# General Relation between Entanglement and Edge Theory in Topological States

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# Outline

- Introduction on quantum entanglement
- Quantum entanglement in many-body systems
- Quantum entanglement in topological states
- General relation between entanglement properties and low energy behavior of topological states

Collaborators: Hosho Katsura (Gakushuin Univeristy, Tokyo), Andreas Ludwig (UCSB), paper in preparation

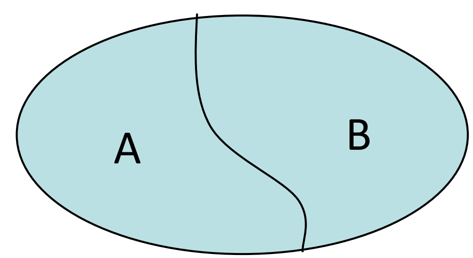
# Quantum Entanglement

- Quantum entanglement is the purely quantum mechanical "correlation" between two parts of a quantum system.
- The state of a quantum system entangled with another system is described by the density matrix
- The entanglement between A and B subsystems is completely determined by the eigenvalues of the density matrix
- Von Neumann entropy $S = -\sum_n \lambda_n \log \lambda_n$

$\mathbf{H}_A$ $\mathbf{H}_B$

$$|\Phi\rangle = |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$$

$$\rho_A = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$$

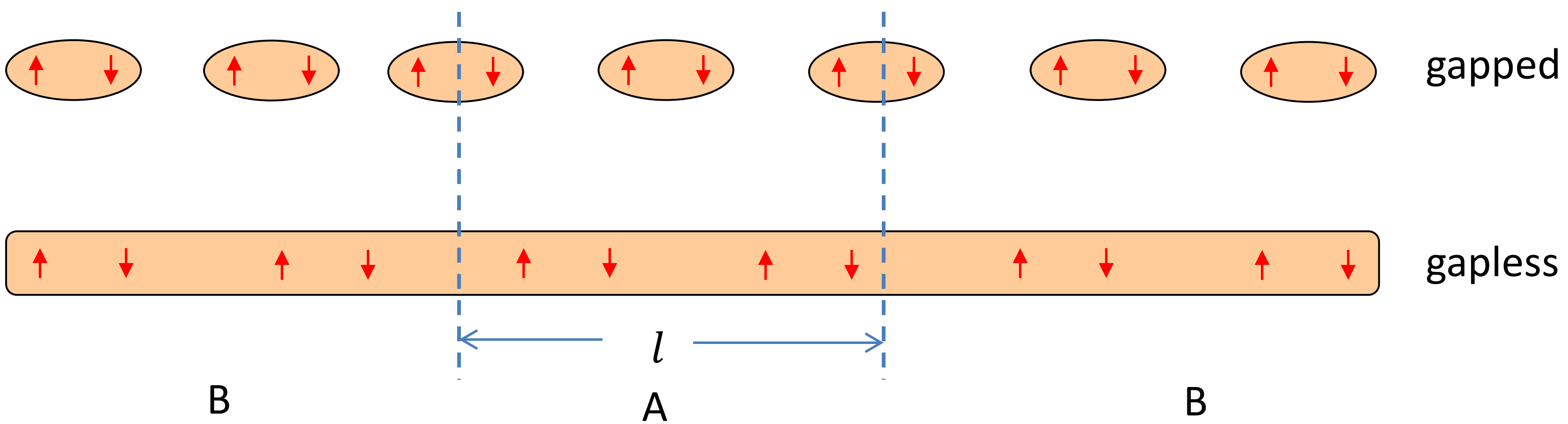



$$|\Phi\rangle = \sum_n \sqrt{\lambda_n}|n\rangle_A|n\rangle_B$$

$$\rho_A = \sum_n \lambda_n |n\rangle_A \langle n\Big|_A$$

# Quantum Entanglement in Many-Body Systems

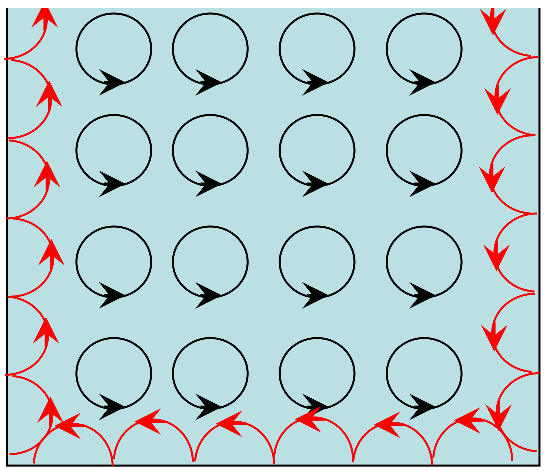
- Quantum entanglement in many-body systems provides a new way to characterize the correlation and orders in many-body systems
- Example: $S_A$=*const*. for gapped 1d system, and $S_A = \frac{c}{3} \log l/a$ for gapless (critical) system
- More low energy states, more entanglement

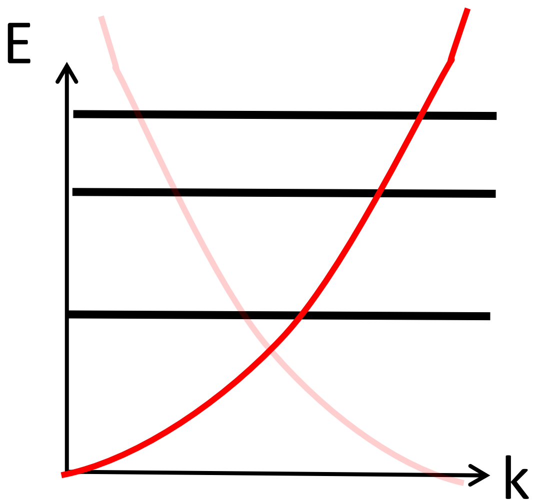



Holzhey et al, Nucl. Phys. **B424**, 443 (1994).

# Topological states of matter

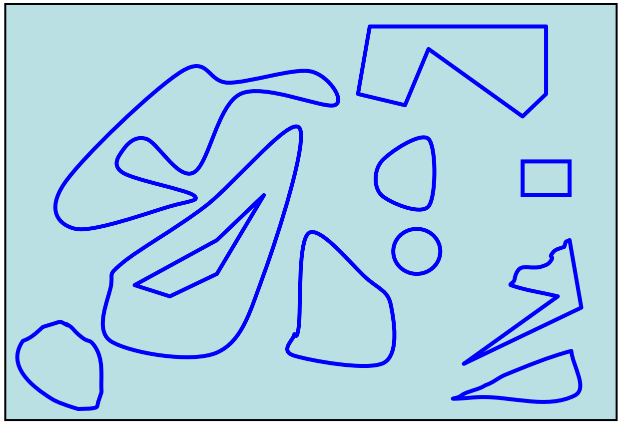
- Topological states of matter are gapped states which cannot be adiabatically deformed to a trivial state while preserving the locality of the Hamiltonian.
- Topological states cannot be characterized by symmetry breaking and can be characterized by topological properties, such as ground state degeneracy (on a manifold without boundary) and/or topological edge states (on a manifold with boundary).
- Example: Quantum Hall States (von Klitzing 1980, Laughlin 1981, Tsui 1982, Laughlin 1983)
- Chiral edge states protected by bulk Chern number (Thouless et al 1982, Niu et al 1983, Avron&Seiler 1983)



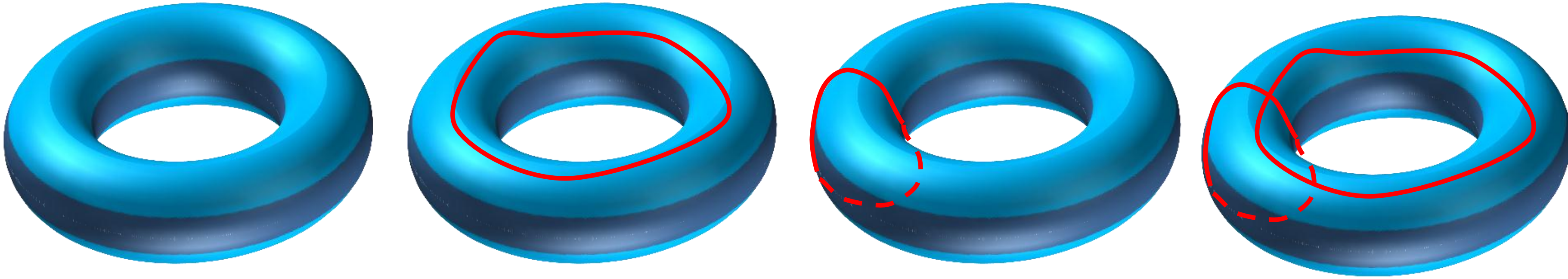


# Topological states of matter

- Another Example: Toric code model, or $Z_2$ gauge theory (Kitaev 2003, Wen 2003). The ground state is a "condensate" of unoriented loops.

$$|G\rangle = \sum_{\text{closed loops}}$$ 

- 4-fold Ground state degeneracy on torus



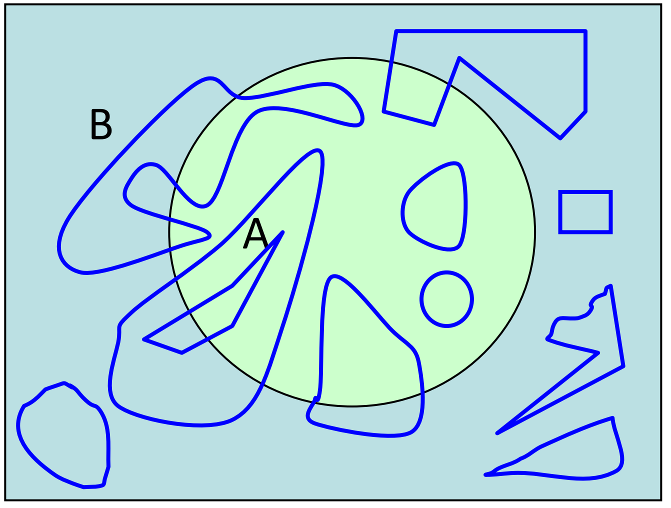
# Entanglement and topological properties

- One can probe the topological properties of a given ground state by studying its entanglement properties.
- Example: Topological entanglement entropy. (Kitaev&Preskill 2006, Levin&Wen 2006)
- $S = \alpha L - S_{topo}$ with $S_{topo} = \log \sqrt{\sum_i d_i^2}$

Toric code model: S=log W with W the possible configurations on the boundary. For N links on the boundary

$$W = 2^{N-1} \Rightarrow S = (N-1)\log 2$$

The -1 comes from the topological "Gauss law" constraint---the loops are closed



# Entanglement spectrum of topological states

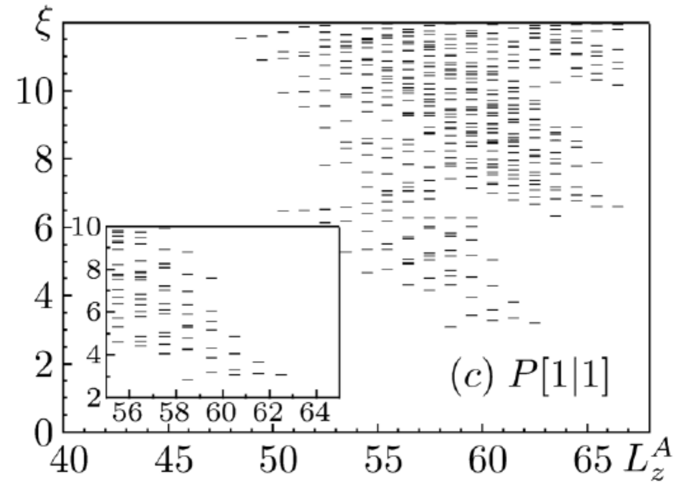
- Topological entanglement entropy provides a characteristics of quantum entanglement in topological states, but it does not contain sufficient information to distinguish different topological states. Different states may have the same topological entropy.
- Complete information on quantum entanglement is encoded in the reduced density matrix

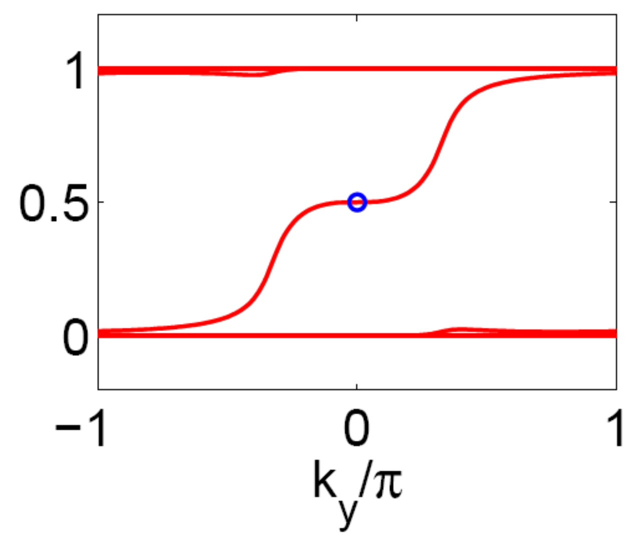
$$\rho_A = \sum_n \lambda_n |n\rangle\langle n| = e^{-H_E}$$

- The "entanglement Hamiltonian" is defined as $H_E = -\log \rho_A$ which plays the role of $\beta H$ in thermodynamic systems. The spectrum of $H_E$ is named as "entanglement spectrum". (Li&Haldane PRL 2008)

# Entanglement spectrum of topological states

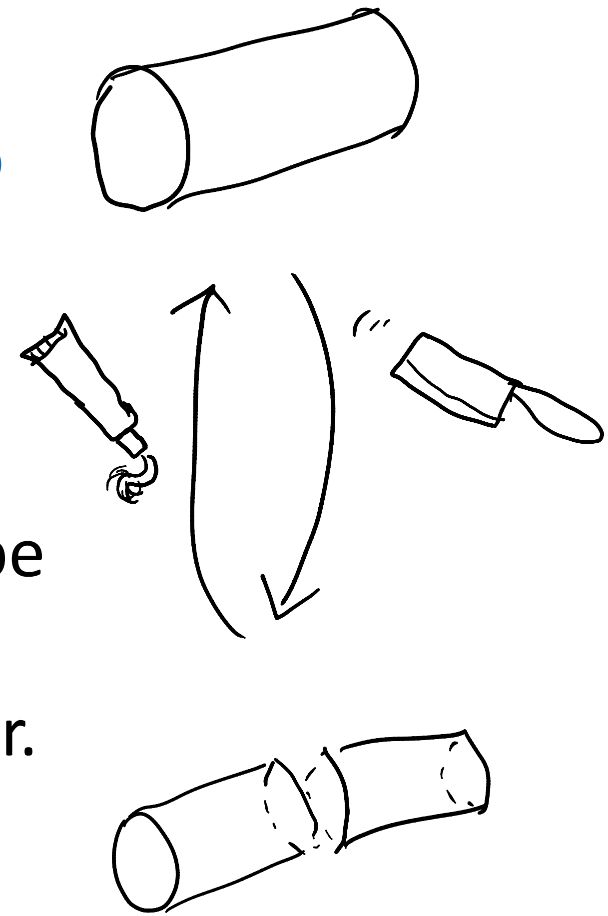
- In some topological states, $H_E$ has been shown to be analogous to the physical Hamiltonian with open boundary.
- Numerical results: on fractional quantum Hall system (Moore-Read state) (Li&Haldane PRL 2008) . Low-lying entanglement spectrum in a given topological sector is described by the same conformal field theory as the edge state (up to a constant energy shift).
- Analytic results on the Topological equivalence between $H_E$ and $H_A$ in (1) noninteracting topological insulators (Turner et al 2010, Fidkowski 2010); (2) the Kitaev model with non-Abelian (Ising) phase (Yao&Qi PRL 2010)



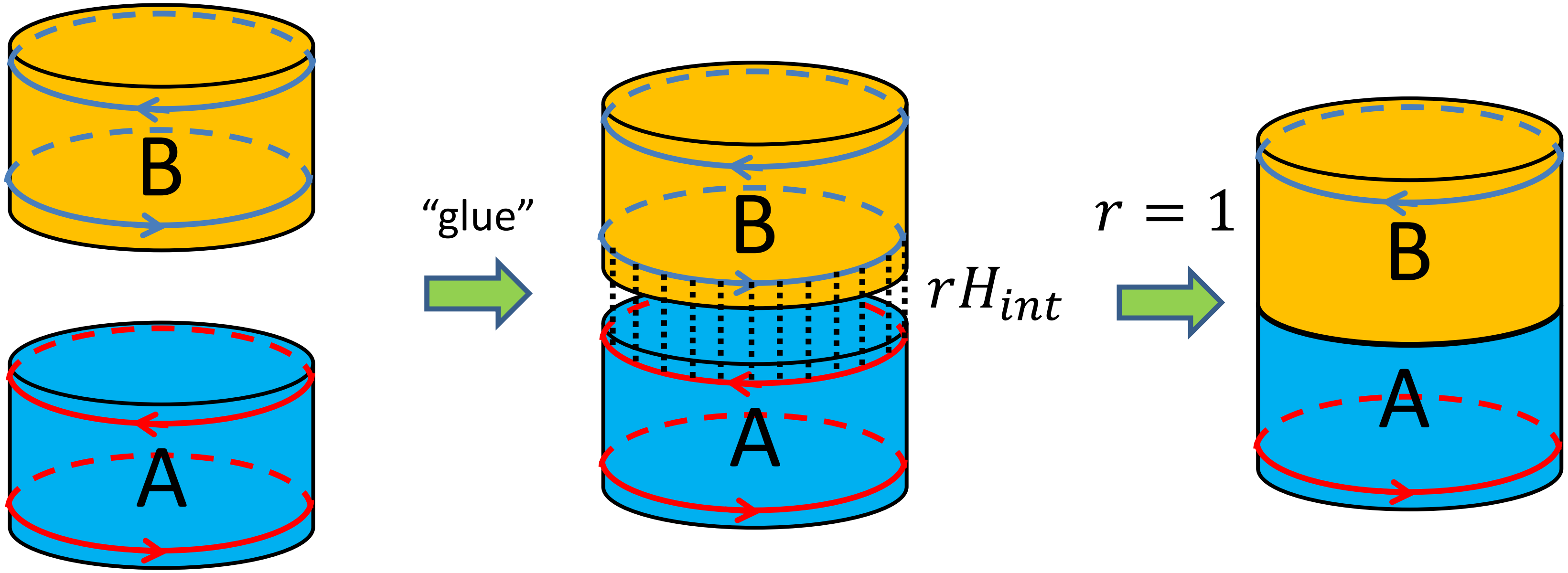



# General relation between entanglement spectrum and edge state energy spectrum

- Physically, why is the entanglement spectrum qualitatively equivalent to the edge state spectrum?
- Intuitive answer:
- 1. ("Cut and glue") The system with bipartition into A and B region can be considered as A and B regions with open boundary being glued together.
- 2. ("Lowest energy states entangle most") The lower is its energy, the more is a state entangled during the glue process with the other region.



# Relating the entanglement spectrum to edge state energy spectrum by "cut and glue" procedure




Open boundary
$H = H_A + H_B$
Edge states described by chiral CFT

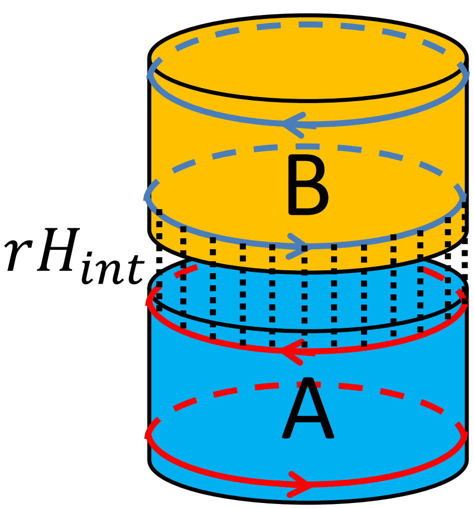
Inter-edge tunneling
$H = H_A + H_B + rH_{int}$

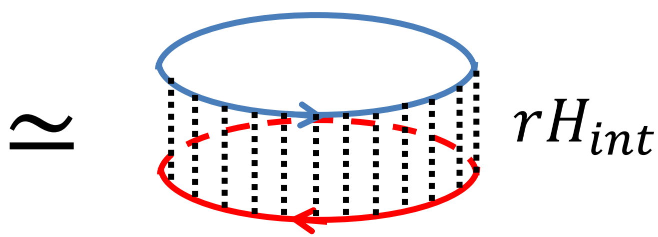
A single cylinder with bipartition
$H = H_A + H_B + H_{int}$

# Reducing the bulk problem to an edge CFT problem

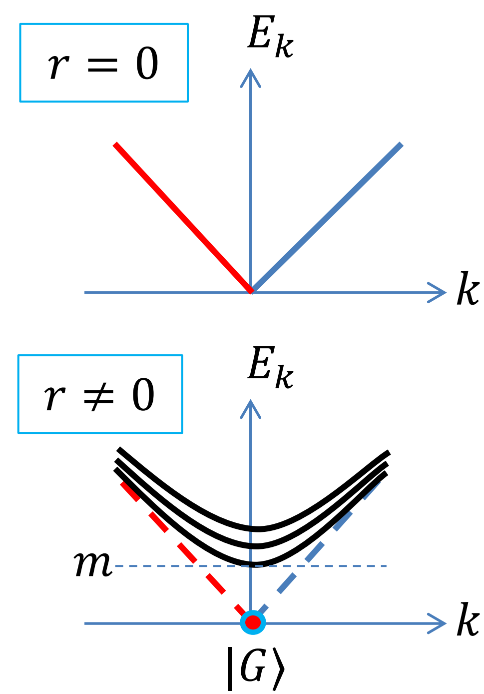
- For small value of $r$, the coupling between A and B is reduced to the coupling between left and right moving edge states. Thus the problem of entanglement spectrum is reduced to the entanglement between left and right movers in the edge CFT induced by a relevant coupling

$$\rho_A \simeq \rho_L = Tr_R |G\rangle\langle G|$$

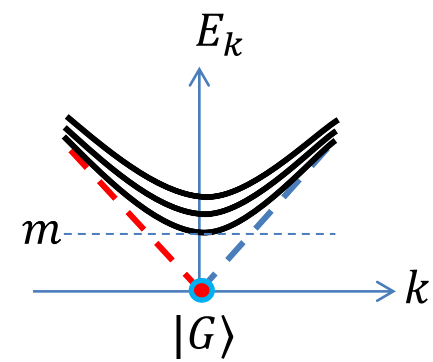


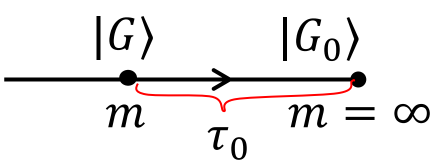

$$H = H_L + H_R + rH_{int}$$



# Obtaining the entanglement spectrum from boundary conformal field theory




- The gapped ground state $|G\rangle$ is not universal.
- However, in RG flow, a generic gapped ground state $|G\rangle$ flows to a universal conformal invariant RG fix point $|G_0\rangle$
- $|G\rangle$ and $|G_0\rangle$ are related by the extrapolation length $\tau_0$ (Diehl 1986, Calabrese&Cardy 2006)

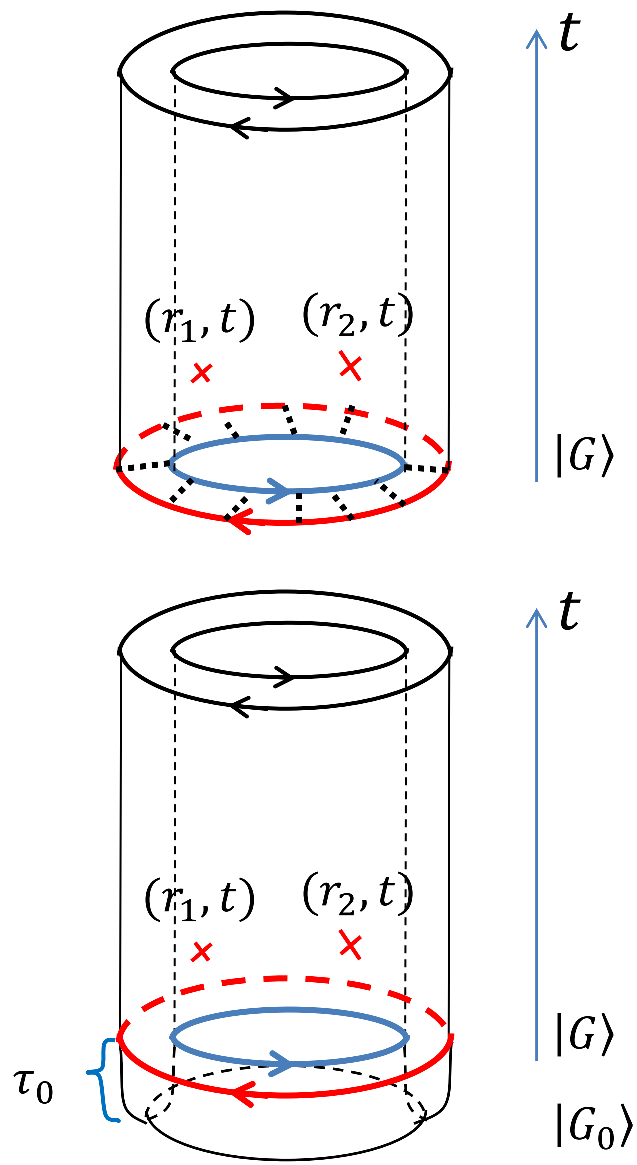



$$\langle O(t)\rangle = \langle G|e^{iHt}\hat{O}e^{-iHt}|G\rangle \simeq \frac{1}{Z}\langle G_0|e^{iHt-\tau_0 H}\ \hat{O}e^{-iHt-\tau_0 H}|G_0\rangle$$

$\Rightarrow |G\rangle \simeq \frac{1}{Z^{1/2}}e^{-\tau_0 H}|G_0\rangle$ with $H = H_L + H_R$ the CFT Hamiltonian

# Relation to the quantum quench problem

- The problem is very related to quantum quench problem studied by Calabrese and Cardy (2006, 2007)
- When a relevant coupling is suddenly switched off, the time evolution of the intitial state $|G\rangle$ is determined by relating $|G\rangle \simeq Z^{-1/2} e^{-\tau_0 H} |G_0\rangle$. Determining the density matrix $\rho_L$ is equivalent to determining all correlation functions of the left mover.

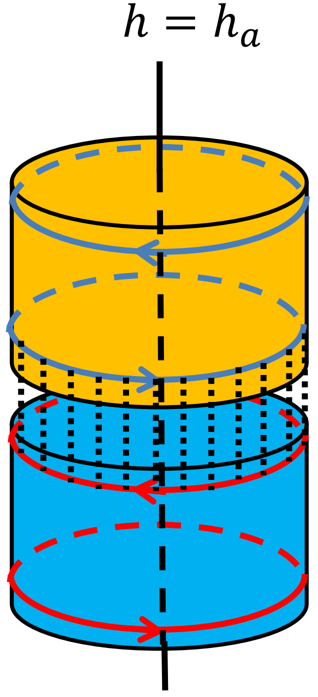


# Ishibashi states and the entanglement spectrum

- For a given CFT, the conformal invariant initial state $|G_0\rangle$ can be obtained explicitly as linear superposition of Ishibashi states (Ishibashi 1987)

$$|G_{0h}\rangle = \sum_{n=0,1,\ldots} \sum_{d_h(n)} |n, d_h(n)\rangle_L \overline{|n, d_h(n)\rangle}_R$$

- $h$ is determined by the representation of conformal symmetry. $p = (n + h)2\pi/L$ determines the momentum eigenvalues. $h$ is determined by the boundary condition of the cylinder, which labels the topological spin of the topological quasi-particles.
- Importantly, Ishibashi state is a **maximally entangled state** in the given subspace $\mathrm{V_h}$ labeled by the primary field $h$.



# General relation between energy spectrum and entanglement spectrum

- For a maximally entangled state, the reduced density matrix is proportional to identity, with maximal entropy, *i.e.,* infinite temperature

$$\rho_{L0} = Tr_R|G_{0h}\rangle\langle G_{0h}| = \sum_{n=0,1,\dots} \sum_{d_h(n)} |n, d_h(n)\rangle_L \langle n, d_h(n)|_L$$

- Moreover, the action of any operator to the right subspace is equivalent to some operator on the left subspace.

$$\begin{aligned}
&O_R|G_{0h}\rangle = O^T{}_L|G_{0h}\rangle \\
&H_R|G_{0h}\rangle = H_L|G_{0h}\rangle \\
&\Rightarrow \quad |G_h\rangle = Z^{-1/2} e^{-\tau_0(H_L+H_R)}|G_{0h}\rangle \\
&\qquad\qquad = \textstyle\sum_{n=0,1,..} \sum_{d_h(n)} e^{-2\tau_0 v(n+h)2\pi/L} |n, d_h(n)\,\rangle_L |n, d_h(n)\rangle_R
\end{aligned}$$

# General relation between energy spectrum and entanglement spectrum

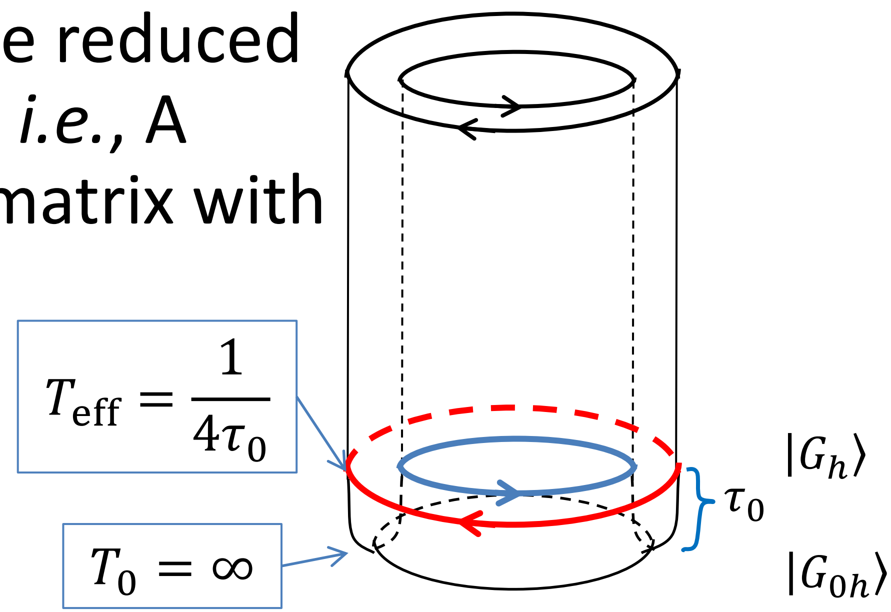
- The reduced density matrix of generic state $|G_h\rangle$ can be obtained which is thermal with a finite temperature

$$\rho_L = Tr_R|G_h\rangle\langle G_h| = \frac{1}{Z}e^{-4\tau_0 H_L}$$

- Thus we have shown how the reduced density matrix of left-mover, *i.e.*, A region, is a thermal density matrix with effective temperature

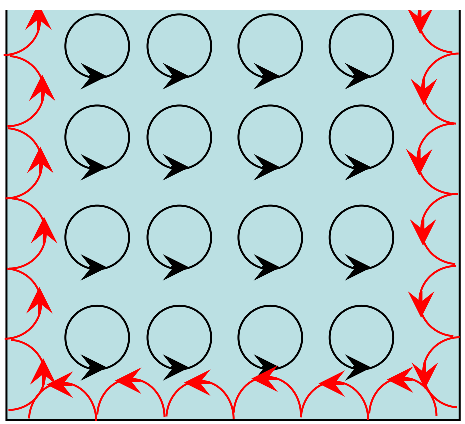
$$T_{\text{eff}} = 1/4\tau_0$$

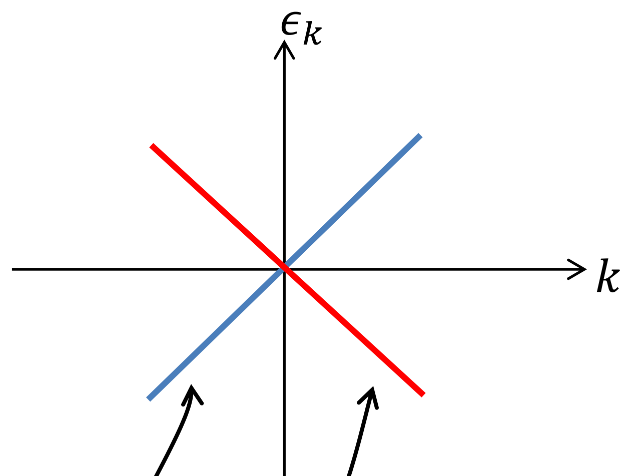
within a given conformal representation.

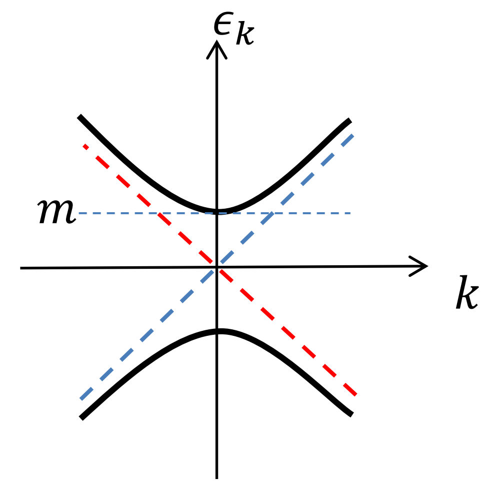


# Example: Free fermion theory

- As an explicit example, we study the free fermion edge states of integer quantum Hall state.






$$H = \sum_k vk(c_k^+ c_k - d_k^+ d_k) + \mathrm{m} \sum_k (c_k^+ d_k + d_k^+ c_k)$$

$$= \sum_{k,s=\pm} E_k \gamma_{ks}^+ \gamma_{ks}$$

- Ground state satisfies $\gamma_{ks}|G\rangle = 0$

# Example: Free fermion theory

- The ground state wavefunction can be written explicitly in an exponential form:

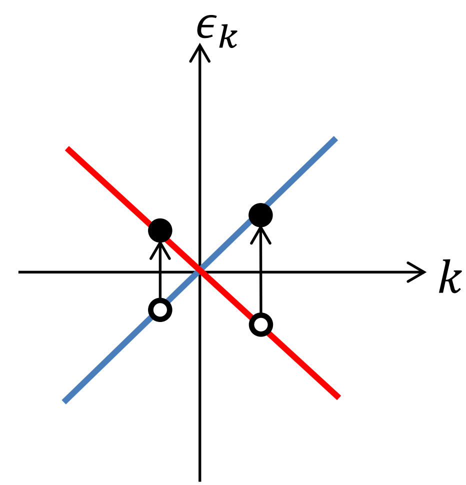
$$|G\rangle = \exp\left\{-\sum_{k>0}\frac{m}{\sqrt{v^2k^2+m^2}+vk}\left(c_k^\dagger d_k + d_{-k}^\dagger c_{-k}\right)\right\}|G_L\rangle\otimes|G_R\rangle$$

Coefficient→ 1 for $k \to 0$

Quasi-particle pair creation operators

- Compared to the Ishibashi state:

$$|G_0\rangle = \exp\left\{-\sum_{k>0}\left(c_k^\dagger d_k + d_{-k}^\dagger c_{-k}\right)\right\}|G_L\rangle\otimes|G_R\rangle$$




- $|G\rangle = Z^{-1/2}e^{-H_E}|G_0\rangle$ with $H_E \simeq \frac{1}{2m}(H_L+H_R)$
- i.e., $\tau_0 = 1/2m$

# Discussion

- According to our result, the entanglement entropy is given by thermal entropy of CFT, which correctly recovers the topological entanglement entropy. (Kitaev&Preskill 2006)
- This approach also applies to the entanglement between coupled one-dimensional systems, each of which is described by a CFT, such as the coupled Heisenburg chain (Poilblanc 1005.2123) and coupled Luttinger liquid (Y.-B. Kim 1009.3016)
- In more generic situation,  non-topological edge states can appear on the edge, such as by edge reconstruction in quantum Hall states. In this case, the entanglement Hamiltonian $H_E$ and the physical edge Hamiltonian $H_{edge}$ are not identical, but still topologically equivalent.

# Conclusion

- By a "cut and glue" approach, we have shown that the entanglement spectrum in a generic (2+1) dimensional topological state with chiral edge states is described by the same conformal field theory as the edge states.
- **Therefore we have demonstrated Li&Haldane's conjecture on the entanglement-edge theory correspondence.**
- When two gapless systems are coupled, there is a "trade off" between **"low energy"** and **"strong entanglement".** The low energy states are coupled and become gapped while the entanglement spectrum becomes gapless.
- Gapless entanglement spectrum can be used as a signature of topological states.
- Such relation between entanglement and dynamics should also apply to higher dimensional topological states.