Dynamics and thermalization in isolated one-dimensional systems

Marcos Rigol

Department of Physics
Georgetown University

Disentangling Quantum Many-body Systems:
Computational and Conceptual Approaches

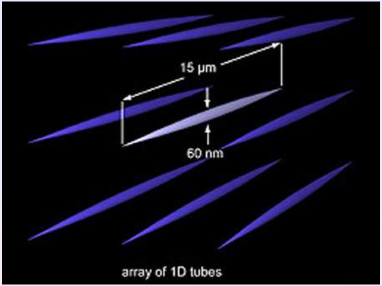
Kavli Institute for Theoretical Physics
November 2, 2010

Outline

1. Introduction
   - Experiments and numerical simulations
2. Non-equilibrium dynamics in one-dimension
   - Quantum mechanics
   - Time evolution vs exact time average
   - Statistical description after relaxation
   - Eigenstate thermalization hypothesis
3. Integrable systems
   - Generalized Gibbs ensemble (GGE)
   - Time evolution, time average, and diagonal ensemble
   - Statistical description after relaxation
   - Eigenstate expectation values and ETH
4. Summary

Outline

Experiments in the 1D regime



Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$

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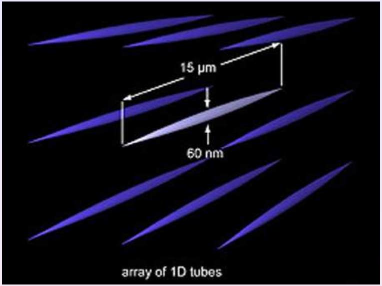

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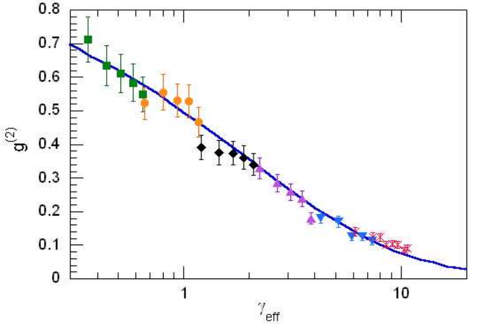
Girardeau '60

T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).

T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).

$$\gamma_{eff} = \frac{m g_{1D}}{\hbar^2 \rho}$$



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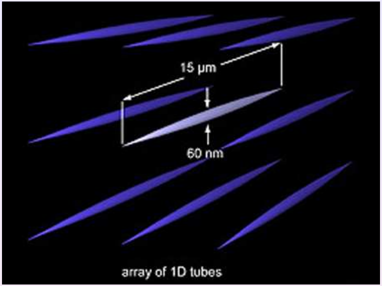

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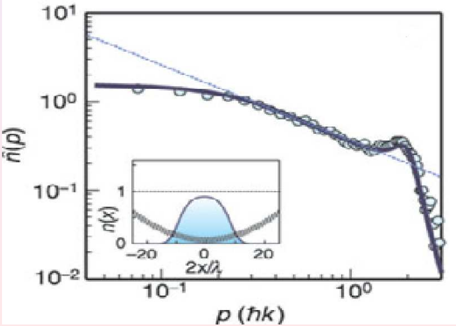
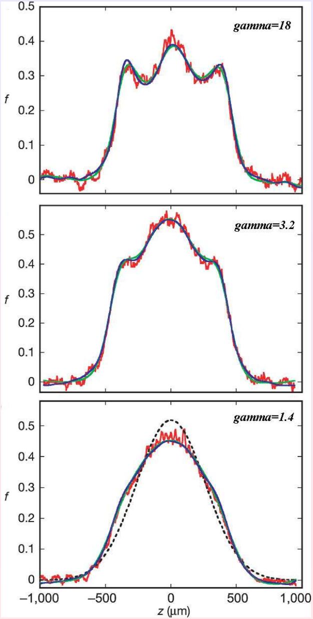
Lieb, Schulz, and Mattis '61

B. Paredes *et al.*,
Nature (London) **429**, 277 (2004).

$n(x)$: Density distribution
$n(p)$: Momentum distribution



Absence of thermalization in 1D?



$$\gamma_{eff} = \frac{m g_{1D}}{\hbar^2 \rho}$$

g_{1D}: 1D scattering length

ρ: Density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

If $\gamma \ll 1$ the system is in the weakly interacting regime

Kinoshita, Wenger, and Weiss, Nature **440**, 900 (2006).

Also in: Hofferberth, Lesanovsky, Fischer, Schumm, and Schmiedmayer, Nature **449**, 324 (2007).

Absence of thermalization in 1D numerical simulations

Hard-core bosons (integrable)

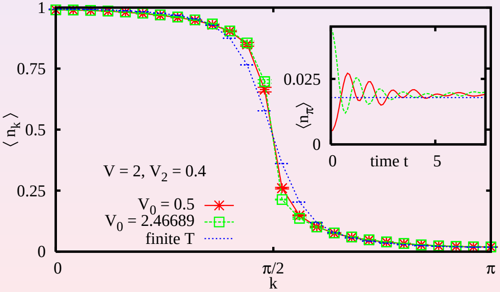
MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Spinless fermions Hamiltonian

$$H = -t \sum_j \left(c^\dagger_{j+1} c_j + \text{H.c.} \right) + V \sum_j n_j n_{j+1} + V_2 \sum_j n_j n_{j+2}$$

S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu, PRL **98**, 210405 (2007).

Momentum distribution function



Soft-core bosons

C. Kollath, A. Läuchli, and E. Altman, PRL **98**, 180601 (2007).

Outline

Thermalization in quantum systems

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_I\rangle \neq |\Psi_\alpha\rangle \quad \text{where} \quad \widehat{H}|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle \quad \text{and} \quad E_0 = \langle\psi_I|\widehat{H}|\psi_I\rangle,$$

then a generic observable O will evolve in time following

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_I\rangle.$$

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Will a generic O in a generic system thermalize?

$$\overline{O(\tau)} = O(E_0) = O(T).$$

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One can rewrite

$$O(\tau) = \sum_{\alpha',\alpha} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'}-E_\alpha)\tau} O_{\alpha'\alpha} \quad \text{where} \quad |\psi_I\rangle = \sum_\alpha C_\alpha|\Psi_\alpha\rangle,$$

and taking the infinite time average (diagonal ensemble)

$$\overline{O(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle = \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{diag}},$$

which depends on the initial conditions through $C_\alpha = \langle\Psi_\alpha|\psi_I\rangle$.

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Relaxation dynamics of hard-core boson in 1D

Hardcore bosons in one dimension

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

MR, Phys. Rev. Lett. **103**, 100403 (2009); Phys. Rev. A **80**, 053607 (2009).

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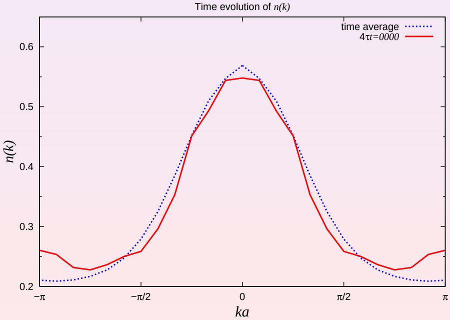
Nonequilibrium dynamics in 1D



$N = 8$ bosons

$L = 24$ lattice sites

Hilbert space: $H = 735,471$
Largest k-sector: $D = 30,667$

Fix $t' = V'$ and quench
$t_{ini} = 0.5, V_{ini} = 2$
$\rightarrow t_{fin} = 1, V_{fin} = 1$

All $k = 0$ states are used!

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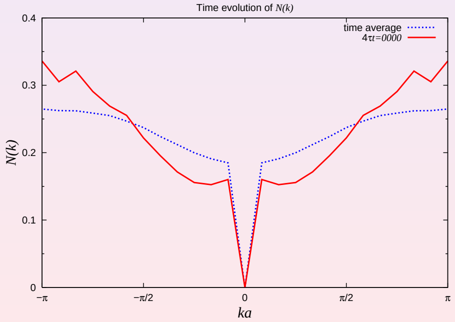

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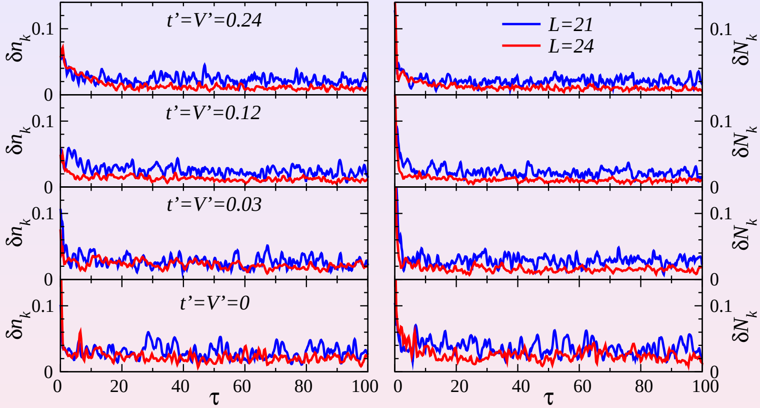
Time evolution and scaling with system size



Relative differences

$$\delta n_k(\tau) = \frac{\sum_k |n(k,\tau) - n_{diag}(k)|}{\sum_k n_{diag}(k)}$$

Effective temperature $T = 3.0$

$$E = Z^{-1}\mathrm{Tr}\left\{\hat{H}\exp(-\hat{H}/k_B T)\right\}$$

Outline

Statistical description after relaxation (nonintegrable)

Canonical calculation

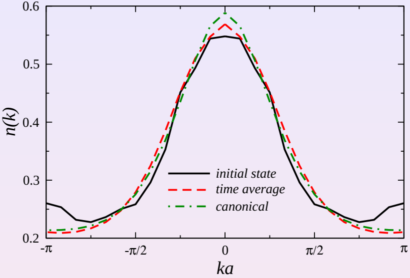
$$A = \mathrm{Tr}\left\{\hat{A}\hat{\rho}\right\}$$

$$\hat{\rho} = Z^{-1}\exp\left(-\hat{H}/k_B T\right)$$

$$Z = \mathrm{Tr}\left\{\exp\left(-\hat{H}/k_B T\right)\right\}$$

$$E_0 = \mathrm{Tr}\left\{\hat{H}\hat{\rho}\right\} \qquad T = 3.0J$$

Momentum distribution



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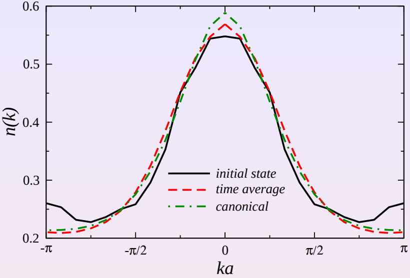
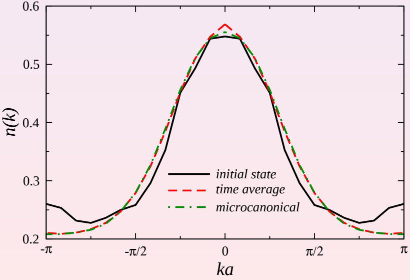




Microcanonical calculation

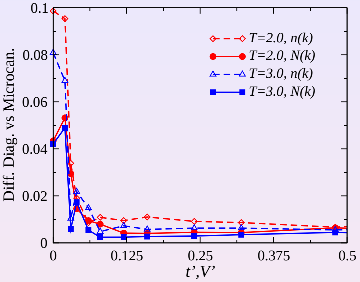
$$A = \frac{1}{N_{states}} \sum_{\alpha} \langle \Psi_\alpha | \hat{A} | \Psi_\alpha \rangle$$

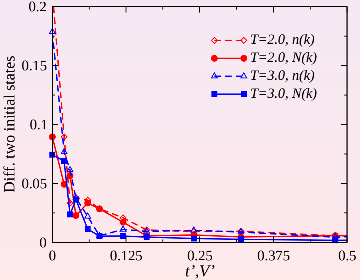
with $E_0 - \Delta E < E_\alpha < E_0 + \Delta E$

N_{states} : # of states in the window

Breakdown of thermalization

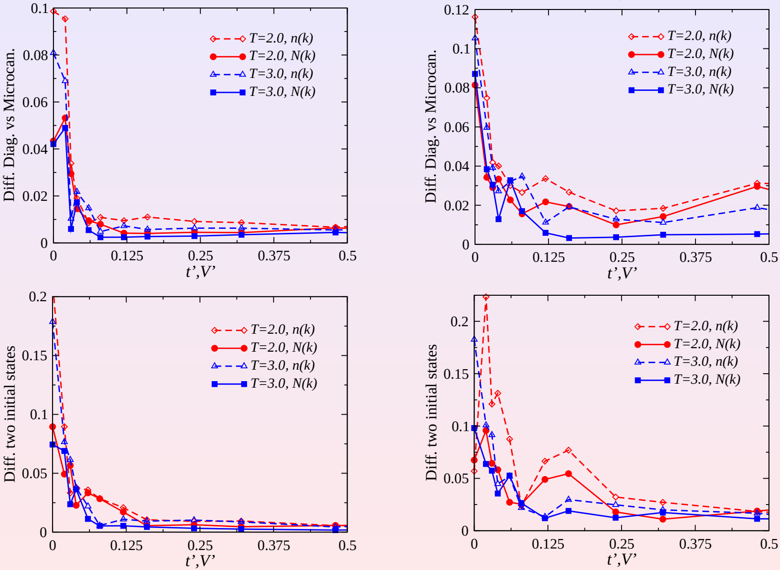
$L = 24,\, N = 8$





Breakdown of thermalization

$L = 24,\ N = 8$ $\qquad$ $L = 21,\ N = 7$



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Eigenstate thermalization hypothesis

Paradox?

$$\sum_{\alpha} |C_\alpha|^2 A_{\alpha\alpha} = \langle A \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_\alpha| < \Delta E} A_{\alpha\alpha}$$

Left hand side: Depends on the initial conditions through $C_\alpha = \langle \Psi_\alpha | \psi_I \rangle$
Right hand side: Depends only on the initial energy

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i) For physically relevant initial conditions, $|C_\alpha|^2$ practically do not fluctuate.

ii) Large (and uncorrelated) fluctuations occur in both $A_{\alpha\alpha}$ and $|C_\alpha|^2$. Any physically relevant initial state performs an unbiased sampling of $A_{\alpha\alpha}$.

Paradox?

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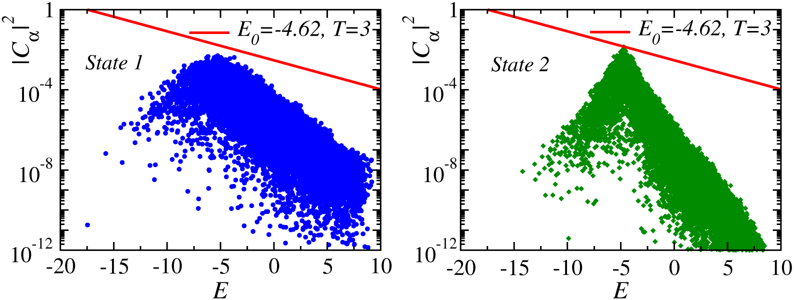

MR, PRA **82**, 037601 (2010).

Eigenstate thermalization hypothesis

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Eigenstate thermalization hypothesis (ETH)

[Deutsch, PRA **43** 2046 (1991); Srednicki, PRE **50**, 888 (1994); Rigol, Dunjko, and Olshanii, Nature **452**, 854 (2008).]

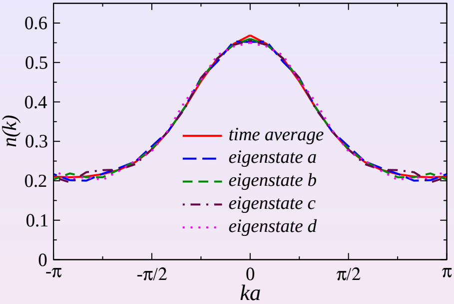
iii) The expectation value $\langle \Psi_\alpha | \hat{A} | \Psi_\alpha \rangle$ of a few-body observable $\widehat{A}$ in an eigenstate of the Hamiltonian $|\Psi_\alpha\rangle$, with energy E_α, of a large interacting many-body system equals the thermal average of $\widehat{A}$ at the mean energy E_α:

$$\langle \Psi_\alpha | \hat{A} | \Psi_\alpha \rangle = \langle A \rangle_{\text{microcan.}}(E_\alpha)$$

ETH – far away from integrability ($t' = V' = 0.24$)

Momentum distribution

Eigenstates $a - d$ are the ones with energies closest to E_0



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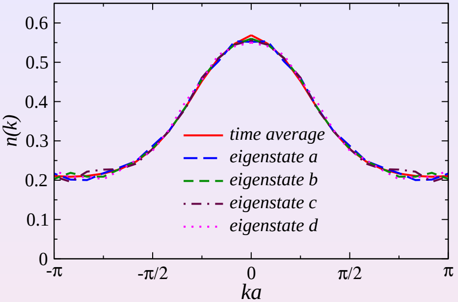

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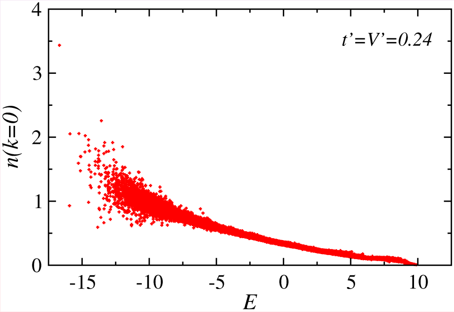


$n(k_x = 0)$ vs energy

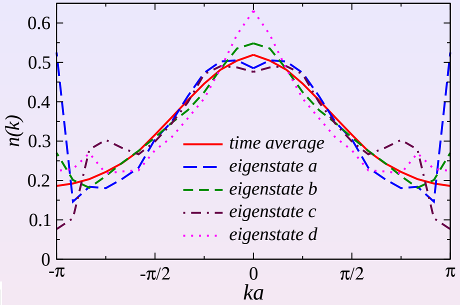
$\rho(E) = \mathrm{P(E)} \times$ dens. stat.

$P(E)_{exact} \rightarrow |C_\alpha|^2$

$P(E)_{mic.} \rightarrow$ constant

$P(E)_{can.} \rightarrow \exp\left(-E/k_B T\right)$

Breakdown of ETH $\longrightarrow$ integrability ($t' = V' = 0.03$)



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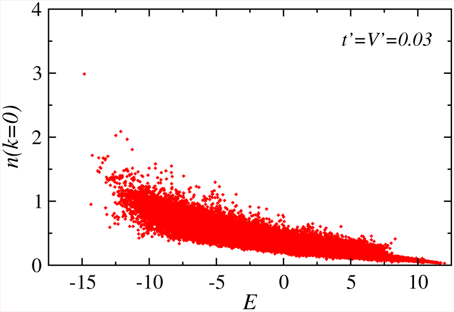

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$P(E)_{mic.} \rightarrow \text{constant}$

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Outline

Bose-Fermi mapping

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i \mu_i \, \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

$$\Downarrow$$

Map to spins and then to fermions (Jordan-Wigner transformation)

$$\sigma_i^+ = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \sigma_i^- = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$

$$\Downarrow$$

Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + \sum_i \mu_i \, \hat{n}_i^f$$

One-particle density matrix

One-particle Green's function

$$G_{ij} = \langle \Psi_{HCB} | \sigma_i^- \sigma_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$

$$\Downarrow$$

Time evolution

$$|\Psi_F(\tau)\rangle = e^{-i\hat{H}_F \tau/\hbar} |\Psi_F^I\rangle = \prod_{\delta=1}^{N} \sum_{\sigma=1}^{L} P_{\sigma\delta}(\tau) \hat{f}_\sigma^\dagger \, |0\rangle$$

$$\Downarrow$$

Exact Green's function

$$G_{ij}(\tau) = \det \left[\left(\mathbf{P}^l(\tau) \right)^\dagger \mathbf{P}^r(\tau) \right]$$

Computation time $\sim L^2 N^3 \rightarrow$ study very large systems

3000 lattice sites, 300 particles

MR and A. Muramatsu, PRL **93**, 230404 (2004); PRL **94**, 240403 (2005).

Finite temperature

One-particle density matrix (grand-canonical ensemble)

$$\rho_{ij} \equiv \frac{1}{Z}\mathrm{Tr}\left\{\hat{b}_i^\dagger \hat{b}_j e^{-\frac{\hat{H}_{HCB}-\mu\sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}}\right\}, \quad Z = \mathrm{Tr}\left\{e^{-\frac{\hat{H}_{HCB}-\mu\sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}}\right\}$$

$$\Downarrow$$

Mapping to noninteracting fermions

$$\rho_{ij} = \frac{1}{Z}\mathrm{Tr}\left\{\hat{f}_i^\dagger \hat{f}_j \prod_{k=1}^{j-1} e^{i\pi \hat{f}_k^\dagger \hat{f}_k} e^{-\frac{\hat{H}_F-\mu\sum_m \hat{f}_m^\dagger \hat{f}_m}{k_B T}} \prod_{l=1}^{i-1} e^{-i\pi \hat{f}_l^\dagger \hat{f}_l}\right\}$$

$$\Downarrow$$

Exact one-particle density matrix

$$\begin{aligned}\rho_{ij} \quad = \quad & \frac{1}{Z}\left\{\det\left[\mathbf{I} + (\mathbf{I}+\mathbf{A})\mathbf{O}_1 \mathbf{U} e^{-(\mathbf{E}-\mu\mathbf{I})/k_B T}\mathbf{U}^\dagger \mathbf{O}_2\right]\right. \\ & \left. -\det\left[\mathbf{I} + \mathbf{O}_1 \mathbf{U} e^{-(\mathbf{E}-\mu\mathbf{I})/k_B T}\mathbf{U}^\dagger \mathbf{O}_2\right]\right\}\end{aligned}$$

Computation time $\sim L^5$: 1000 sites

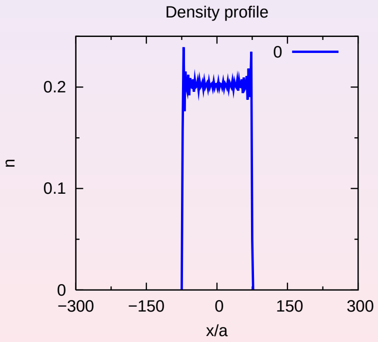
MR, PRA **72**, 063607 (2005).

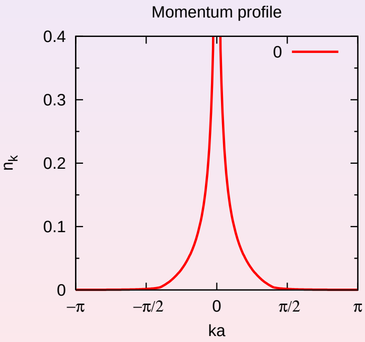
Relaxation dynamics in an integrable system

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i \mu_i \, \hat{n}_i$$

Constraints on the bosonic operators: $\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$





MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Thermal equilibrium

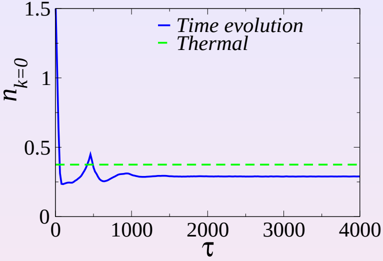
$$\hat{\rho} = Z^{-1} \exp\left[-\left(\hat{H} - \mu\hat{N}\right)/k_B T\right]$$

$$Z = \mathrm{Tr}\left\{\exp\left[-\left(\hat{H} - \mu\hat{N}\right)/k_B T\right]\right\}$$

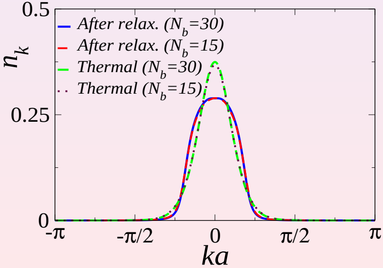
$$E = \mathrm{Tr}\left\{\hat{H}\hat{\rho}\right\}, \quad N = \mathrm{Tr}\left\{\hat{N}\hat{\rho}\right\}$$

MR, PRA **72**, 063607 (2005).

Evolution of $n_{k=0}$



n_k after relaxation



Statistical description after relaxation

Thermal equilibrium

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MR, PRA **72**, 063607 (2005).

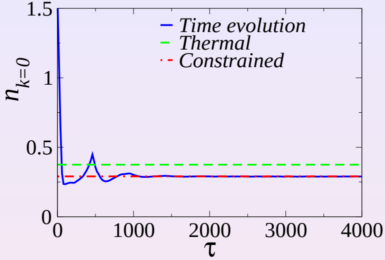
Constrained equilibrium

$$\hat{\rho}_c = Z_c^{-1} \exp\left[-\sum_m \lambda_m \hat{I}_m\right]$$

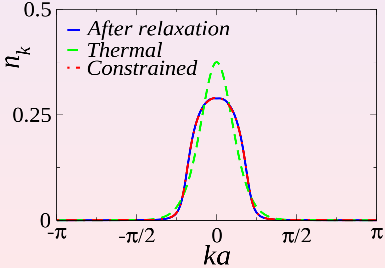
$$Z_c = \mathrm{Tr}\left\{\exp\left[-\sum_m \lambda_m \hat{I}_m\right]\right\}$$

$$\langle \hat{I}_m \rangle_{\tau=0} = \mathrm{Tr}\left\{\hat{I}_m \hat{\rho}_c\right\}$$

Evolution of $n_{k=0}$



n_k after relaxation



Generalized thermalization in integrable systems

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_0\rangle \neq |\alpha\rangle \quad \text{where} \quad \widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E_0 = \langle\psi_0|\widehat{H}|\psi_0\rangle,$$

then a generic observable O will evolve in time following

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_0\rangle.$$

What is it that we call generalized thermalization?

$$\overline{O(\tau)} = O(I_1, \dots, I_L).$$

One can rewrite

$$O(\tau) = \sum_{\alpha',\alpha} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'} - E_\alpha)\tau} O_{\alpha'\alpha} \quad \text{where} \quad |\psi_0\rangle = \sum_\alpha C_\alpha|\alpha\rangle,$$

and taking the infinite time average (diagonal ensemble?)

$$\overline{O(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle \stackrel{?}{=} \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{diag}},$$

which depends on the initial conditions through $C_\alpha = \langle\alpha|\psi_0\rangle$.

Outline

Hamiltonian and numerical calculations

Hard-core boson Hamiltonian

$$H = -J\sum_{i=1}^{L-1}\left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.}\right) + V(\tau)\sum_{i=1}^{L}(i - L/2)^2 \hat{n}_i$$

A. C. Cassidy, C. W. Clark, and MR, arXiv:1008.4794.

Hamiltonian and numerical calculations

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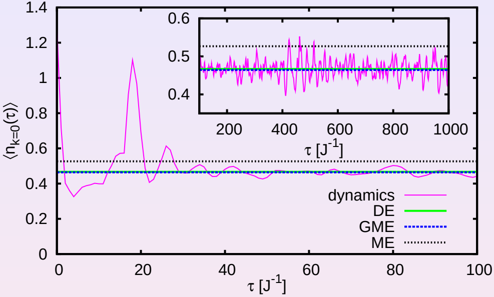
A. C. Cassidy, C. W. Clark, and MR, arXiv:1008.4794.

Numerical calculations

- $V(\tau < 0) = V_0$, $V(\tau \geq 0) = 0$, and $n = N/L = 0.2$.
- The initial state $|\Psi_0\rangle$ is the ground state for $\tau < 0$.
- The dynamics and GGE results are obtained in polynomial time.
- For all other calculations, the $|\alpha\rangle$'s are generated from single particle states ($L = 50$ and $N = 10 \Rightarrow 10^{10}$ states).
- For the microcanonical and canonical ensembles the usual weights are used.
- For the diagonal ensemble $C_\alpha = \det \left[\mathbf{P}_\alpha^\dagger \mathbf{P}_0 \right]$.

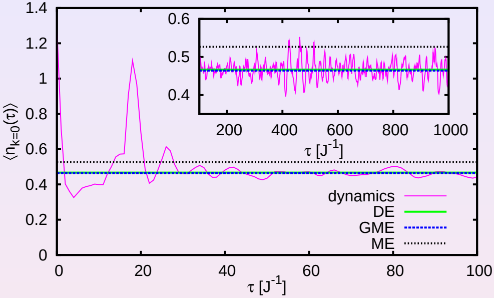
Time evolution, time average, and diagonal ensemble

Evolution of $\langle \hat{n}_{k=0} \rangle$



Time evolution, time average, and diagonal ensemble

Evolution of $\langle \hat{n}_{k=0} \rangle$



Error

$$(\Delta n_k)_\tau = \frac{\sum_k |\langle \hat{n}_k \rangle_{\text{DE}} - \overline{n_k}|}{\sum_k \langle \hat{n}_k \rangle_{\text{DE}}}$$

where

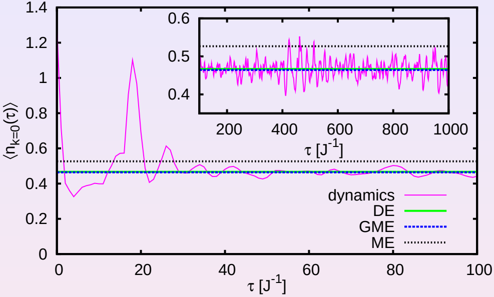
$$\overline{n_k} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} d\tau \langle \hat{n}_k(\tau) \rangle$$

Fluctuations

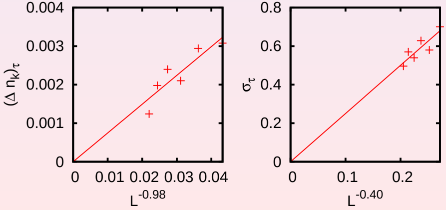
$$\sigma_\tau = \sum_k \sqrt{\overline{n_k^2} - \overline{n_k}^2}$$

Time evolution, time average, and diagonal ensemble

Evolution of $\langle \hat{n}_{k=0} \rangle$



Scaling



Error

$$(\Delta n_k)_\tau = \frac{\sum_k |\langle \hat{n}_k \rangle_{\text{DE}} - \overline{n_k}|}{\sum_k \langle \hat{n}_k \rangle_{\text{DE}}}$$

where

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Fluctuations

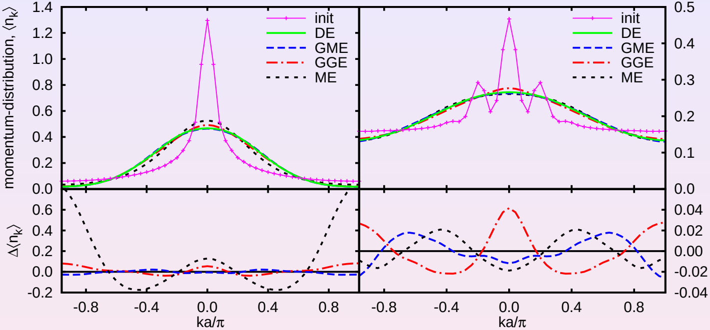
$$\sigma_\tau = \sum_k \sqrt{\overline{n_k^2} - \overline{n_k}^2}$$

See also: Kollar and Eckstein, PRA **78**, 013626 (2008).

Outline

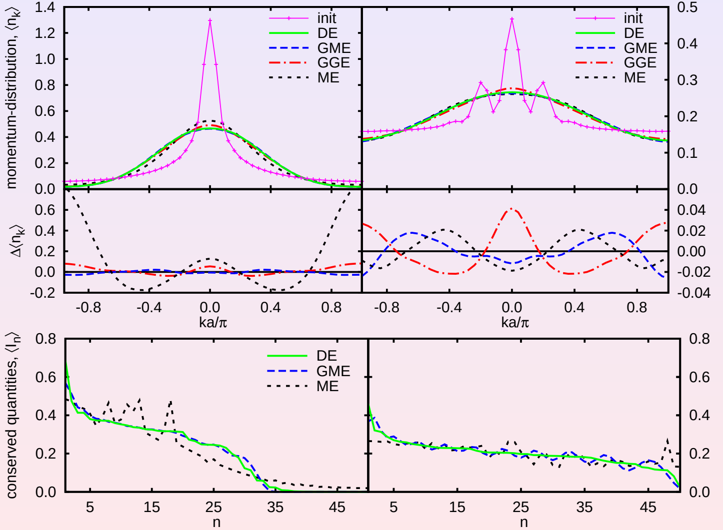
Statistical description after relaxation

Results for n_k and the integrals of motion



Statistical description after relaxation

Results for n_k and the integrals of motion



Generalized microcanonical ensemble (GME)

- GME idea, assign equal weight to all eigenstates whose values of the conserved quantities are close to a target distribution

- GME ingredients

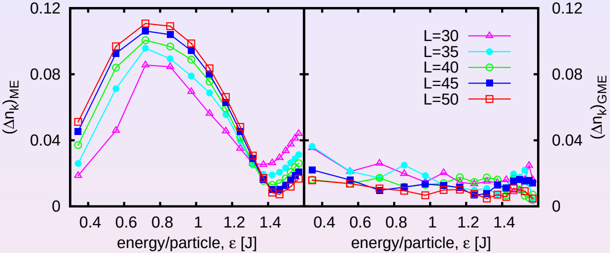
  1. Ordered distribution of conserved quantities in the initial state I_n
  2. A target distribution of conserved quantities $\{I^*_{n^*_i} = 1\}$, with n^*_i ($i = 1, \ldots, N$) computed to describe I_n in a coarse grained sense
  3. The distance of each individual many-body eigenstate from the target state $\delta^\alpha = \left[\frac{1}{N}\sum_{i=1}^{N} I_{n^*_i}(n^\alpha_i - n^*_i)^2\right]^{1/2}$

- Target distribution (the n^*_i's are not restricted to integer values)

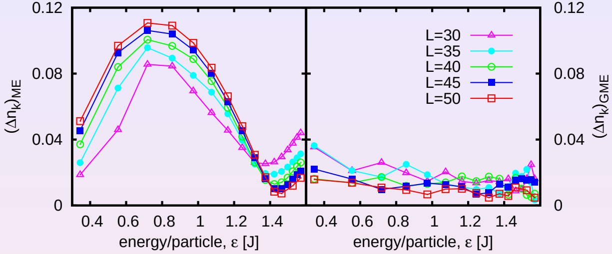
  1. n^*_1 is computed so that $\int_{0.5}^{n^*_1} I(x)dx = 0.5$, where $I(x) = I_n, x \in (n - 1/2, n + 1/2]$
  2. All other values of n^*_i are computed so that $\int_{n^*_{i-1}}^{n^*_i} I(x)dx = 1$

Finite size scaling

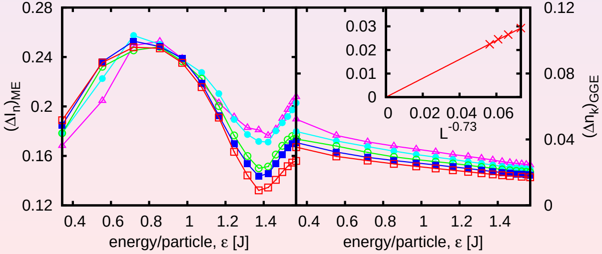
Results for Δn_k in the microcanonical ensembles



Results for Δn_k in the microcanonical ensembles



Integrals of motion and the GGE

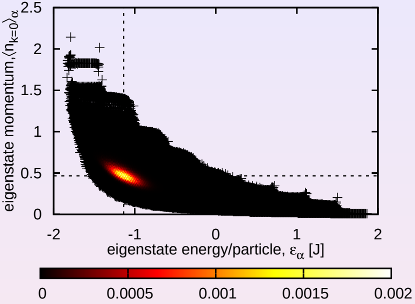


Outline

1. Introduction
   - Experiments and numerical simulations
2. Non-equilibrium dynamics in one-dimension
   - Quantum mechanics
   - Time evolution vs exact time average
   - Statistical description after relaxation
   - Eigenstate thermalization hypothesis
3. Integrable systems
   - Generalized Gibbs ensemble (GGE)
   - Time evolution, time average, and diagonal ensemble
   - Statistical description after relaxation
   - Eigenstate expectation values and ETH
4. Summary

Eigenstate expectation values and scaling

Weights in the diagonal ensemble



Eigenstate expectation values and scaling

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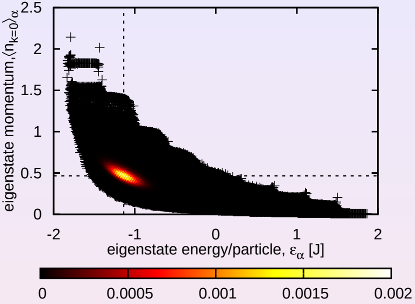
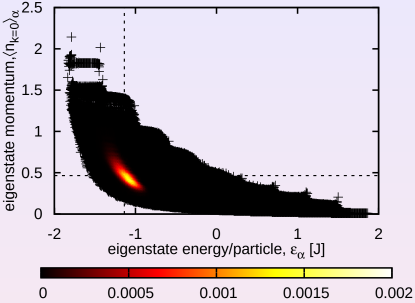
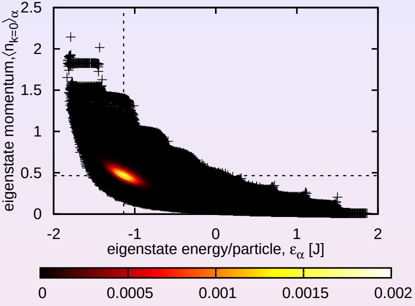


Weights in the GME

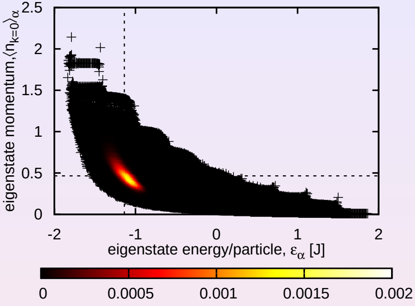


Eigenstate expectation values and scaling

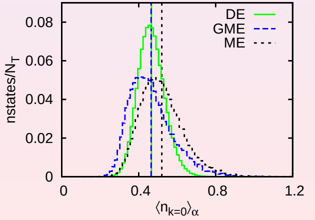
Weights in the diagonal ensemble



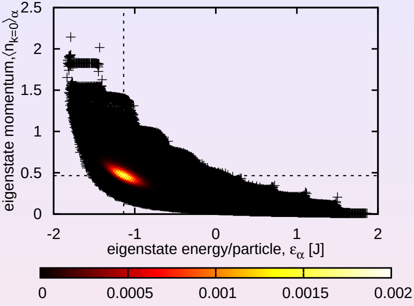
Weights in the GME



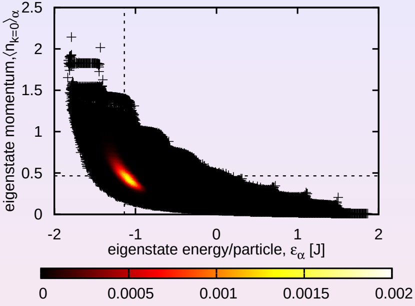
Histogram



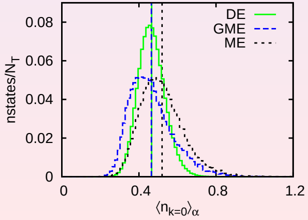
Weights in the diagonal ensemble



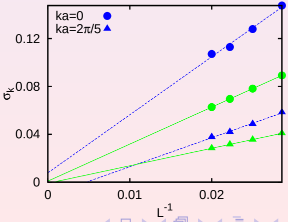
Weights in the GME



Histogram



Scaling of σ_k



Eigenstate thermalization hypothesis

Eigenstate thermalization hypothesis (ETH)

[Deutsch, PRA **43** 2046 (1991); Srednicki, PRE **50**, 888 (1994); Rigol, Dunjko, and Olshanii, Nature **452**, 854 (2008).]

- The expectation value $\langle\alpha|\widehat{O}|\alpha\rangle$ of a few-body observable $\widehat{O}$ in an eigenstate of the Hamiltonian $|\alpha\rangle$, with energy E_α, of a many-body system equals the thermal average of $\widehat{O}$ at the mean energy E_α:

$$\langle\alpha|\widehat{O}|\alpha\rangle = \langle\widehat{O}\rangle_{\text{microcan.}}(E_\alpha)$$

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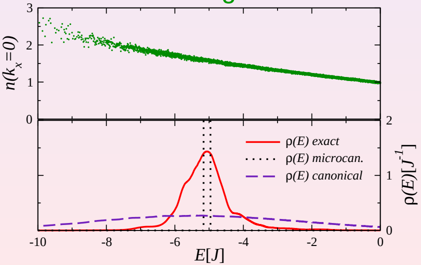
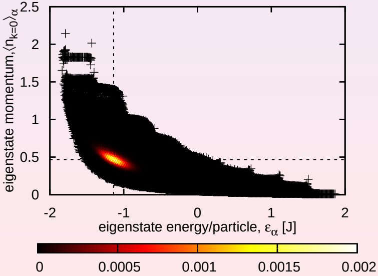
Nonintegrable



Integrable



Summary

- There in thermalization far away from integrability
  - ★ Finite size effects
- Eigenstate thermalization hypothesis
  - ★ $\langle \Psi_\alpha | \hat{A} | \Psi_\alpha \rangle = \langle A \rangle_{\text{microcan.}}(E_\alpha)$
- Thermalization and ETH break down close integrability (finite system)
  - ★ KAM in the thermodynamic limit?
- In integrable systems observables after relaxation can be described by generalized statistical ensembles (GGE and GME)
  - ★ The number of constraints increases polynomially with system size while the Hilbert space increases exponentially with system size
- "Updated" ensembles have their origin in a generalized view of ETH: eigenstates with similar integrals of motion have similar observables
  - ★ Typicality and thermodynamics for isolated integrable systems?

Collaborators

- Amy C. Cassidy (NIST)
- Charles W. Clark (NIST)
- Vanja Dunjko (U Mass Boston)
- Alejandro Muramatsu (U Stuttgart)
- Maxim Olshanii (U Mass Boston)
- Lea F. Santos (Yeshiva U)
- Vladimir Yurovsky (Tel Aviv U)

Time fluctuations

Are they small because of dephasing?

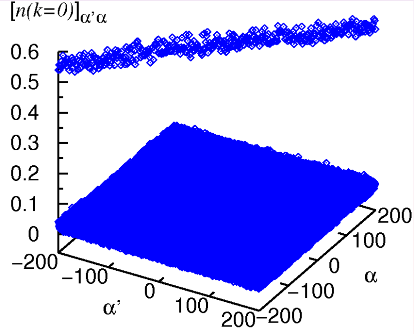
$$\langle \hat{A}(t) \rangle - \overline{\langle \hat{A}(t) \rangle} = \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} C^{\star}_{\alpha'} C_{\alpha} e^{i(E_{\alpha'}-E_{\alpha})t} A_{\alpha'\alpha} \sim \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} \frac{e^{i(E_{\alpha'}-E_{\alpha})t}}{N_{\text{states}}} A_{\alpha'\alpha}$$

$$\sim \frac{\sqrt{N^2_{\text{states}}}}{N_{\text{states}}} A^{\text{typical}}_{\alpha'\alpha} \sim A^{\text{typical}}_{\alpha'\alpha}$$

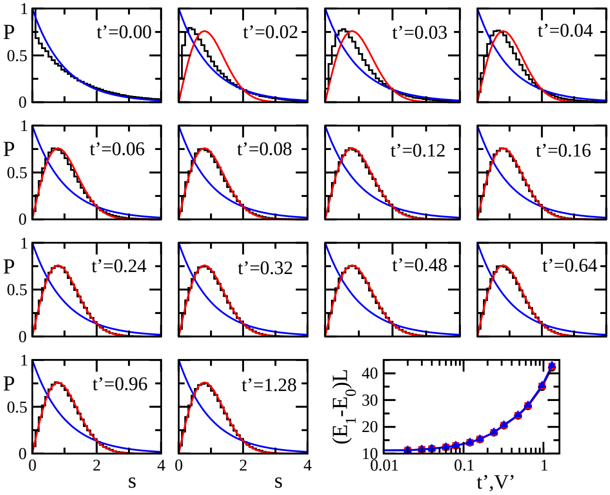
Time average of $\langle \hat{A} \rangle$

$$\overline{\langle \hat{A} \rangle} = \sum_{\alpha} |C_{\alpha}|^2 A_{\alpha\alpha}$$

$$\sim \sum_{\alpha} \frac{1}{N_{\text{states}}} A_{\alpha\alpha} \sim A^{\text{typical}}_{\alpha\alpha}$$

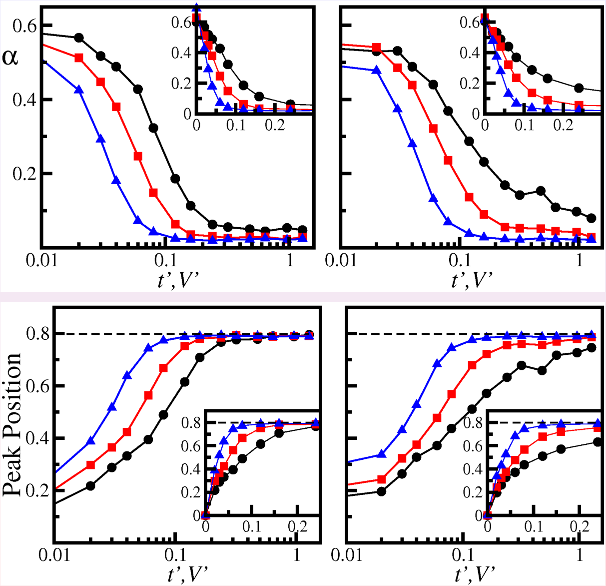


Level spacing distribution



L.F. Santos and MR, Phys. Rev. E **81**, 036206 (2010).

Scaling of the level spacing distribution



Left panels (bosons)

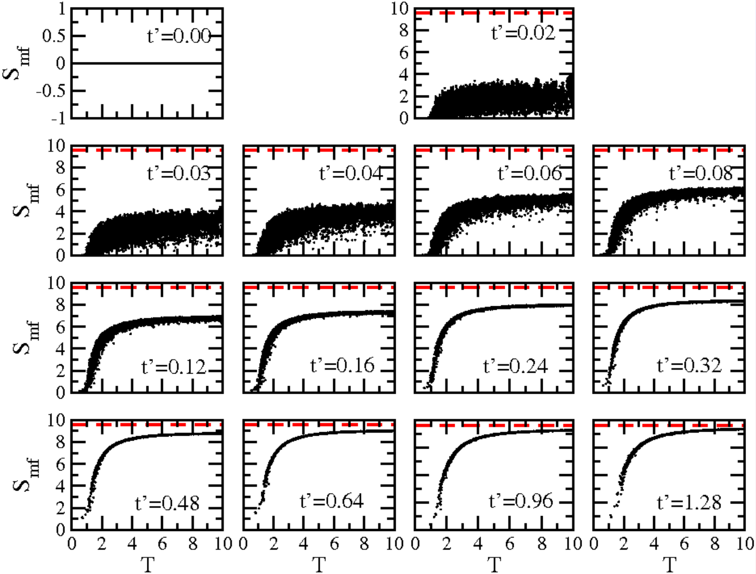
Right panels (fermions)

Black circles
$L = 18, \ N_p = 6$

Red squares
$L = 21, \ N_p = 7$

Blue triangles
$L = 24, \ N_p = 8$

Information entropy ($S_j \equiv -\sum_{k=1}^{D} |c_j^k|^2 \ln |c_j^k|^2$)



L.F. Santos and MR, Phys. Rev. E **81**, 036206 (2010).

Relation between thermalization and ETH

Quantifying ETH

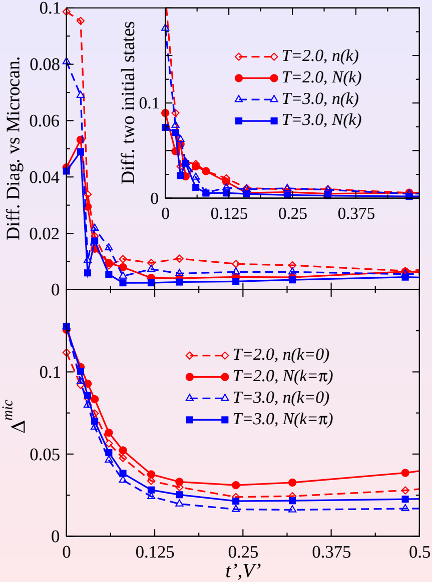
$$\Delta^{mic} O = \frac{\sum_\alpha |O_{\alpha\alpha} - O_{mic}|}{N_{states}\, O_{mic}}$$

$O_{\alpha\alpha}$: eigenstate expectation values of $\hat{O}$

O_{mic}: microcanonical expectation values of $\hat{O}$

The sum over α contains all states with energies in the window $[E - \Delta E, E + \Delta E]$, and N_{states} is the number of states in the sum ($\Delta E = 0.1$).

Observables of interest:
$n(k = 0)$ and $N(k = \pi)$



Statistical description after relaxation

Integrals of motion

(underlying noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger}|0\rangle = E_m \hat{\gamma}_m^{f\dagger}|0\rangle$$

$$\left\{\hat{I}_m^f\right\} = \{\hat{\gamma}_m^{f\dagger}\hat{\gamma}_m^f\}$$

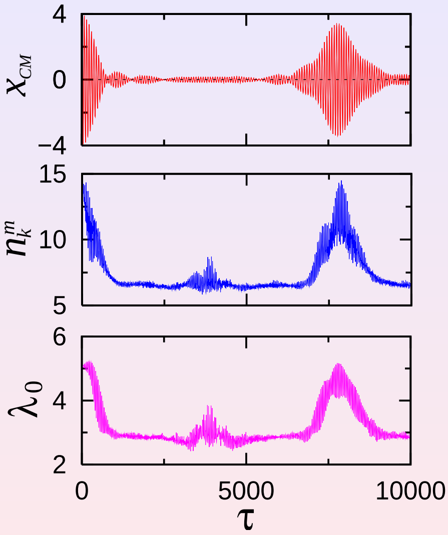
Lagrange multipliers

$$\lambda_m = \ln\left[\frac{1-\langle\hat{I}_m\rangle_{\tau=0}}{\langle\hat{I}_m\rangle_{\tau=0}}\right]$$

Other examples in:

- M. A. Cazalilla, PRL **97**, 156403, (2006).
- P. Calabrese and J. Cardy, J. Stat. Mech.: Theory Exp., P06008 (2007).
- M. Cramer *et al.*, PRL **100**, 030602 (2008).
- T. Barthel and U. Schollwöck, PRL **100**, 100601 (2008).
- M. Eckstein and M. Kollar, PRL **100**, 120404 (2008).
- M. Kollar and M. Eckstein, PRA **78**, 013626 (2008).
- A. Flesch *et al.*, PRA **78**, 033608 (2008).
- A. Iucci and M. A. Cazalilla, PRA **80**, 063619 (2009).
- D. Fioretto and G. Mussardo, NJP **12**, 055015 (2010).
- J. Mossel and J.-S. Caux, NJP **12**, 055028 (2010).

Poincaré recurrences?



MR *et al.*, Phys. Rev. Lett. **95**, 110402 (2005).