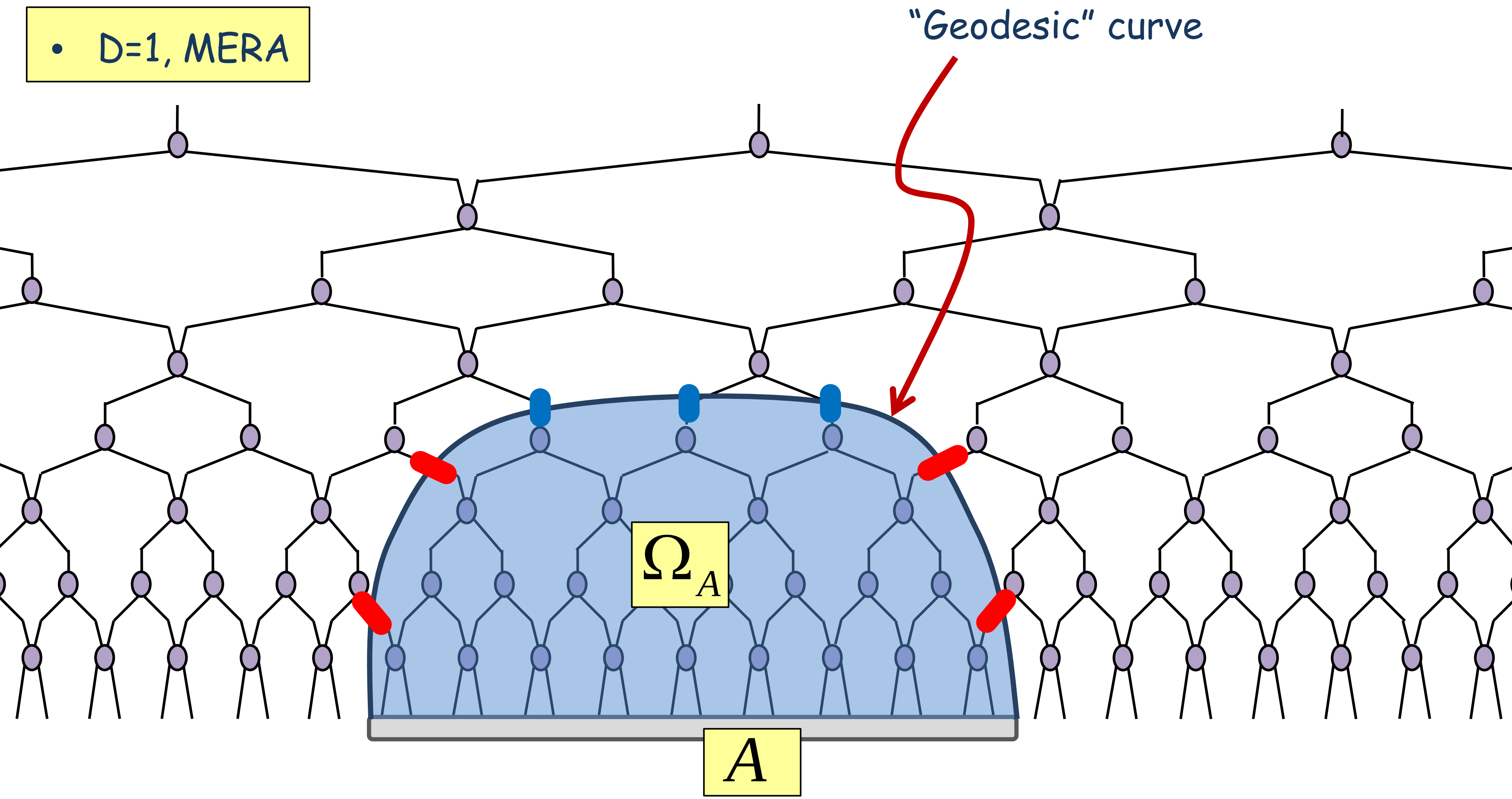
1D MERA





Computation of entanglement entropy

- D=1, MERA

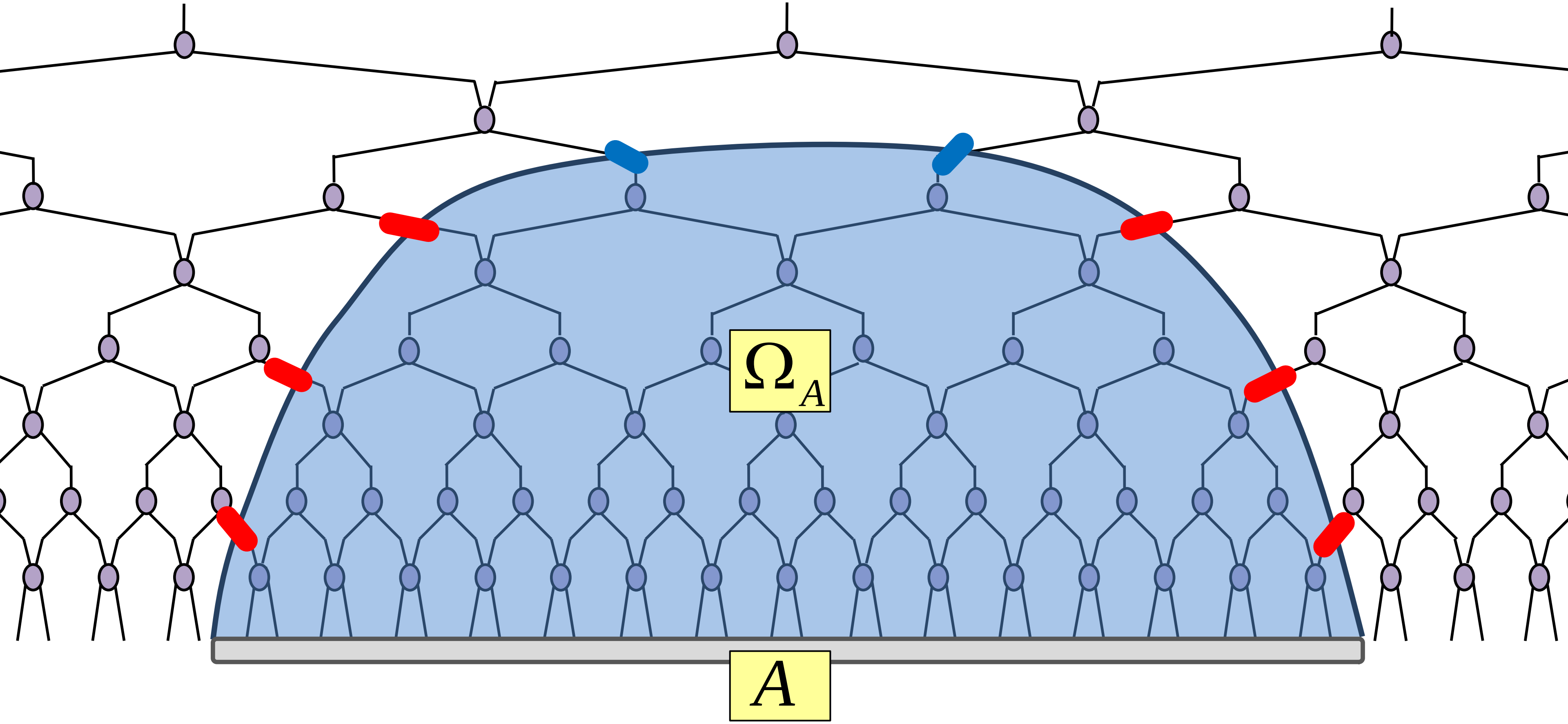


- Entanglement entropy as boundary in holographic geometry: $S(A) \sim |\partial \Omega_A|$

Holographic Entanglement Entropy, e.g. Nishioka, Ryu, Takayanagi, J. Phys. A 2009

Computation of entanglement entropy

- D=1, MERA



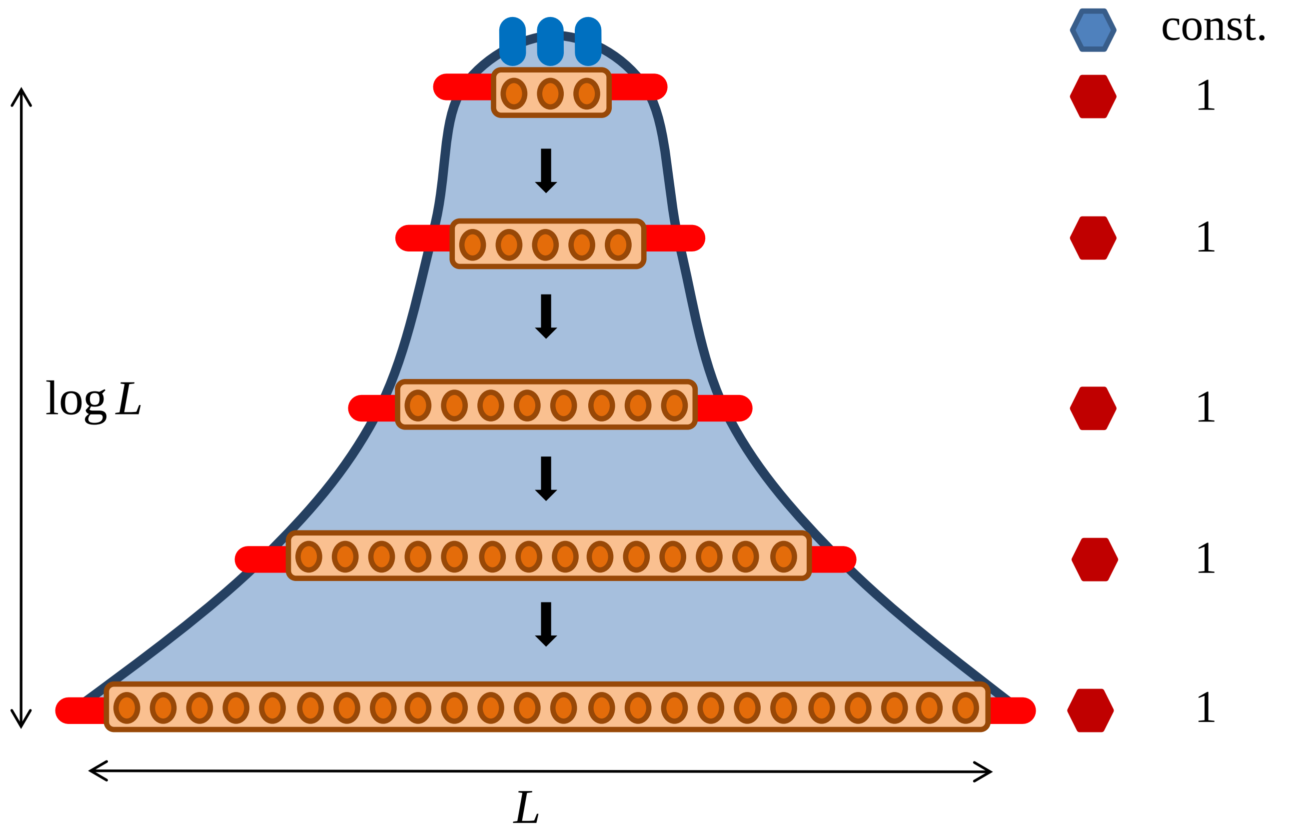
- Entanglement entropy as boundary in holographic geometry: $S(A) \propto |\partial\Omega_A|$

Holographic Entanglement Entropy, e.g. Nishioka, Ryu, Takayanagi, J. Phys. A 2009

MERA for D=1 spatial dimensions

- Entanglement entropy as boundary in holographic geometry: $\boxed{S(A) \sim \left|\partial\Omega_A\right|}$

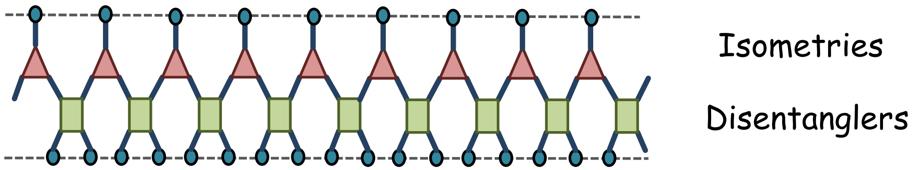
contributions to entropy



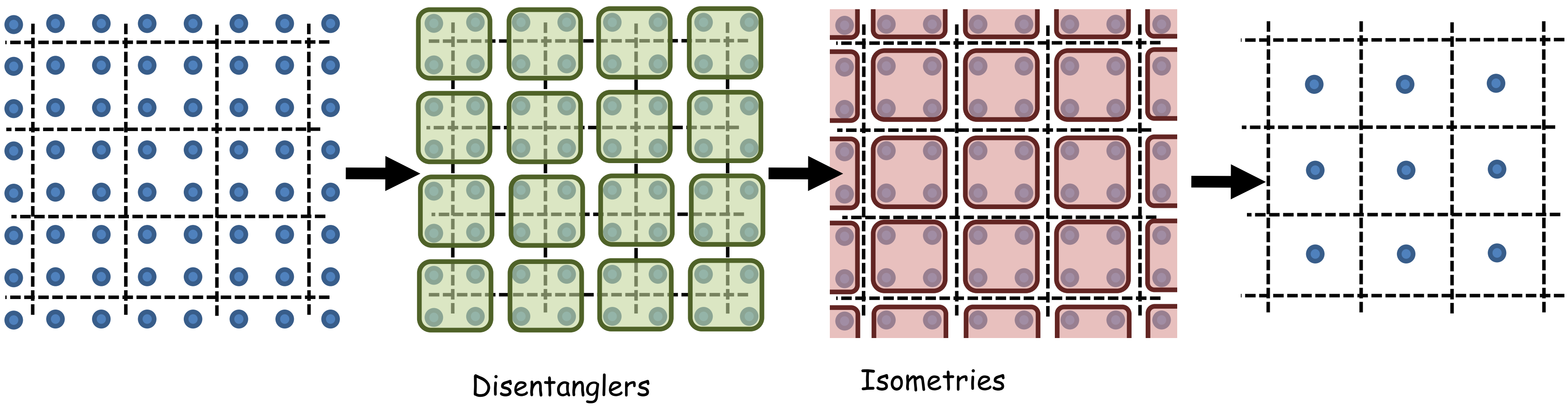
- Tensor network (MERA) based on holographic geometry can produce violations of boundary law, logarithmic violation: $\boxed{S(L) \approx \log L}$

MERA for D=2 spatial dimensions

D=1 spatial dimensions



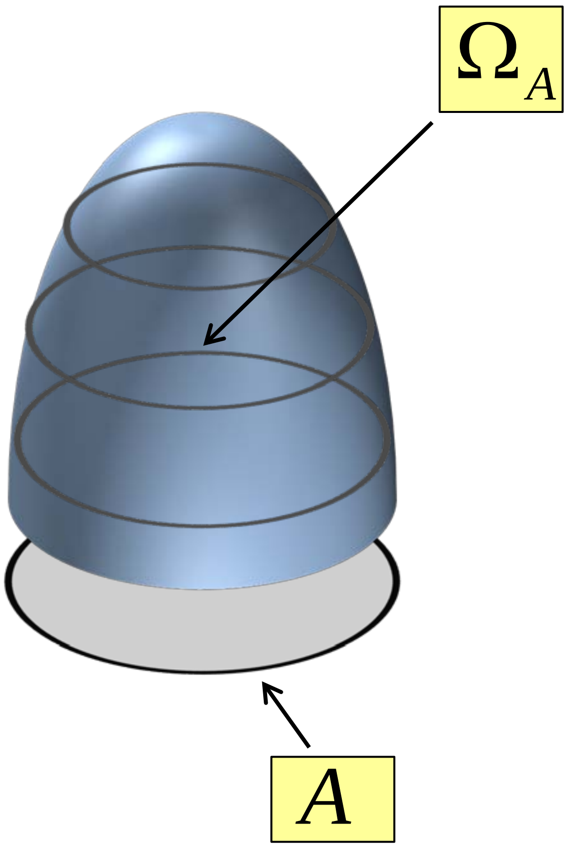
D=2 spatial dimensions

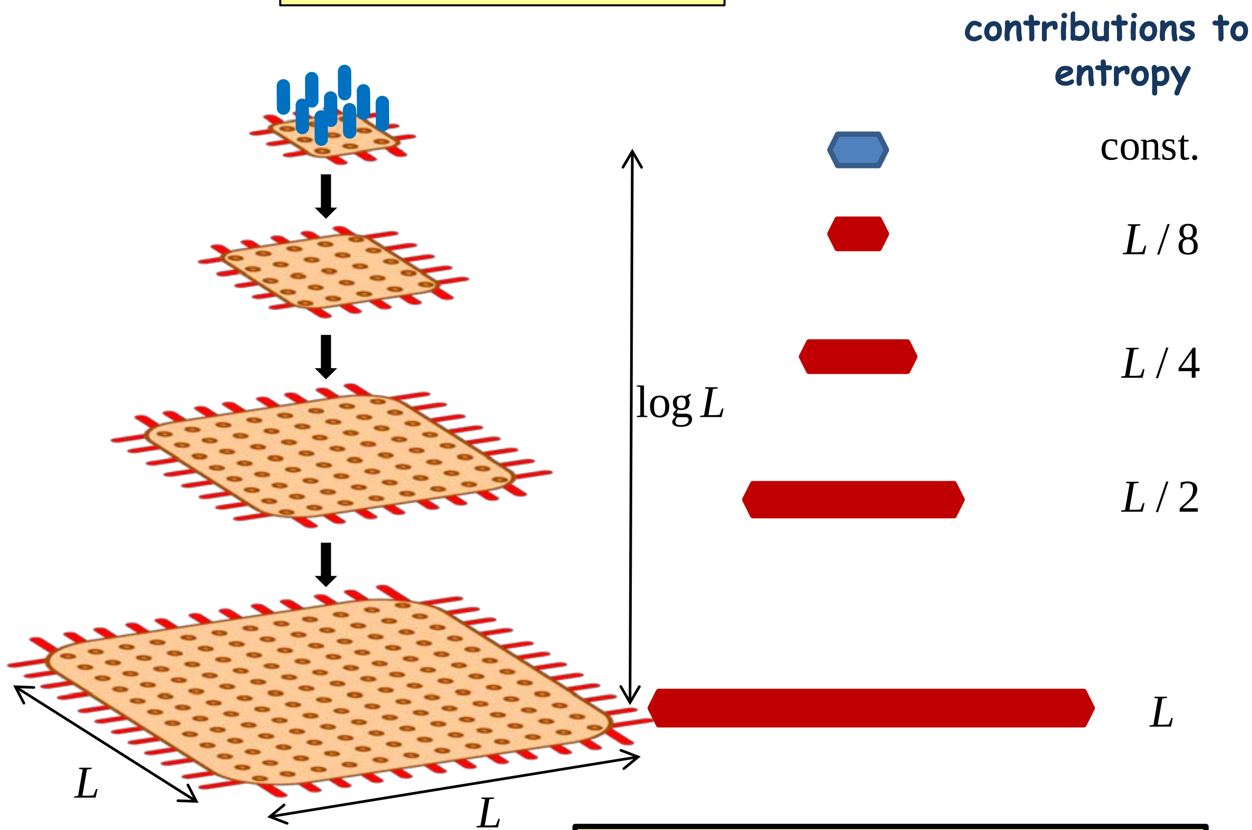


MERA for D=2 spatial dimensions

- Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim \left|\partial\Omega_A\right|$$





$$S_L \approx L\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \approx L$$

boundary law for entropy scaling!

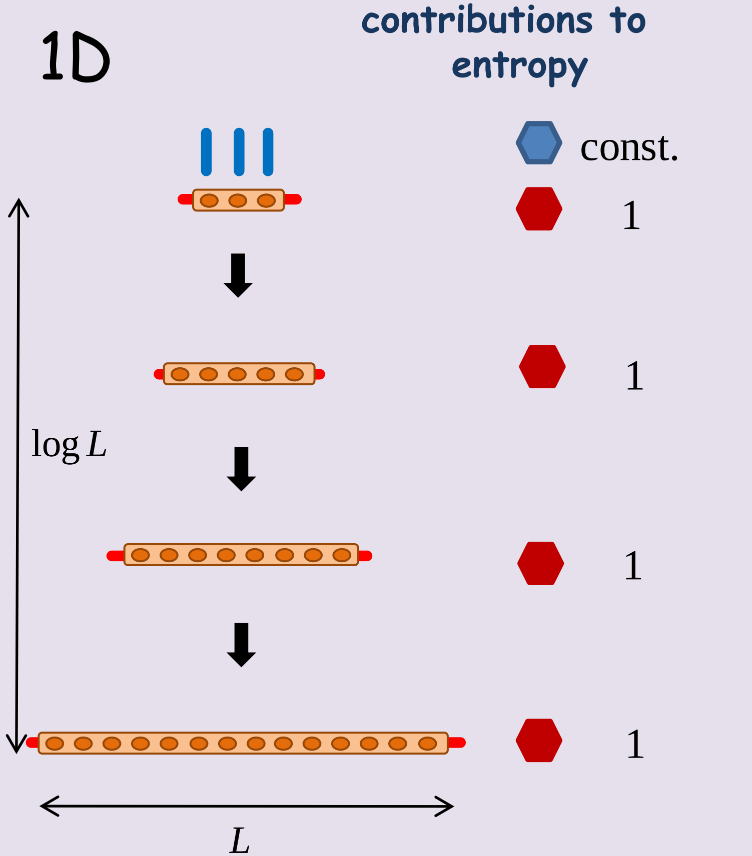
Entanglement entropy in the MERA

Vidal, quant-ph/0610099

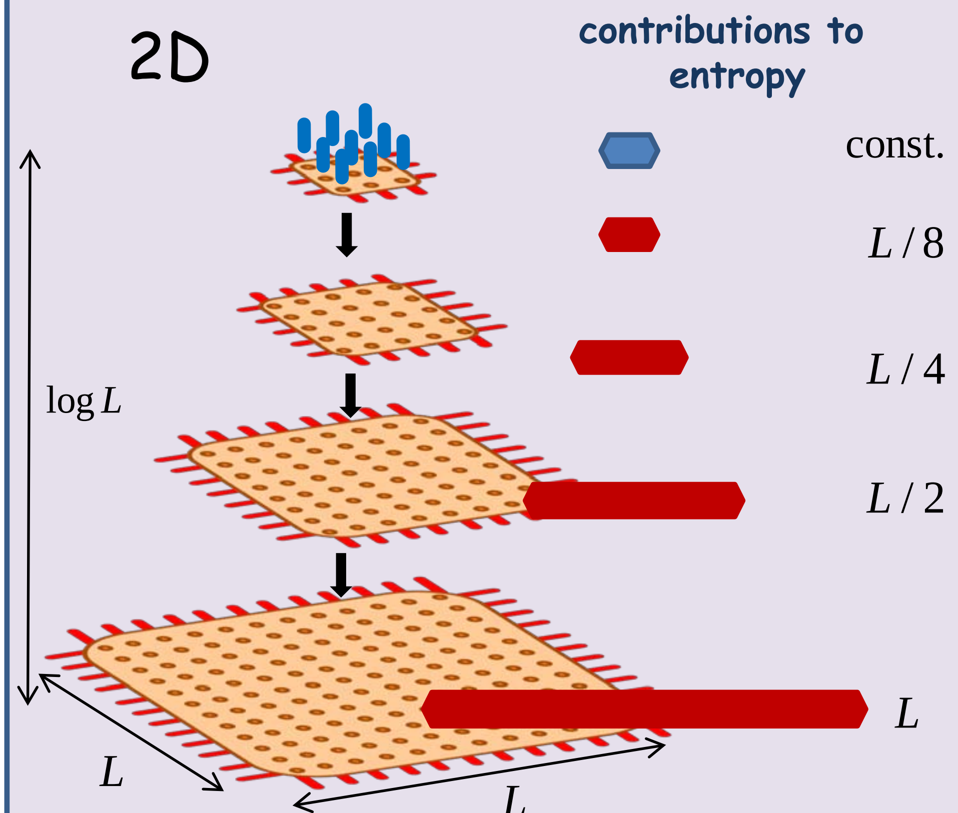
left out of PRL 101, 110501 (2008) !!!

- Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim \left|\partial\Omega_A\right|$$



$$S_L \approx \underbrace{(1+1+\cdots+1)}_{\log L} \approx \log L$$



$$S_L \approx L\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \approx L$$

Entanglement entropy and Tensor Networks

- Can Tensor Network methods reproduce the proper entanglement entropy?

Tensor networks in physical geometry:

1D	MPS:	$S_L = \text{const.}$
2D	PEPS:	$S_L = L$

	Gap.	Crit.	
1D S_L	const.	$\log(L)$	

	Gap.	Crit.I	Crit.II
2D S_L	L	L	$L\log(L)$

- First and second order phase transitions
- Frustrated Magnets
- Interacting Fermions

All on large (or infinite) 2D lattices!

Entanglement entropy and Tensor Networks

- Can Tensor Network methods reproduce the proper entanglement entropy?

Tensor networks in holographic geometry:

1D MERA: $S_L = \log L$

2D MERA: $S_L = L$

	Gap.	Crit.	
1D S_L	const.	$\log(L)$	

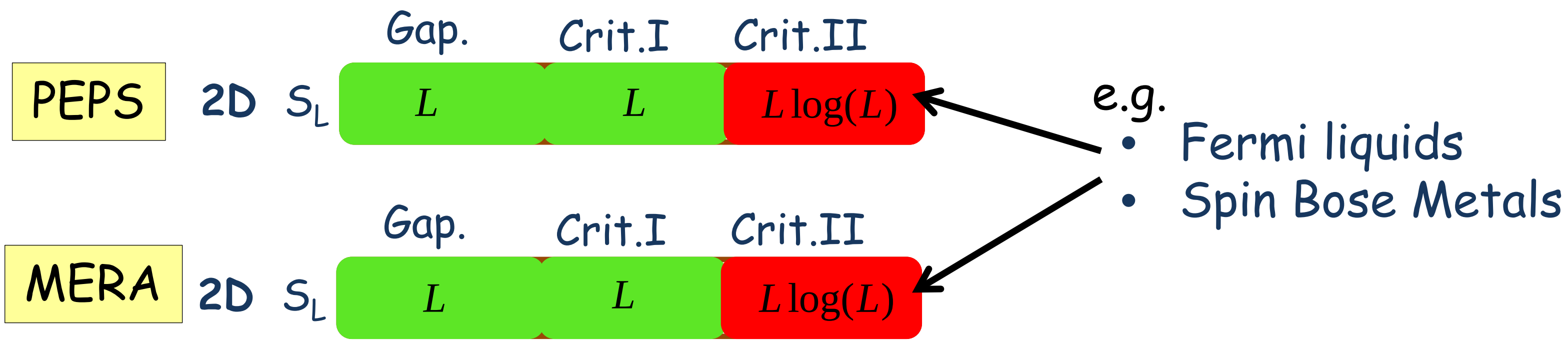
	Gap.	Crit.I	Crit.II
2D S_L	L	L	$L\log(L)$

- Natural description of scale-invariant 1D systems

- First and second order phase transitions
- Frustrated Magnets
- Interacting Fermions

All on large (or infinite) 2D lattices!

Entanglement entropy and Tensor Networks



- Certain types of critical 2D phases cannot be properly addressed (even in principle) with current tensor-network techniques

- The **entanglement structure** of these systems is not properly understood

- The Crit.II phases are not fixed points of the **RG flow**

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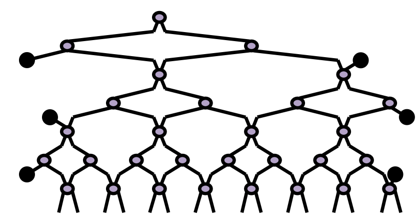


- Work in progress...
  - **Classification** of many-body entanglement via **holographic geometry**
  - Generalised **RG flow / RG fixed points**

Entanglement entropy and Tensor Networks

- Entanglement entropy in MERA as boundary in holographic geometry:

$$\boxed{S(A) \sim \left|\partial\Omega_A\right|}$$

- By considering exotic holographic geometries we obtain a more general class of MERA that reproduces more entanglement entropy

➡ Branching MERA

Evenbly, Vidal, in preparation

		Gap.	Crit.I	Crit.II
2D	S_L	L	L	$L\log(L)$

- Potentially a good ansatz for 2D systems with a 1D Fermi surface?

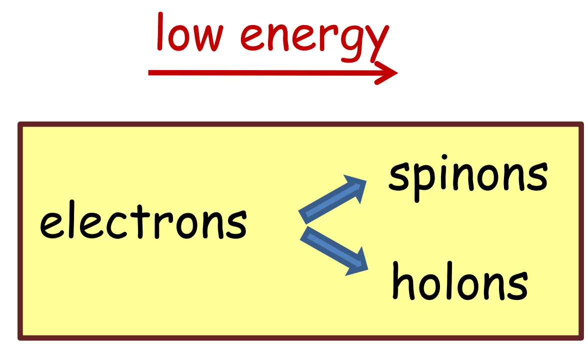
Low energy decoupling and holographic branching

MOTIVATION:

at low energies, sometimes sets of degrees of freedom decouple

Examples:

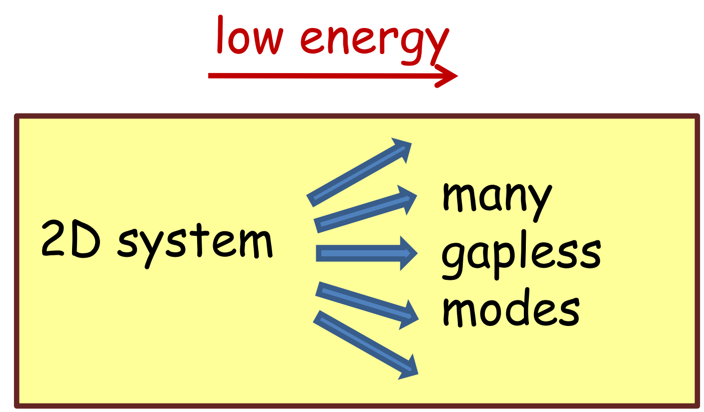
- 1D system: spin-charge separation



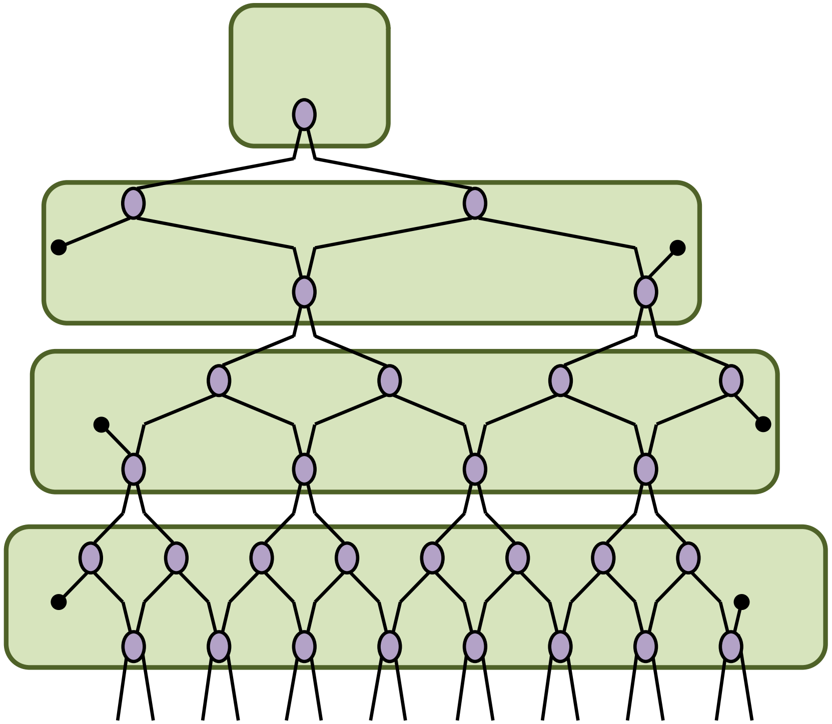
- 2D systems with 1D Fermi surface (or 1D Bose surface)

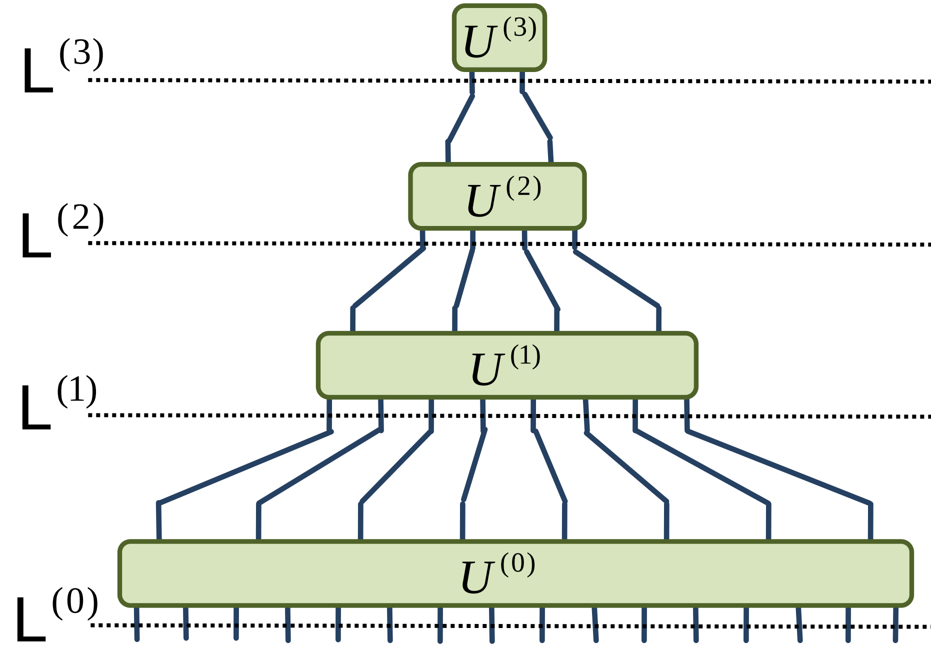
  - free fermions
  - Fermi liquids
  - spin Bose metal

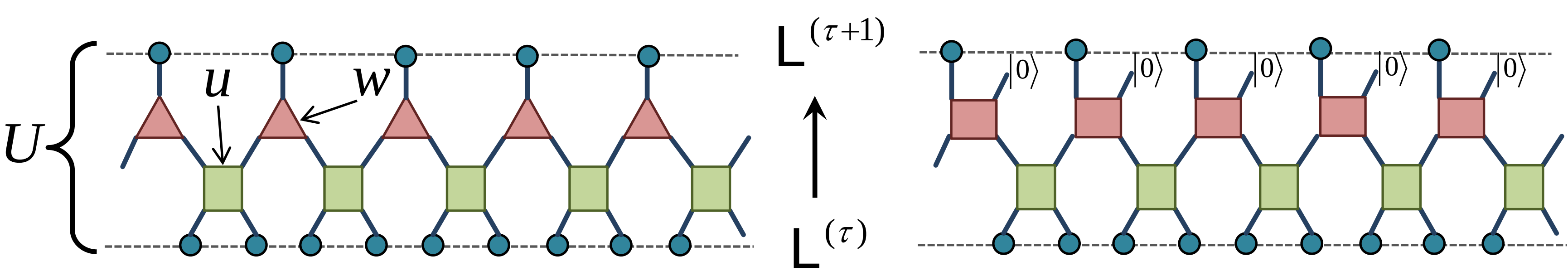
  e.g. Block, Sheng, Motrunich, Fisher, arXiv:1009.1179.
  Sheng, Motrunich, Fisher, arXiv:0902.4210.



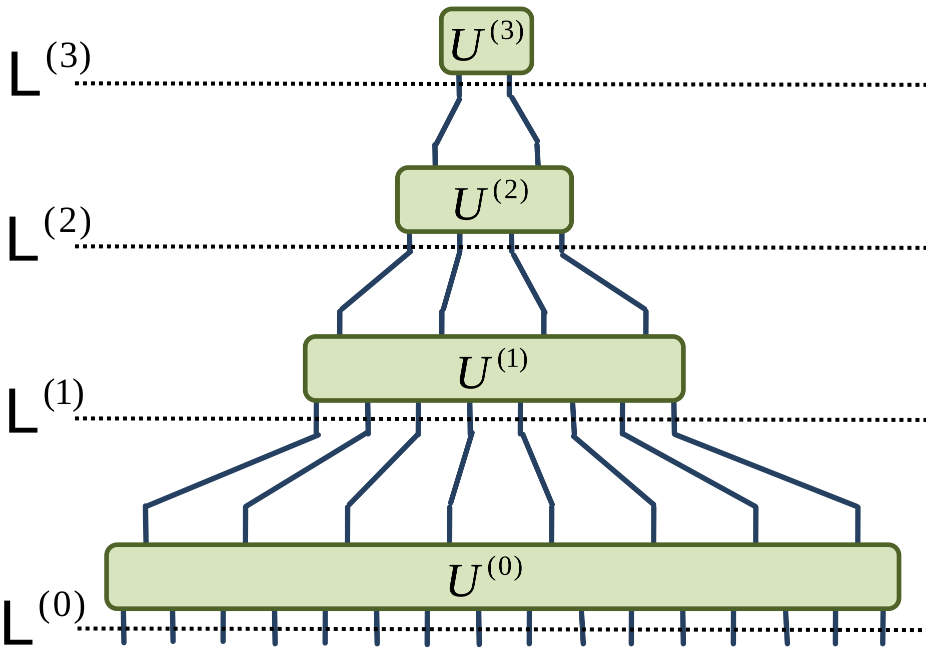
- simplified diagrammatic representation for the MERA



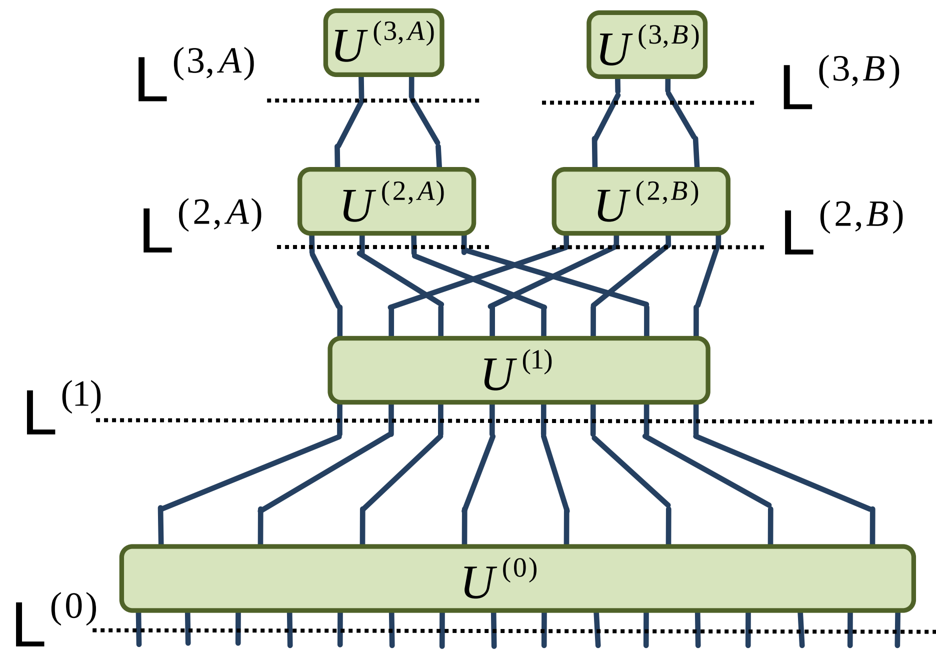


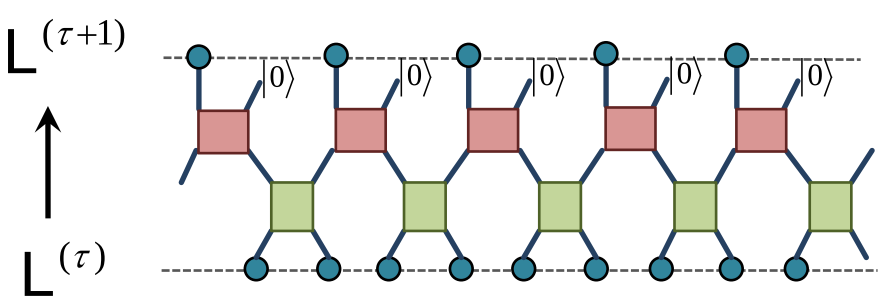


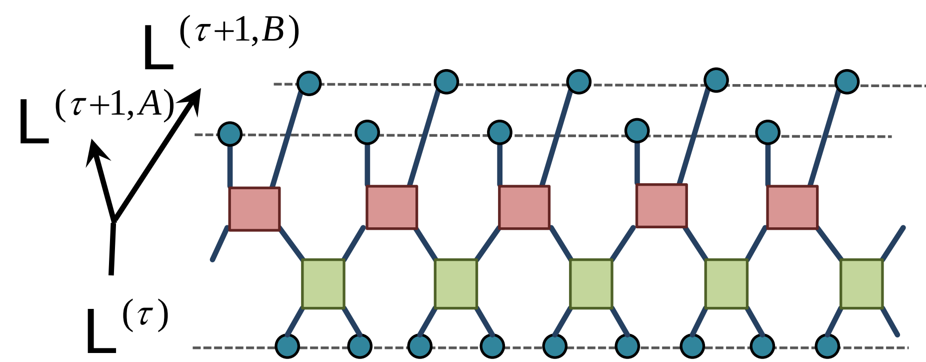
- Entanglement renormalization in the presence of decoupling



branching MERA



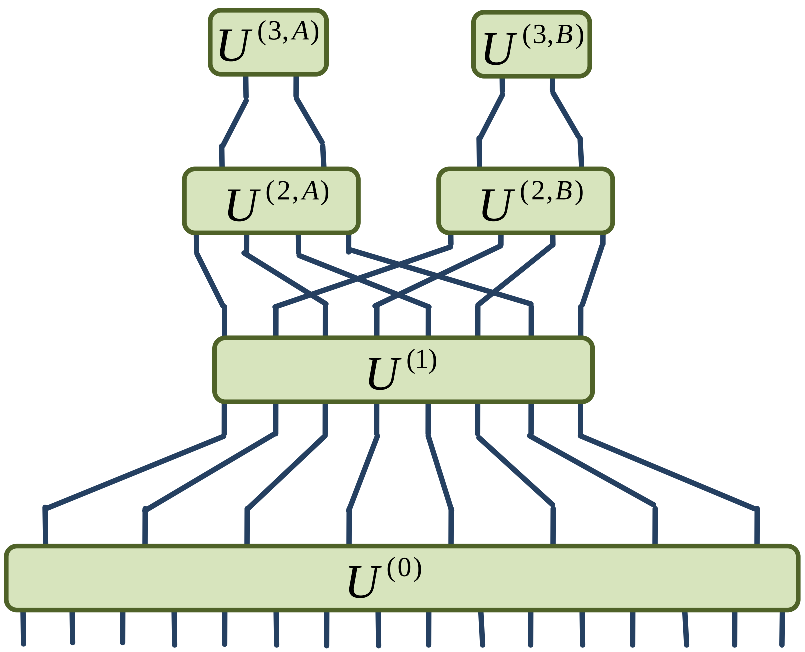


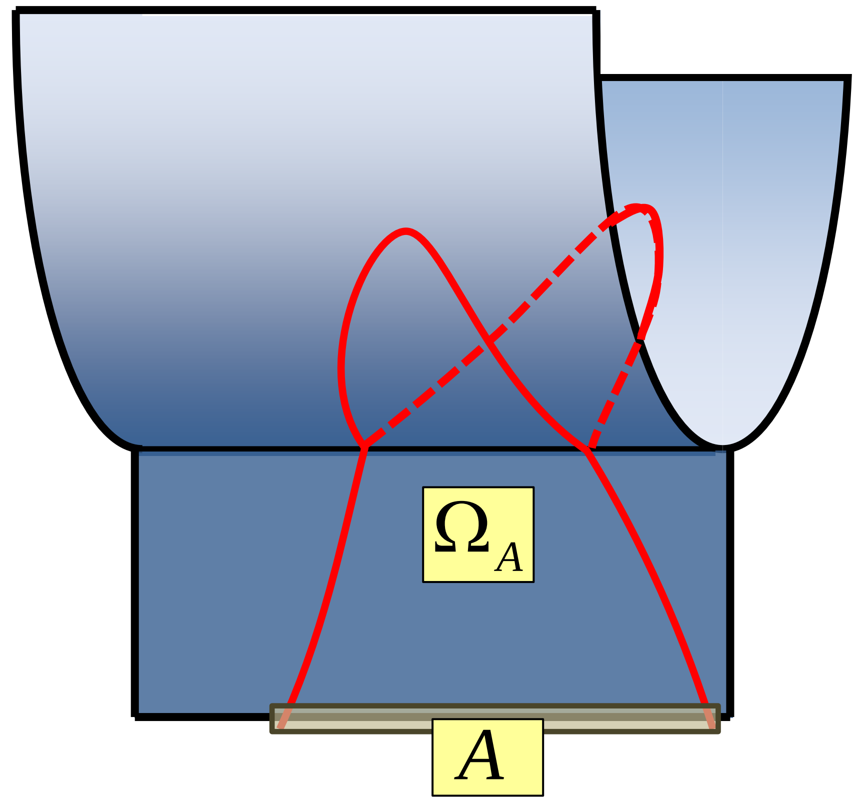


Entanglement entropy?

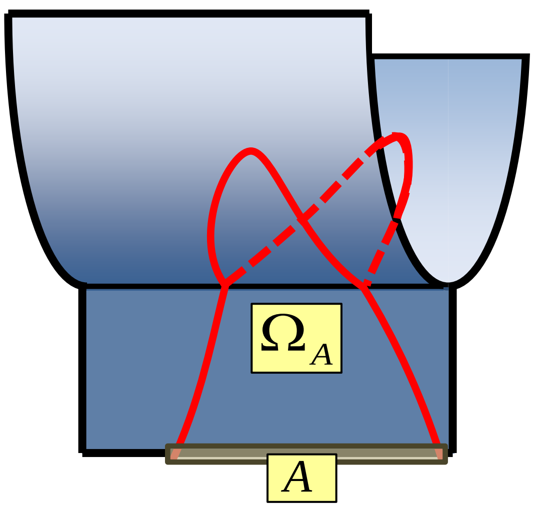
- given by boundary in exotic holographic geometry: $S(A) \square \left|\partial\Omega_A\right|$

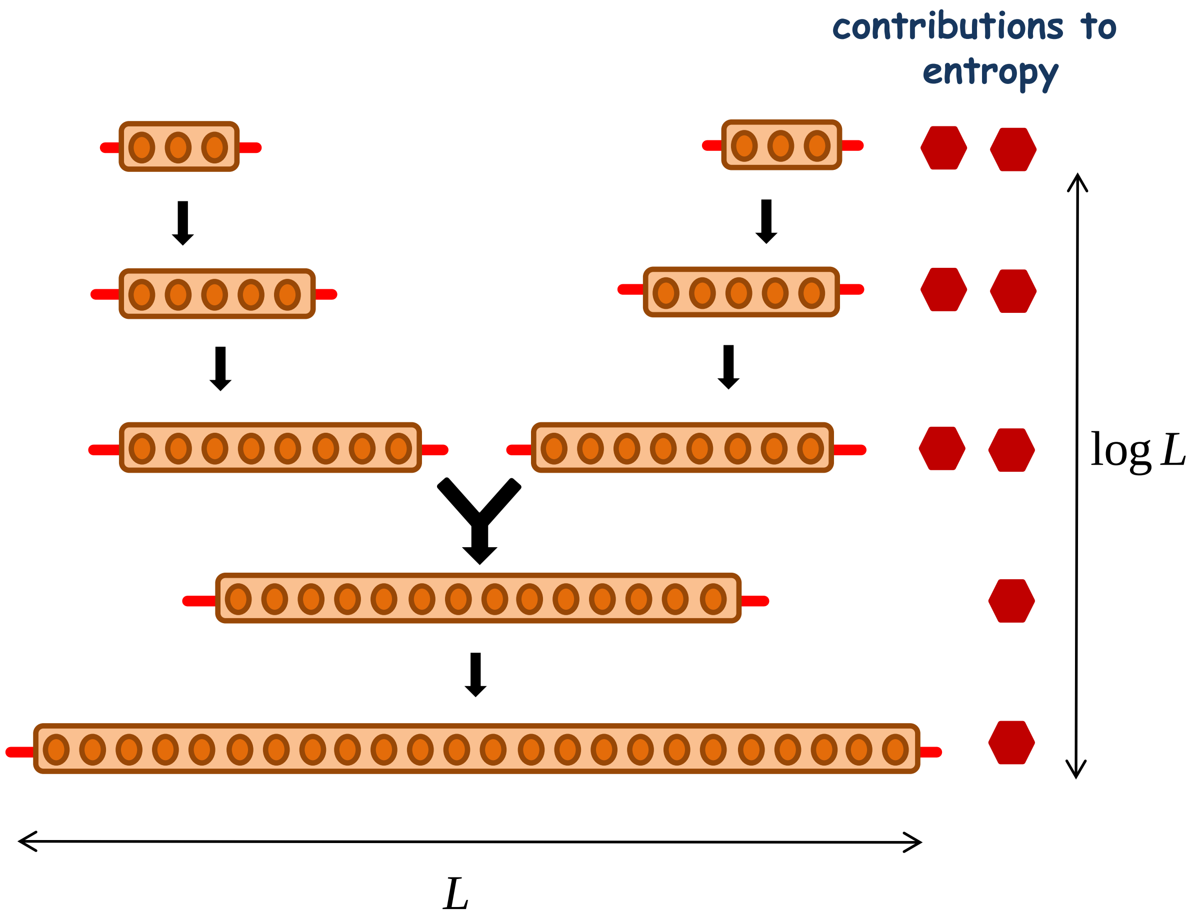
branching MERA





Entanglement entropy?





- Holographic branching can increase size of Ω_A

branching MERA can reproduce more entanglement entropy!

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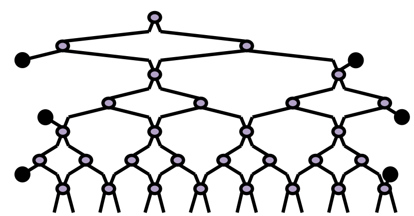

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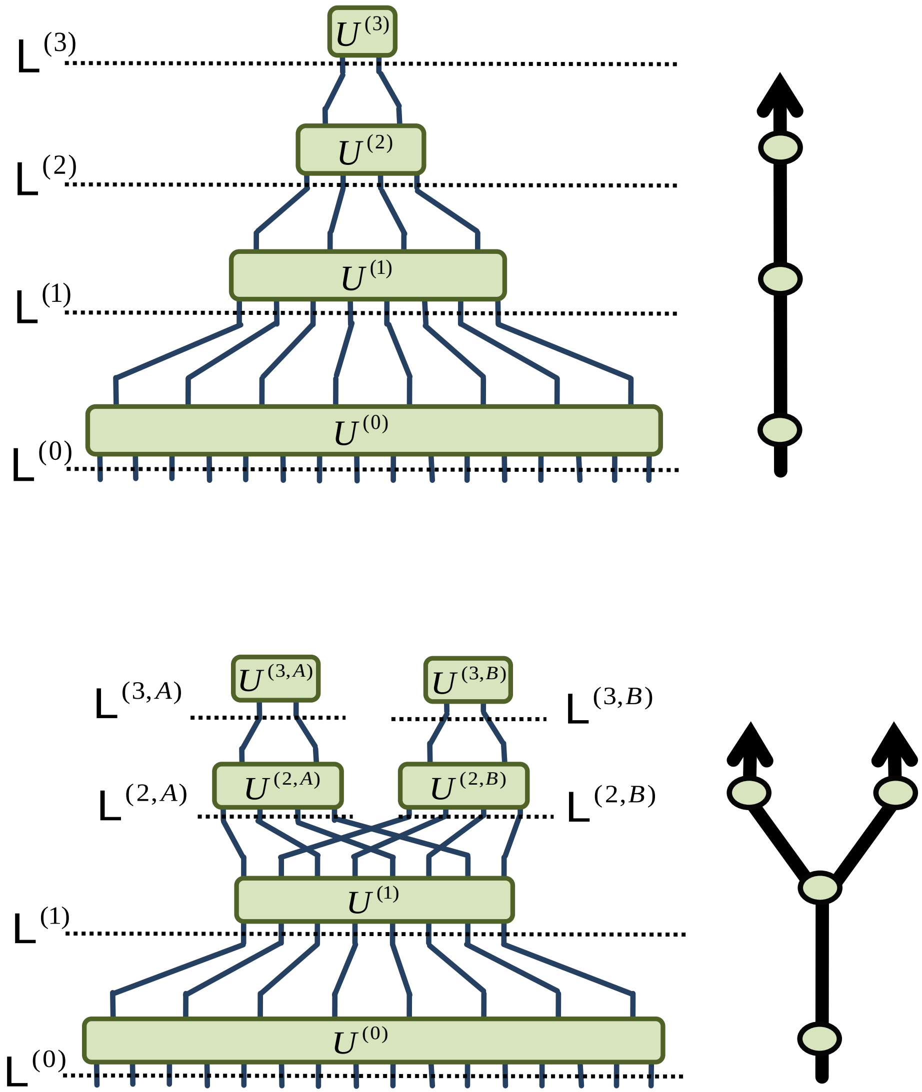
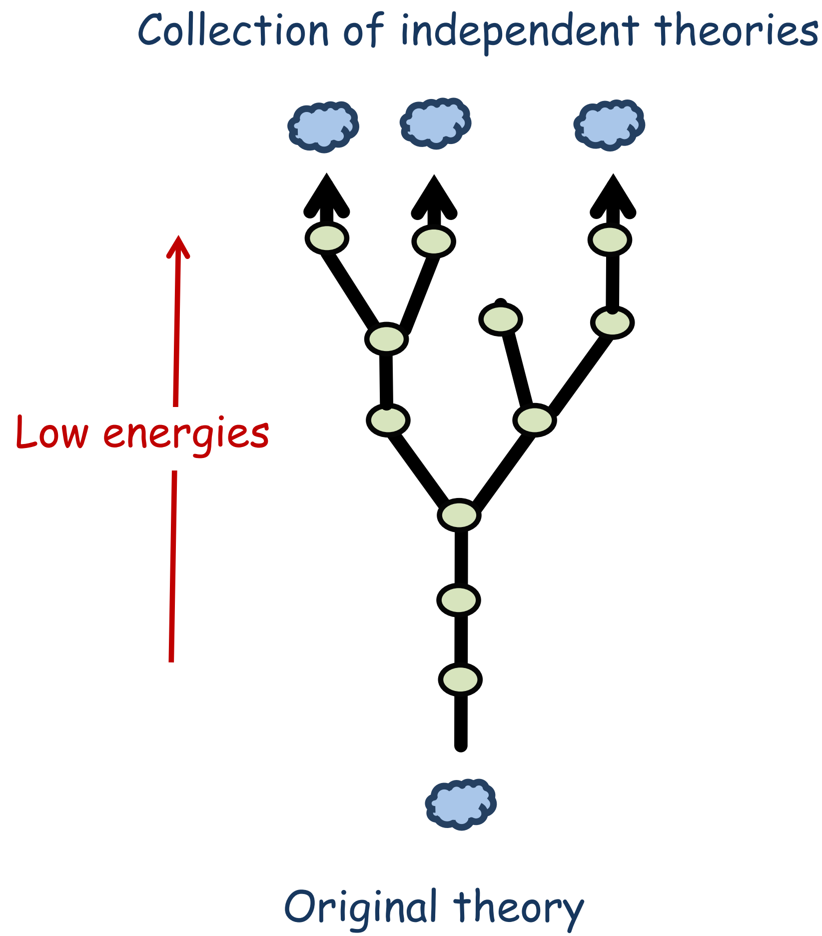
  - **Classification** of many-body entanglement via **holographic geometry**
  - Generalised **RG flow / RG fixed points**

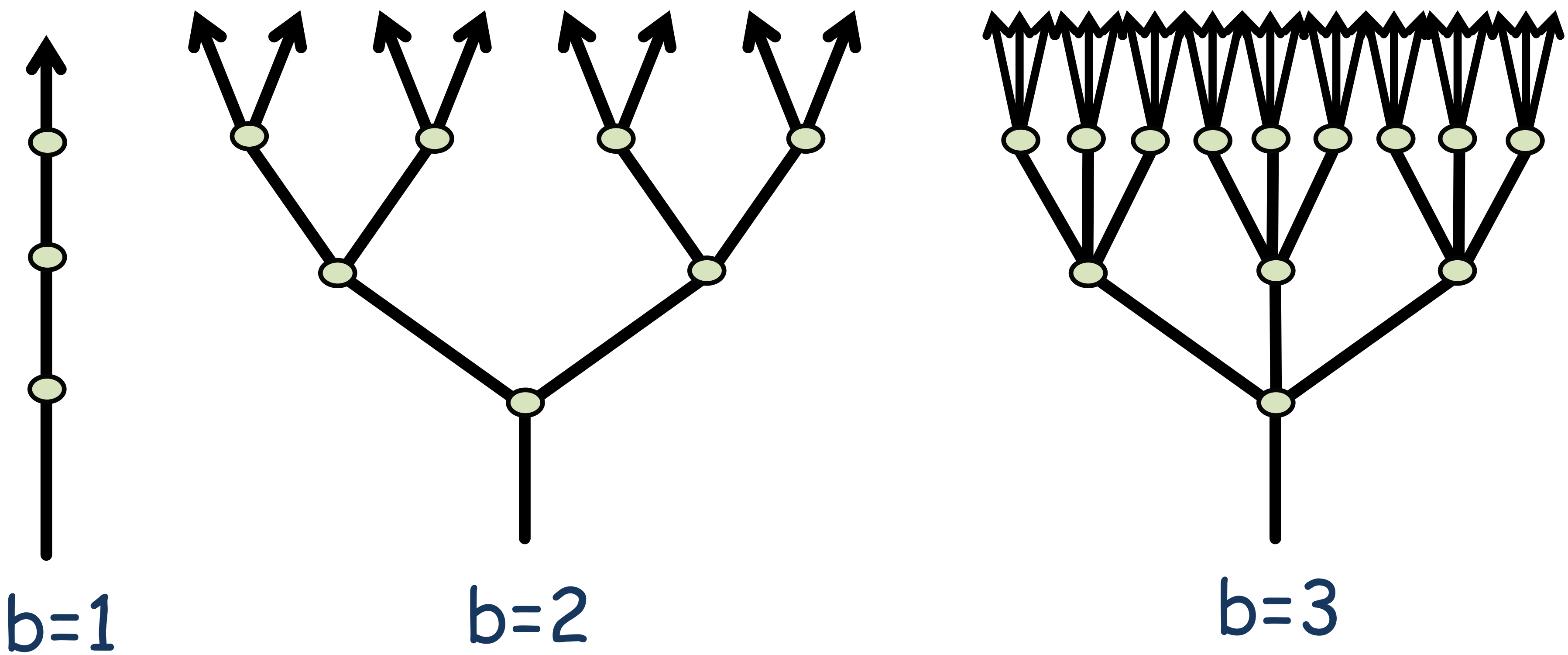
Holographic tree





Holographic Trees

- regular b-ary holographic trees:



Branching Parameter, b

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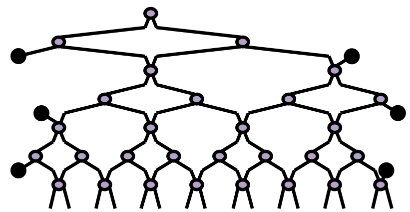

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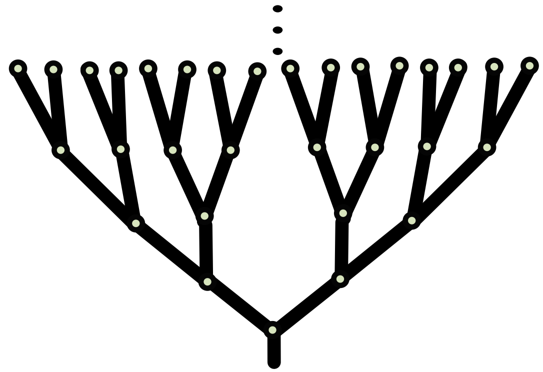
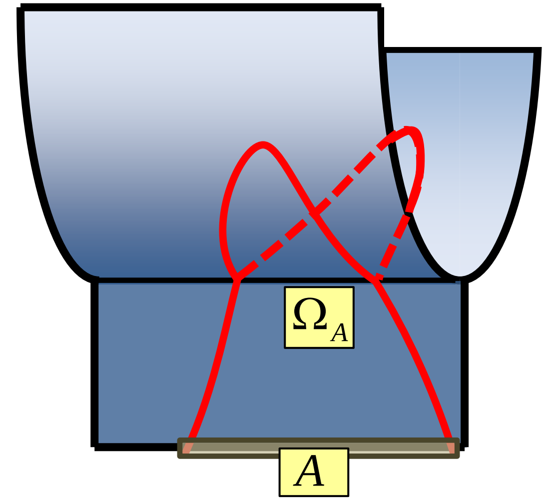
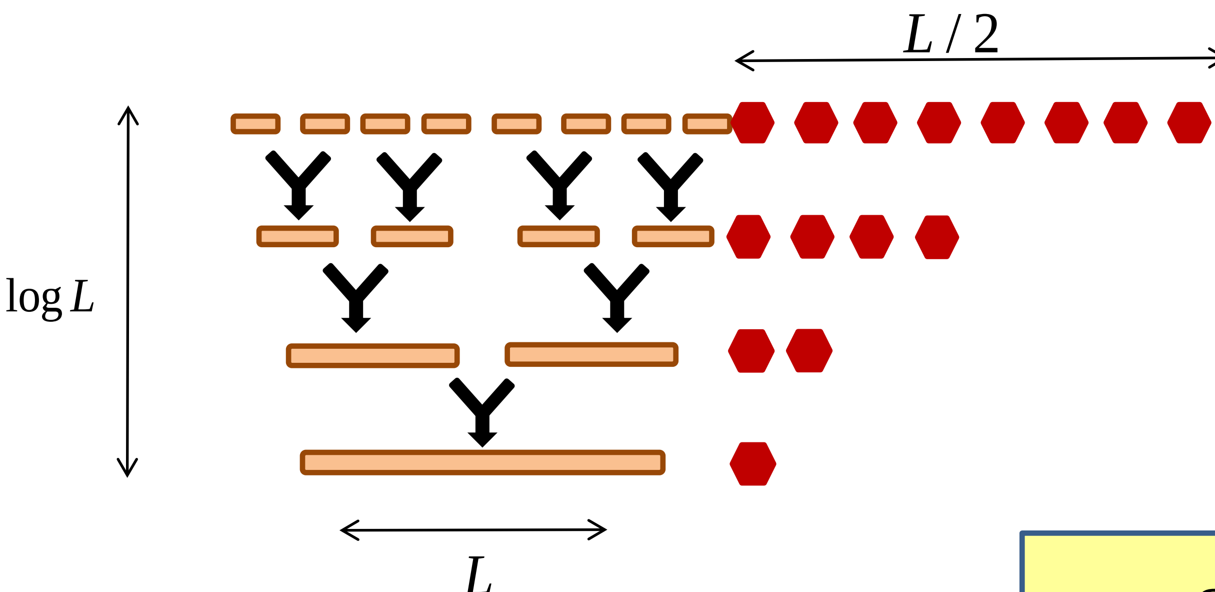
Holographic trees and entanglement entropy

b=2 branching MERA in
D=1 spatial dimensions







contributions to entropy

$8 = 2^{\log_2 L - 1} = L/2$

$4 = 2^2 = L/4$

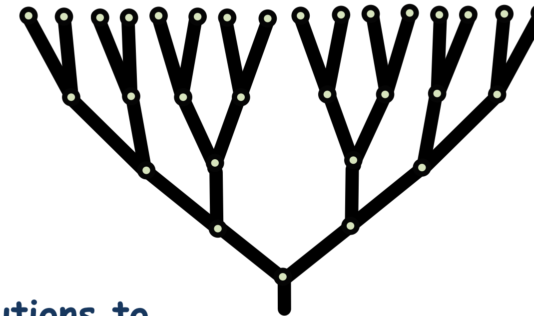
$2 = 2^1 = L/8$

$1 = 2^0 = L/16$

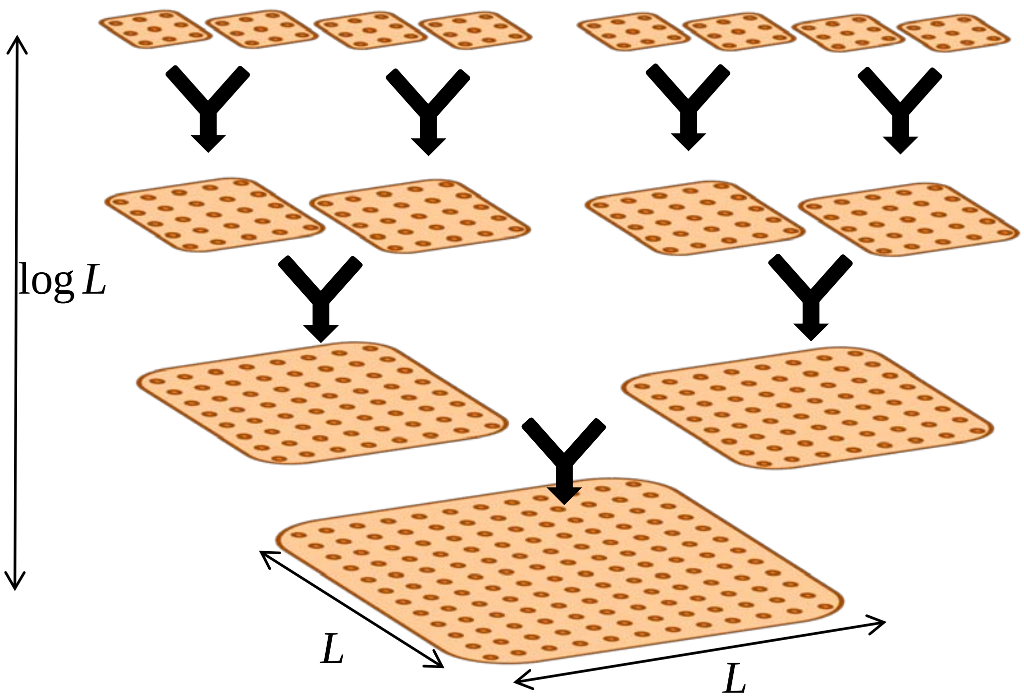
$$S_L \approx L$$

entropic “bulk” law (!)

Holographic tree and entanglement entropy



b=2 branching MERA in D=2 spatial dimensions



contributions to entropy

$$S_L \approx \underbrace{L + L + \cdots + L}_{\log L}$$

$$S_L \approx L \log L$$

logarithmic violation (!)

Is the (b=2) branching MERA a good ansatz for $S_L = L\log L$ phase??

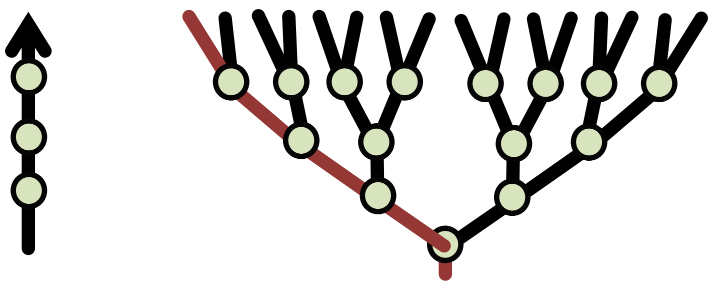
Yes!

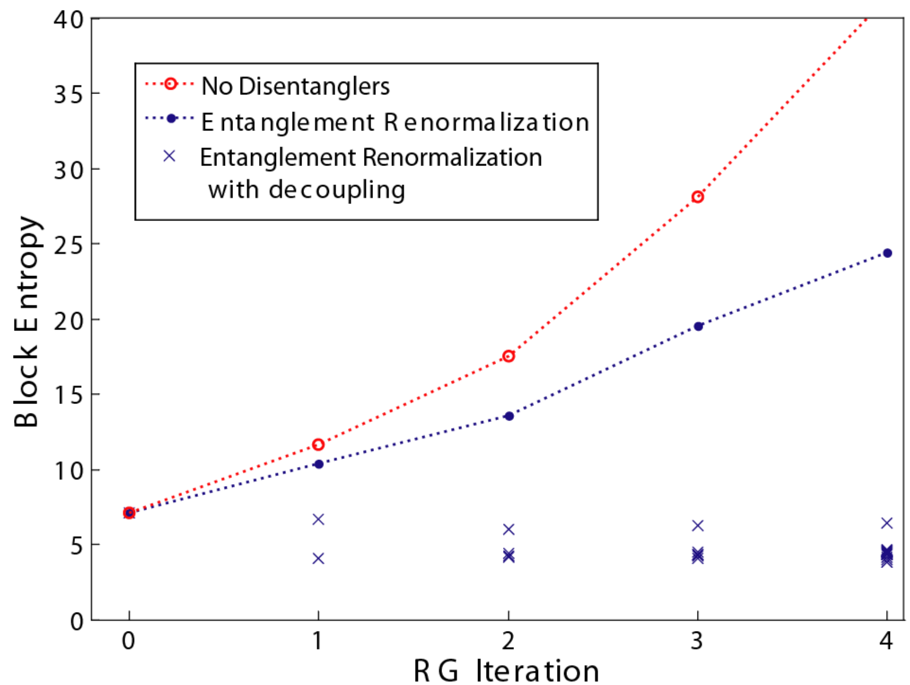
Example: free fermions in 2D

$$H = \sum_{\langle x,y \rangle} (a_x^\dagger a_y + h.c.)$$

critical model type II
(1D Fermi surface)

$$S_L \approx L \log L$$





Branching MERA

Evenbly, Vidal, in preparation

	Gap.	Crit.I	Crit.II
2D S_L	L	L	$L \log(L)$

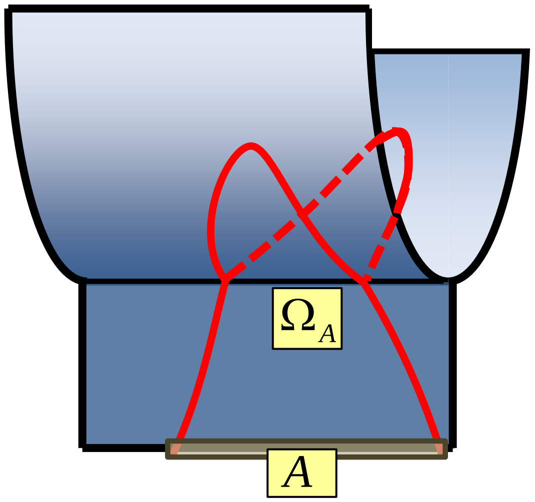
Scaling of entanglement: free fermions vs branching MERA

- Free Fermions:

Dimension of Fermi Surface, Γ

Spatial dimension

	Γ=0	Γ=1	Γ=2	
1D	$\log(L)$			
2D	L	$L\log(L)$		
3D	L^2	L^2	$L^2\log(L)$	

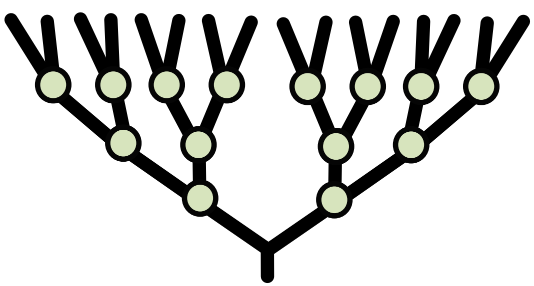


- Regular branching MERA:

Branching Parameter, b

Spatial dimension

	b=1	b=2	b=4	b=8
1D	$\log(L)$	L		
2D	L	$L\log(L)$	L^2	
3D	L^2	L^2	$L^2\log(L)$	L^3



- Proposed relation between dimensionality of Fermi surface and branching parameter:

$$\boxed{b = 2^{\Gamma}}$$

Corrections to the Boundary Law for Entanglement Entropy

$$S_L = L^{D-1} f(L)$$

Boundary Law

Correction

Arbitrary Corrections!

$f(L) = 1$

Boundary Law

$b < 2^{D-1}$

$f(L) = \log L$

Logarithmic Violation

$b = 2^{D-1}$

$f(L) = L$

Bulk Law

$b = 2^{D}$

Sub-logarithmic

e.g. $f(L) = \log\left(\log L\right)$

Super-logarithmic

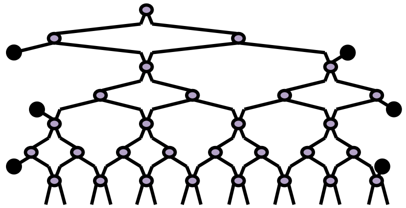
e.g. $f(L) = \left(\log L\right)^{\alpha}$

$f(L) = L^{\alpha}$

Example: 1D branching MERA with $S_L \approx (\log L)^2$

Outline

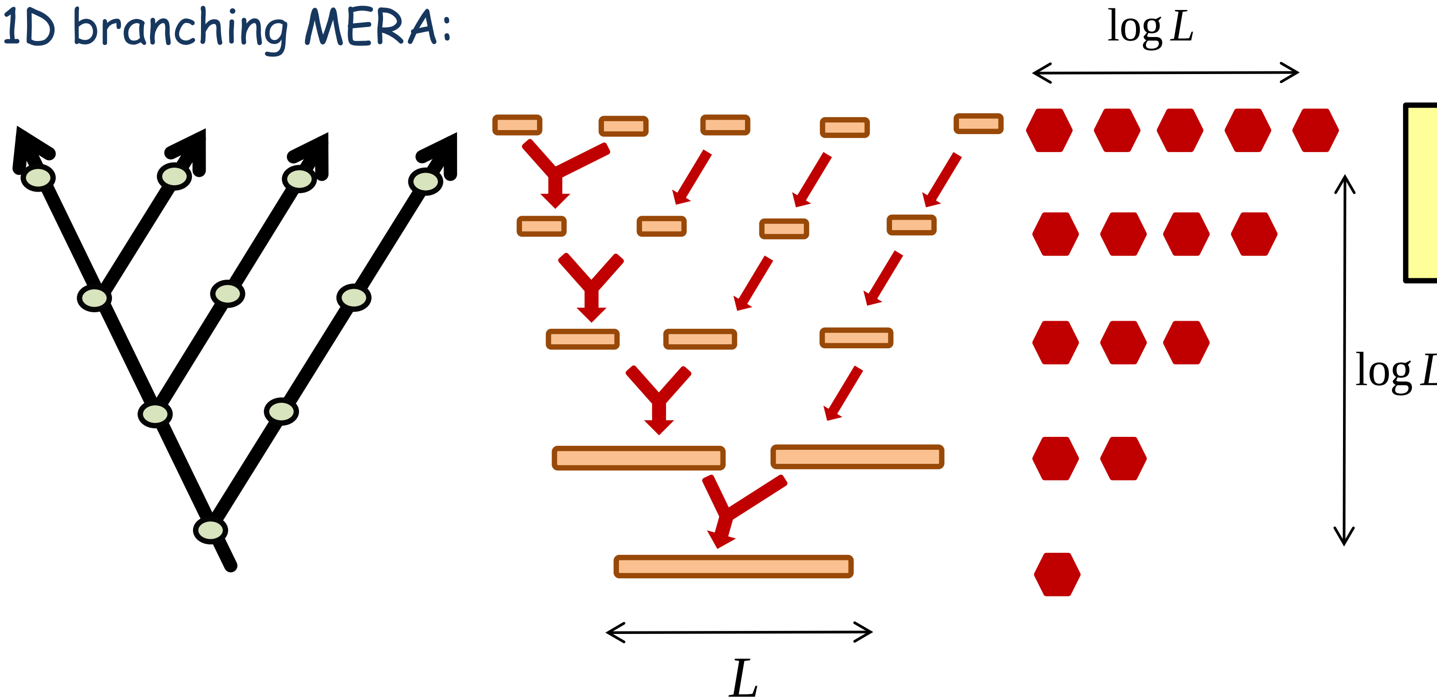
- Entanglement and tensor network methods
    - Scaling of entanglement entropy in ground states
    - Scaling of entanglement entropy in tensor network ansatz
      - physical geometry vs holographic geometry
    - Comparison of entropy scaling:
      - ground states vs tensor network ansatz
- The branching MERA
    - Decoupling a many-body theory
    - Holographic trees
    - Scaling of entropy in the branching MERA
    - Example: $S_L = L\log L$ entropy scaling in 2D fermions
    - Example: $S_L = (\log L)^2$ entropy scaling in 1D fermions



- Work in progress...
    - **Classification** of many-body entanglement via **holographic geometry**
    - Generalised **RG flow / RG fixed points**

Branching MERA beyond Regular Holographic Trees

1D branching MERA:



$$S_L \approx (\log L)^2$$

- Can we find a Hamiltonian that has this ground state entropy scaling?

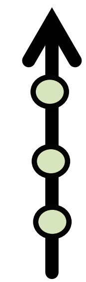
Yes!

$$H = \sum_{r=-\infty}^{\infty} \left(\sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{\phi(d)}{d^2} \left(\hat{a}_{r+d}^{\dagger} \hat{a}_r + \hat{a}_r^{\dagger} \hat{a}_{r+d} \right) \right) - \mu \sum_r \hat{a}_r^{\dagger} \hat{a}_r$$

$$\phi(d) \approx \cos\left(\log_2 |d| \right)$$

Branching MERA beyond Regular Holographic Trees

Holographic Tree:



Hamiltonian:

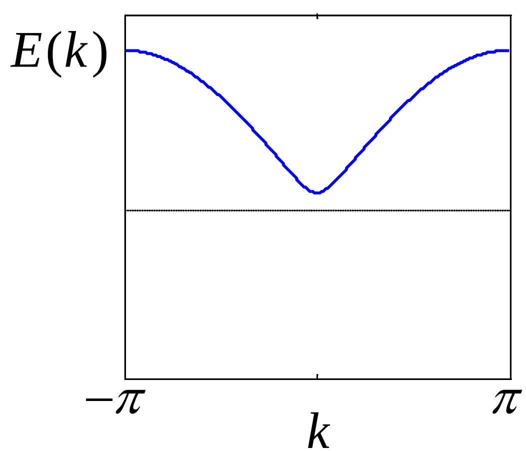
Gapped Ising

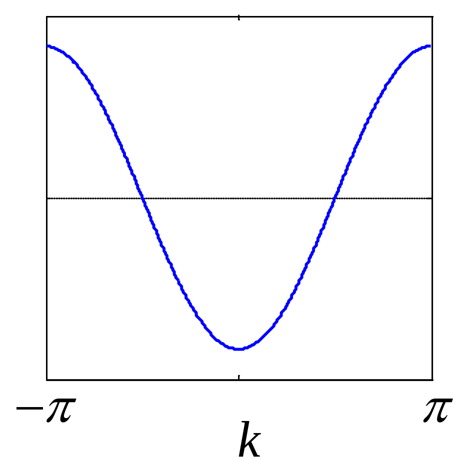
$$H = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + \hat{a}^\dagger_r\hat{a}^\dagger_{r+1} + h.c.\right) - \lambda \sum_r \hat{a}^\dagger_r \hat{a}_r$$

Critical XX

$$H = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + \hat{a}^\dagger_r\hat{a}_{r+1}\right)$$

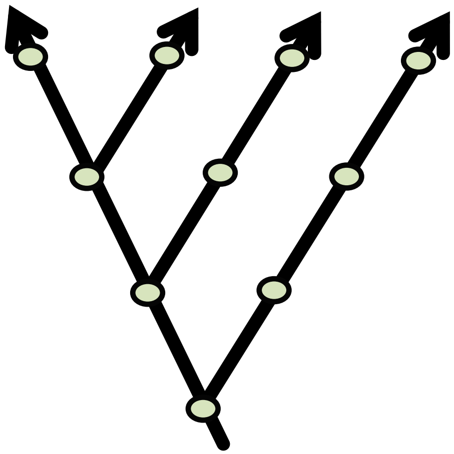
Dispersion:





Branching MERA beyond Regular Holographic Trees

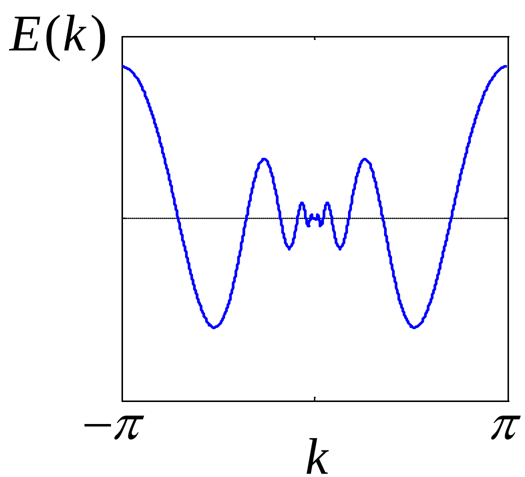
Holographic Tree:



Hamiltonian:

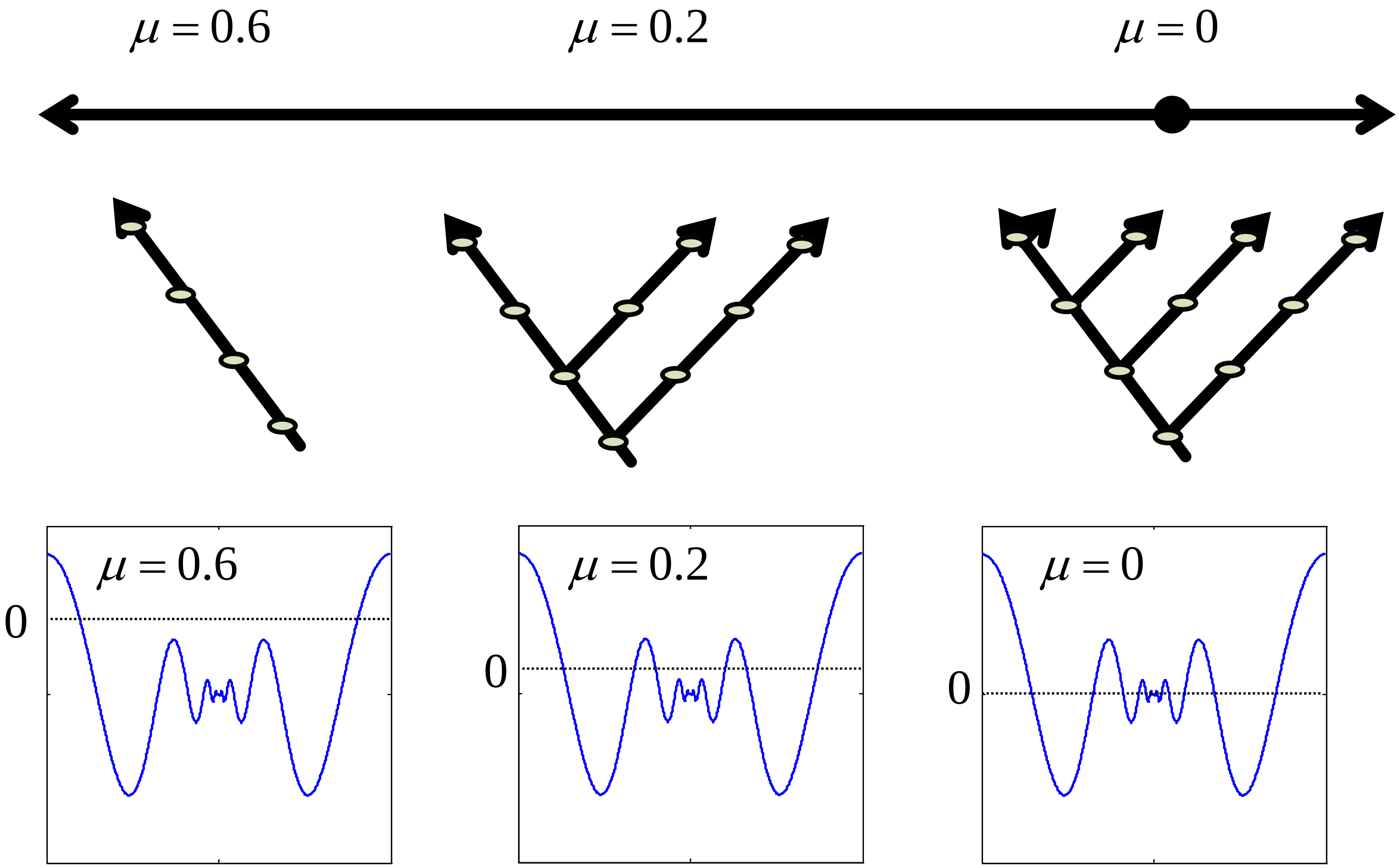
$$H=\sum_{r=-\infty}^{\infty}\left(\sum_{\substack{d=-\infty\\ d\neq 0}}^{\infty}\frac{\phi(d)}{d^2}\left(\hat{a}_{r+d}^{\dagger}\hat{a}_r+h.c.\right)\right)-\mu\sum_r \hat{a}_r^{\dagger}\hat{a}_r$$

Dispersion:

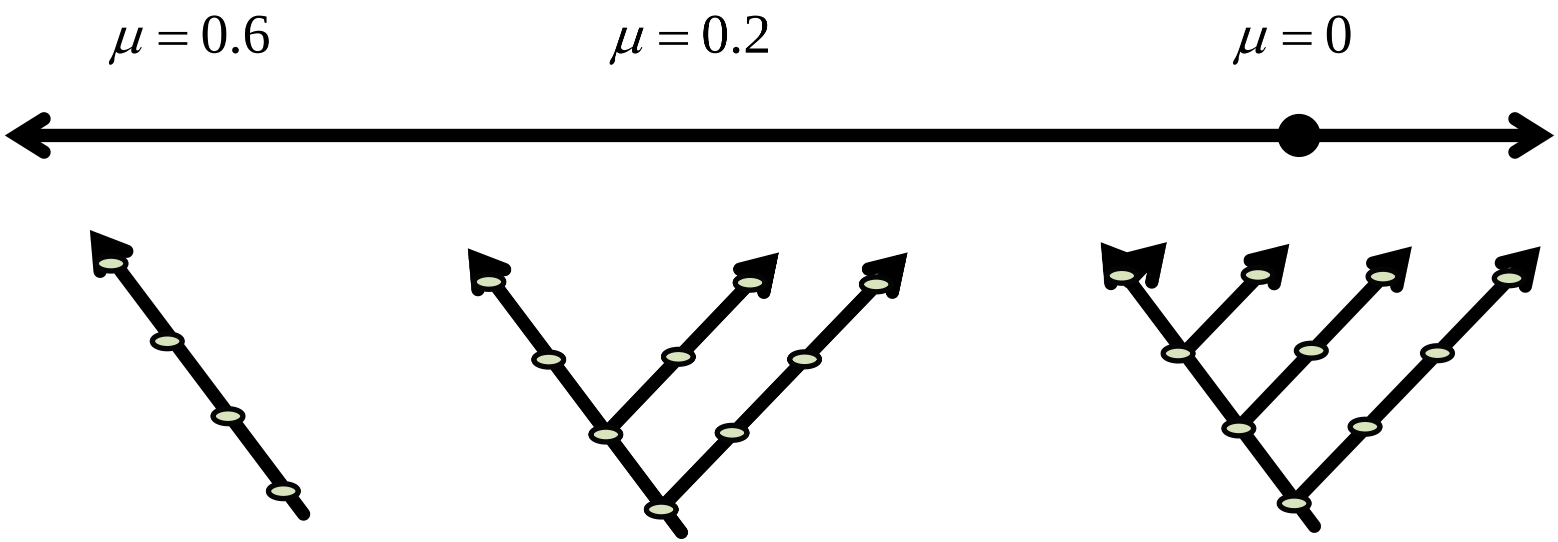


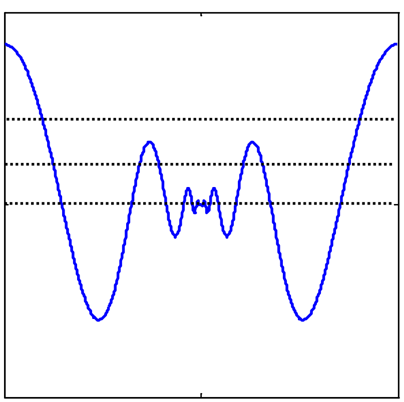
$$E(k)=\left|\sin\left(\frac{k}{2}\right)\cos\left(\pi\log_2\left|\frac{\pi}{k}\right|\right)\right|$$

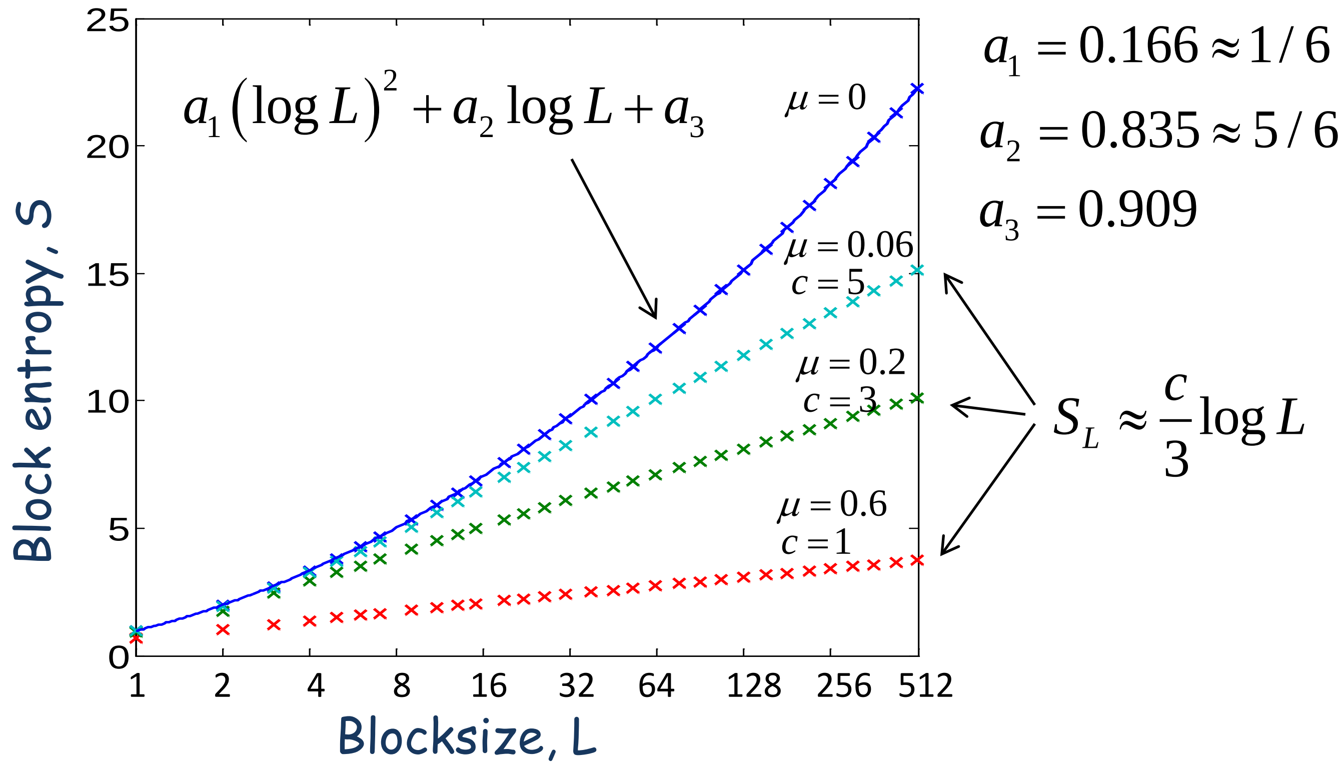
Chemical Potential:



Chemical Potential:







Phase Transition in D=1 Free Fermions

Chemical Potential:

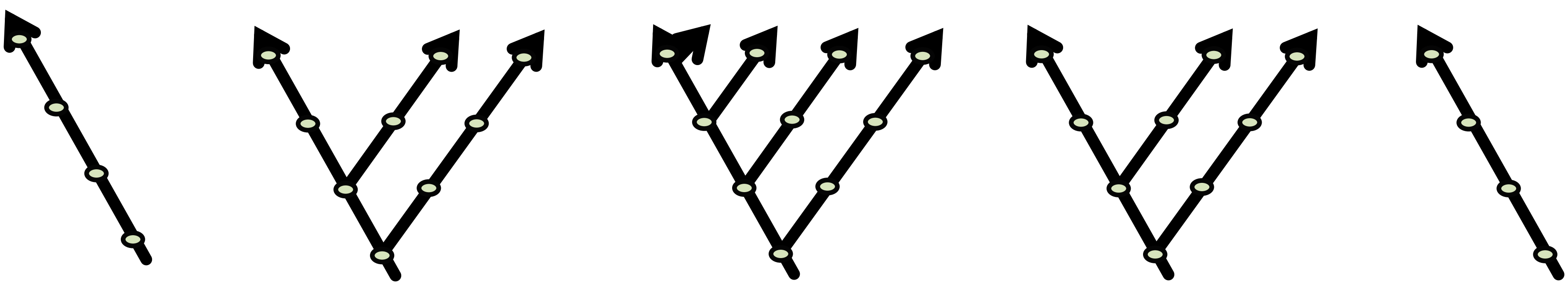
$\mu < 0$ | $\mu = 0$ | $\mu > 0$

Crit: $S_L \approx \log L$

Crit.II: $S_L \approx (\log L)^2$

Crit: $S_L \approx \log L$

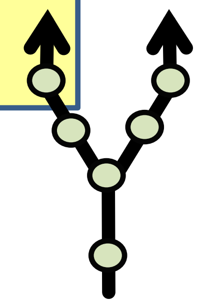
Holographic Geometry:



entanglement renormalization (real space RG) → MERA ⇔ holographic geometry (=spatial pattern of entanglement)

new! ↓

decoupling → branching MERA ⇔ holographic tree



Motivation:

- to find an ansatz that naturally decsribes 2D phases: **S ~ L log L**

branching MERA:

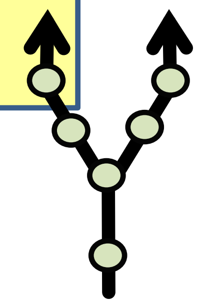
- admits a holographic interpretation of entropy scaling (many different violations of boundary law)

- an efficient ansatz for critical phases beyond reach of MPS/PEPS

- (further on...) basis of an algorithm to simulate highly entangled critical phases of matter entanglement renormalization (real space RG) → MERA ↔ holographic geometry (=spatial pattern of entanglement)

new! ↓

decoupling → branching MERA ↔ holographic tree



Work in progress...

- **classification** of entanglement in many-body ground states via **holographic geometry**
- systems with a surface of gapless modes as **fixed points** of a **generalised RG flow.**

Outline

- Entanglement and tensor network methods
  - Scaling of entanglement entropy in ground states
  - Scaling of entanglement entropy in tensor network ansatz
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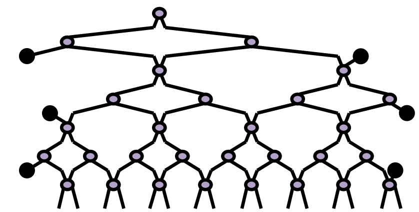
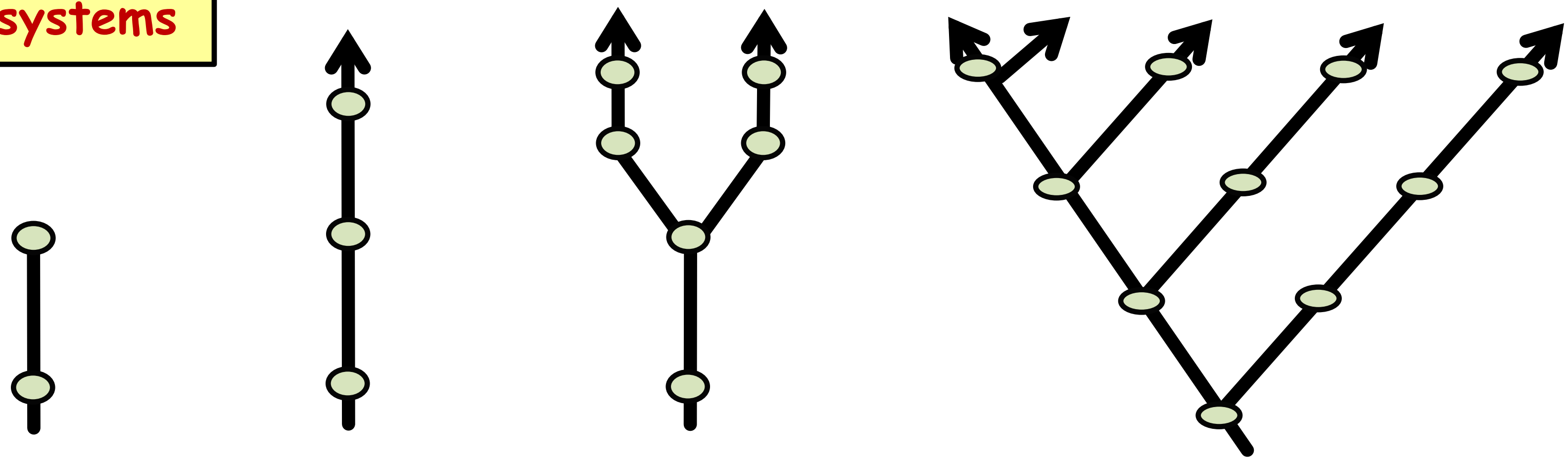


- Work in progress…
  - **Classification** of many-body entanglement via **holographic geometry**
  - Generalised **RG flow / RG fixed points**

Classification of entanglement in ground states

1D systems



gapped — $S_L \approx \text{const.}$

crit. — $S_L \approx \log L$

crit. — $S_L \approx \log L$

crit.II — $S_L \approx (\log L)^2$

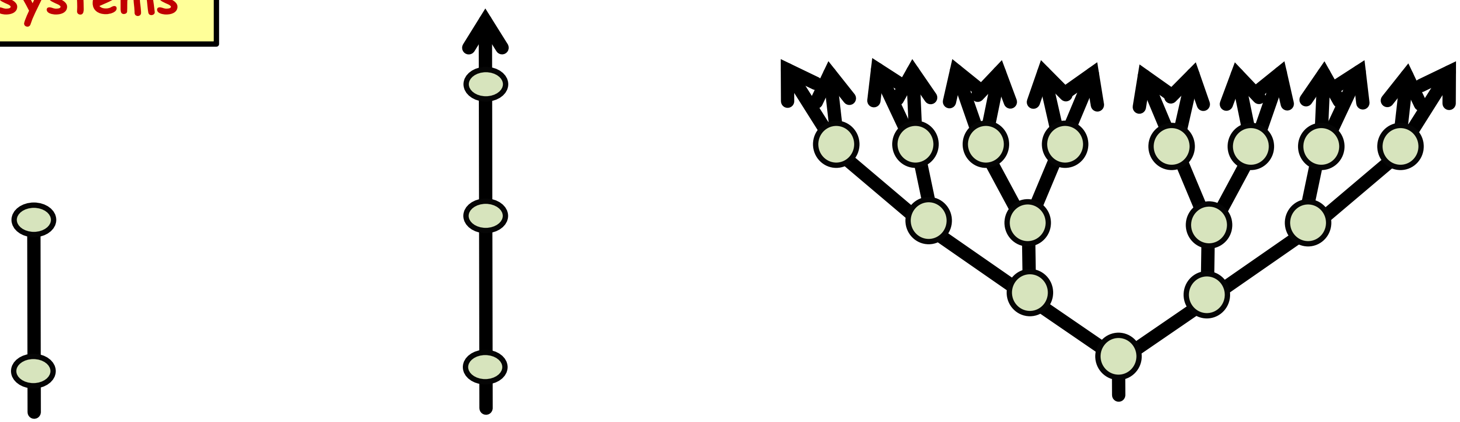
holographic geometry $\Longrightarrow$ block entanglement entropy scaling

Classification via holographic geometry:

- entanglement contributions from different length scales
- describes how different sets of degrees of freedom become decoupled at different length scales

Classification of entanglement in ground states

2D systems



gapped $S_L \approx L$

crit.I $S_L \approx L$

crit.II $S_L \approx L \log L$

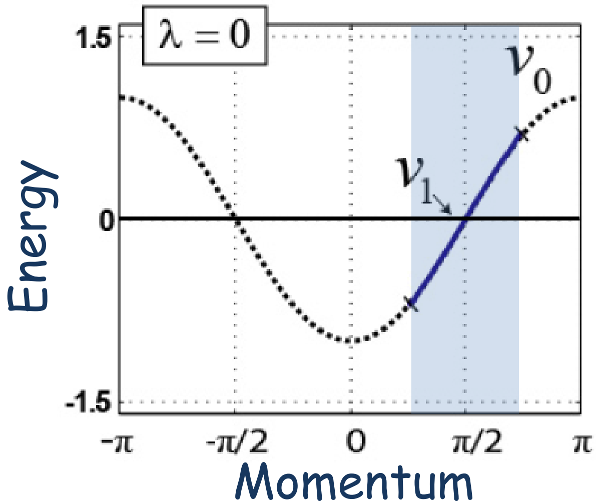
holographic geometry $\Longrightarrow$ **block entanglement entropy scaling**

Classification via holographic geometry:

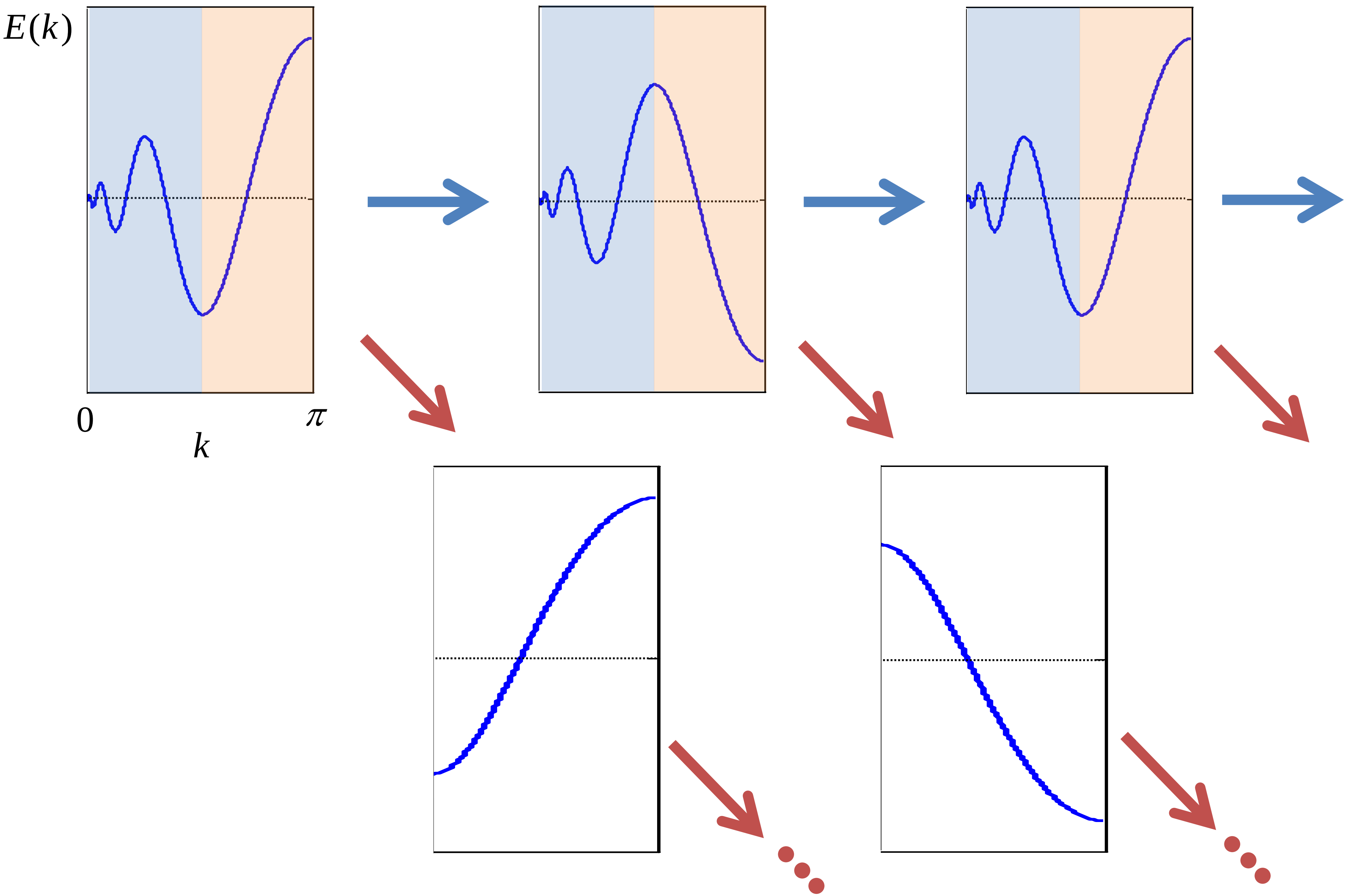
- entanglement contributions from different length scales
- describes how different sets of degrees of freedom become decoupled at different length scales

Generalised RG flow

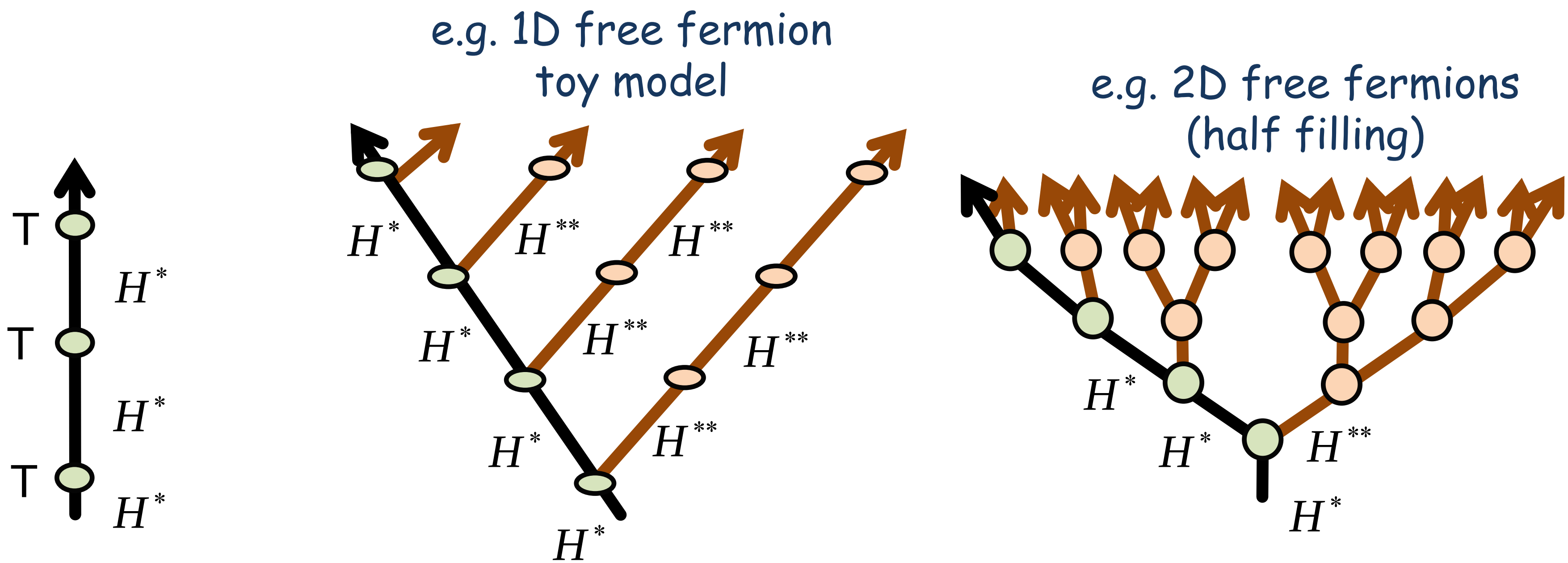
Renormalization Flow



Generalised RG flow: Toy example



Generalised Scale Invariance



Scale Invariance

finite number of gapless modes

Generalised Scale Invariance

infinite (discrete) set of gapless modes	1D surface of gapless modes

scale-invariant properties:

- scaling dimensions / scaling operators?
- correlators?

branching MERA

(Evenbly, Vidal, in preparation)

- generalization of the MERA to more complex **holographic geometries** in order to produce **violations of the boundary law** for entanglement entropy scaling

  - e.g. 2D branching MERA with entanglement: $\boxed{S_L \propto L \log L}$

  an ansatz for 2D systems with a 1D Fermi surface?

- we propose to use **holographic geometry** to **classify entanglement** in many-body ground states

  - geometric understanding of violations to the boundary law

- new notions of **RG flow** / **RG fixed points**

  - 2D fermions (with 1D fermi surface) as RG fixed points