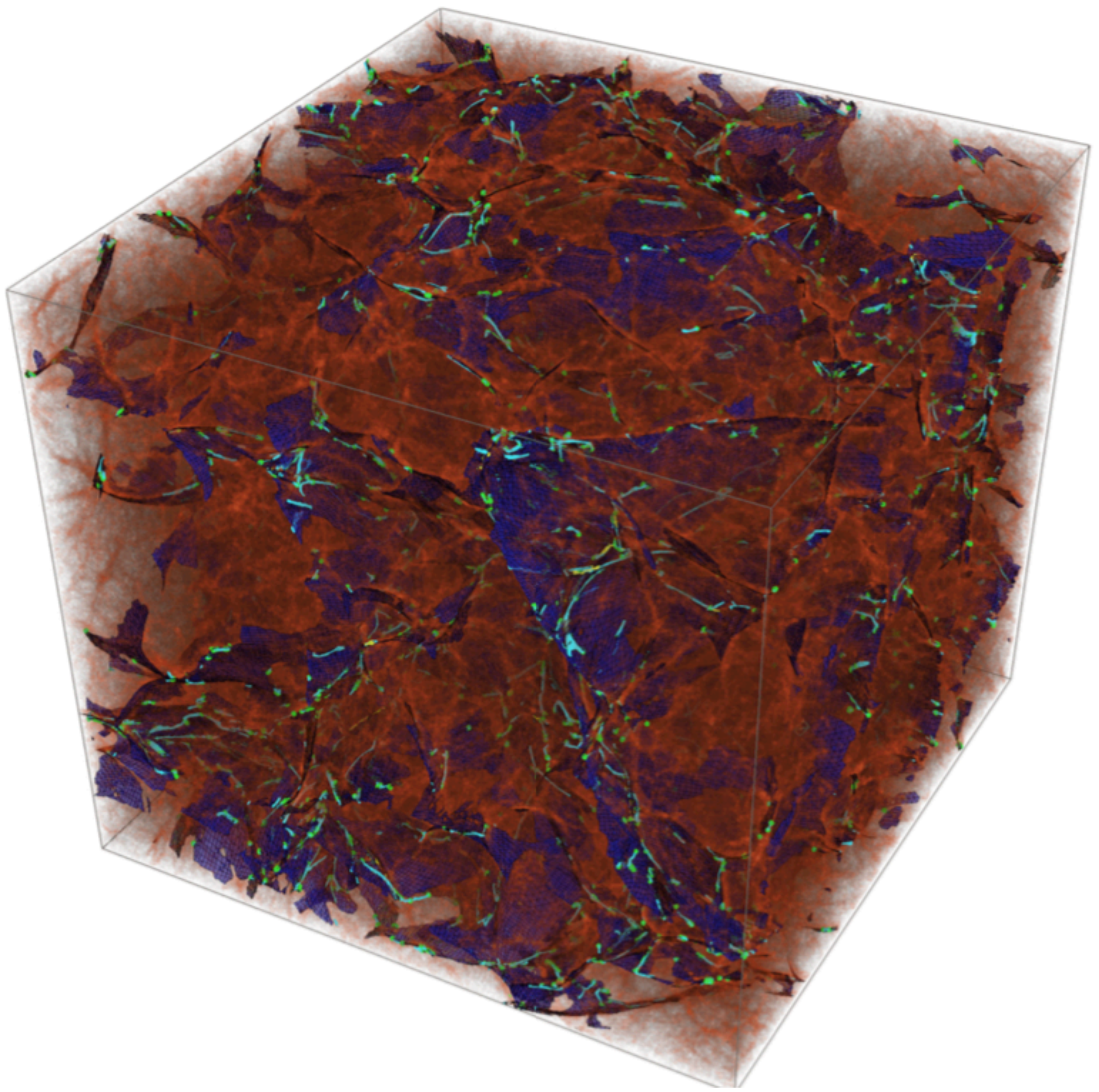
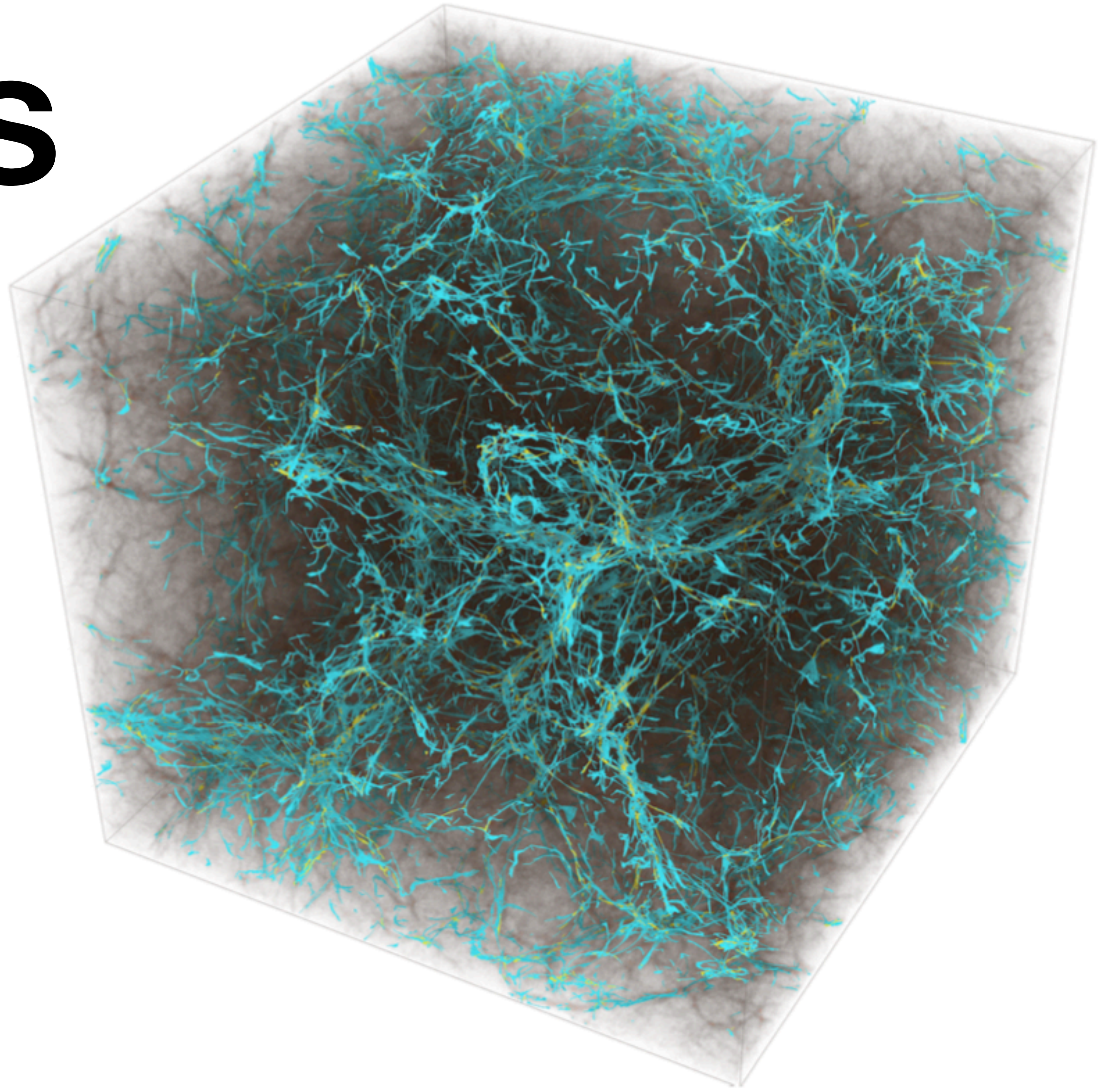
Dissecting the cosmic web with caustics





Job Feldbrugge
University of Edinburgh

6th of February 2023

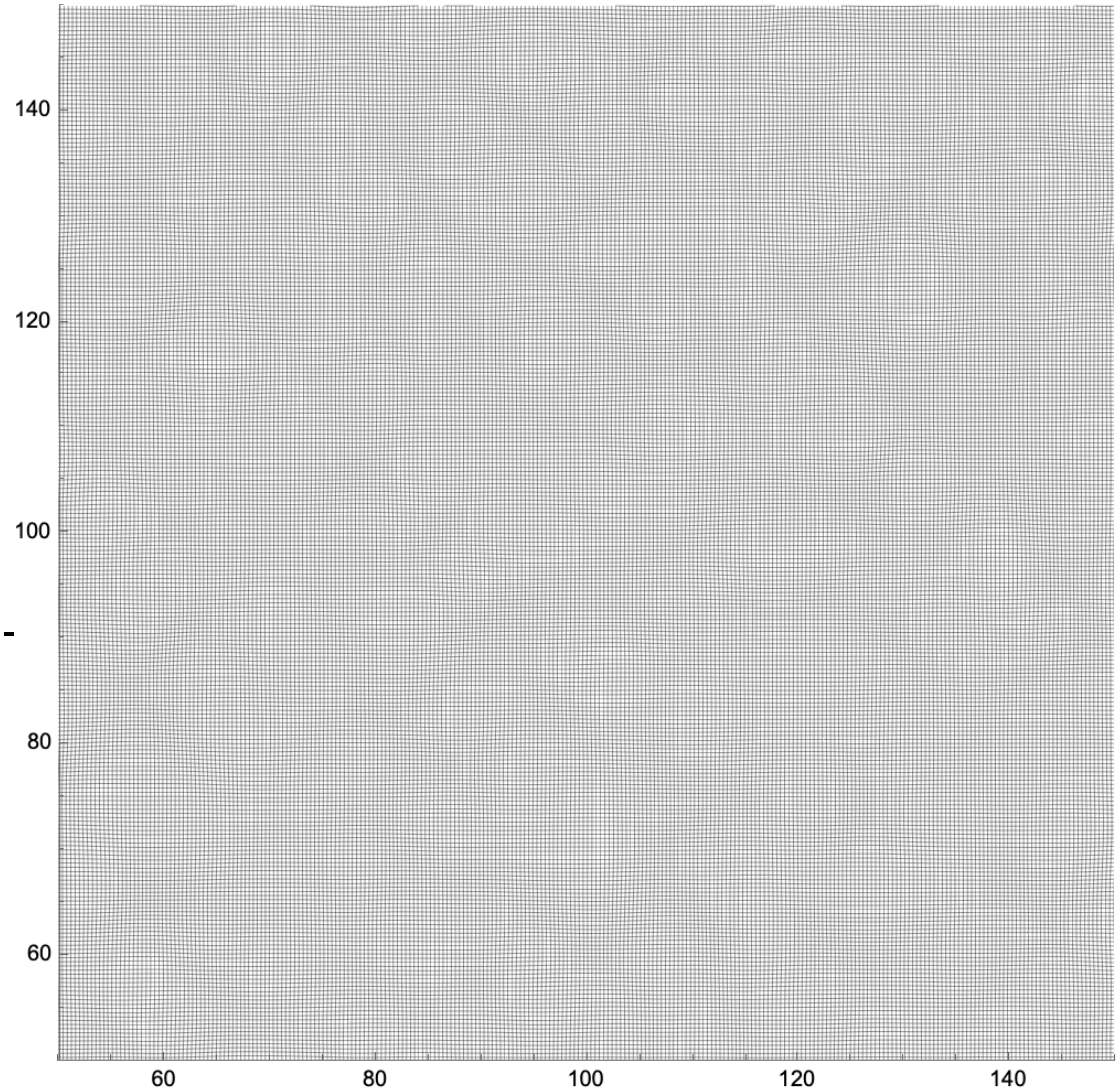
Co-evolution of the Cosmic Web and Galaxies across Cosmic Time

Caustic conditions

Caustics

- Dark matter forms the geometric structure of the cosmic web through formation of multi-stream regions

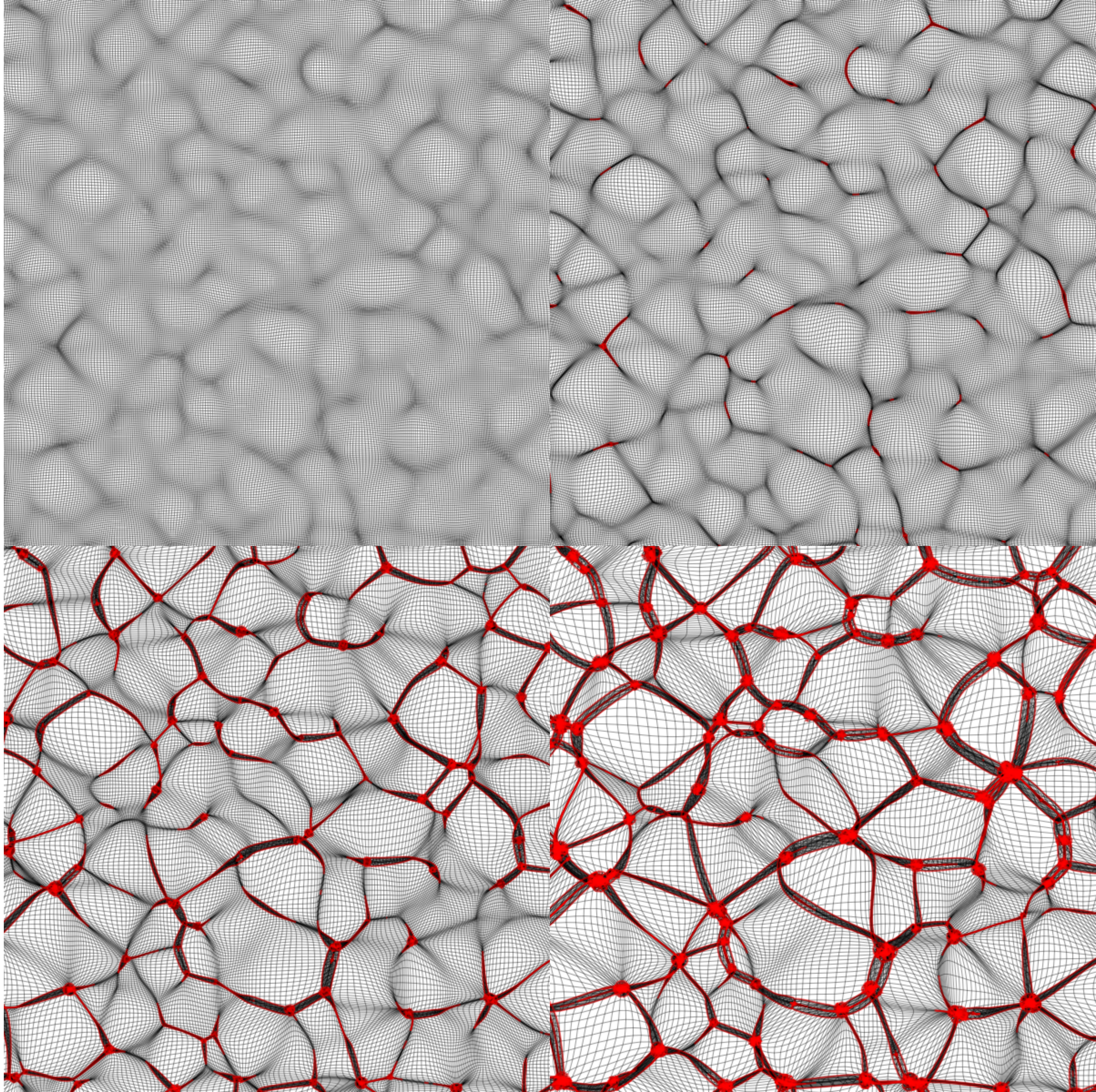
- The caustics bound the multi-stream regions



Caustics

- Dark matter forms the geometric structure of the cosmic web through formation of multi-stream regions

- The caustics bound the multi-stream regions



Caustics

- *Vladimir Arnol'd* extended *René Thom's* classification of stable degenerate critical points to **Lagrangian catastrophe theory**
- The **classification of caustics** was applied to *large-scale structure formation* to predict the geometric structure of the *cosmic web*
- Renewed interest, Hidding et al (2013)

1972 NORMAL FORMS FOR FUNCTIONS NEAR DEGENERATE CRITICAL POINTS, THE WEYL GROUPS OF A_k, D_k, E_k AND LAGRANGIAN SINGULARITIES

V. I. Arnol'd

1980 EVOLUTION OF SINGULARITIES OF POTENTIAL FLOWS IN COLLISION-FREE MEDIA AND THE METAMORPHOSIS OF CAUSTICS IN THREE-DIMENSIONAL SPACE

V. I. Arnol'd

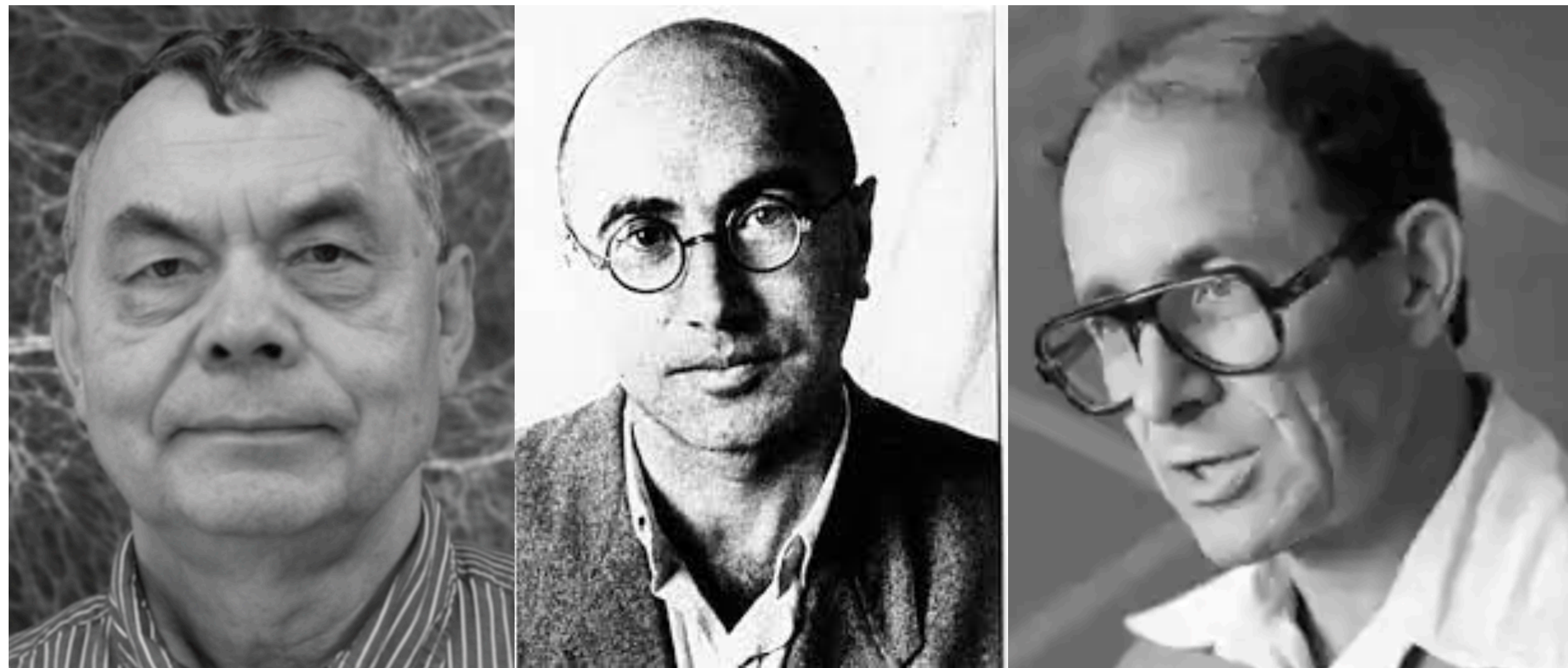
1982 The Large Scale Structure of the Universe I. General Properties. One- and Two-Dimensional Models

V. I. ARNOLD
Moscow State University, U.S.S.R.

and

S. F. SHANDARIN and YA. B. ZELDOVICH
Institute of Applied Mathematics, Moscow, U.S.S.R.

(*Received August* 11, 1981)



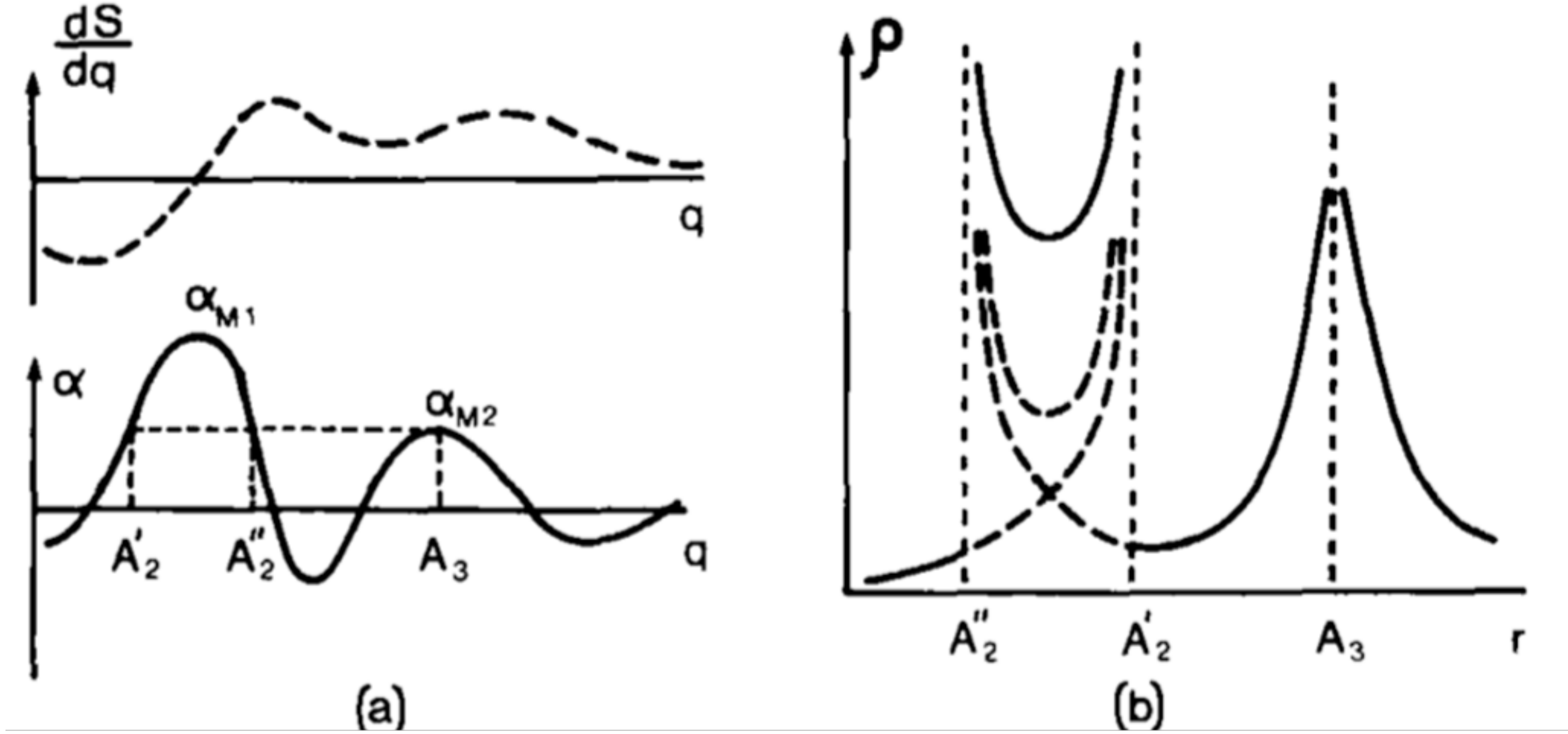
Caustics

Arnol'd, Shandarin, Zel'dovich (1982)

Lagrangian fluid dynamics

$$\mathbf{x}_t(\mathbf{q}) = \mathbf{q} + \mathbf{s}_t(\mathbf{q})$$

where the displacement map solves the Euler equation and the Poisson equation, while implementing conservation of mass. The density follows as the reciprocal of the Jacobian



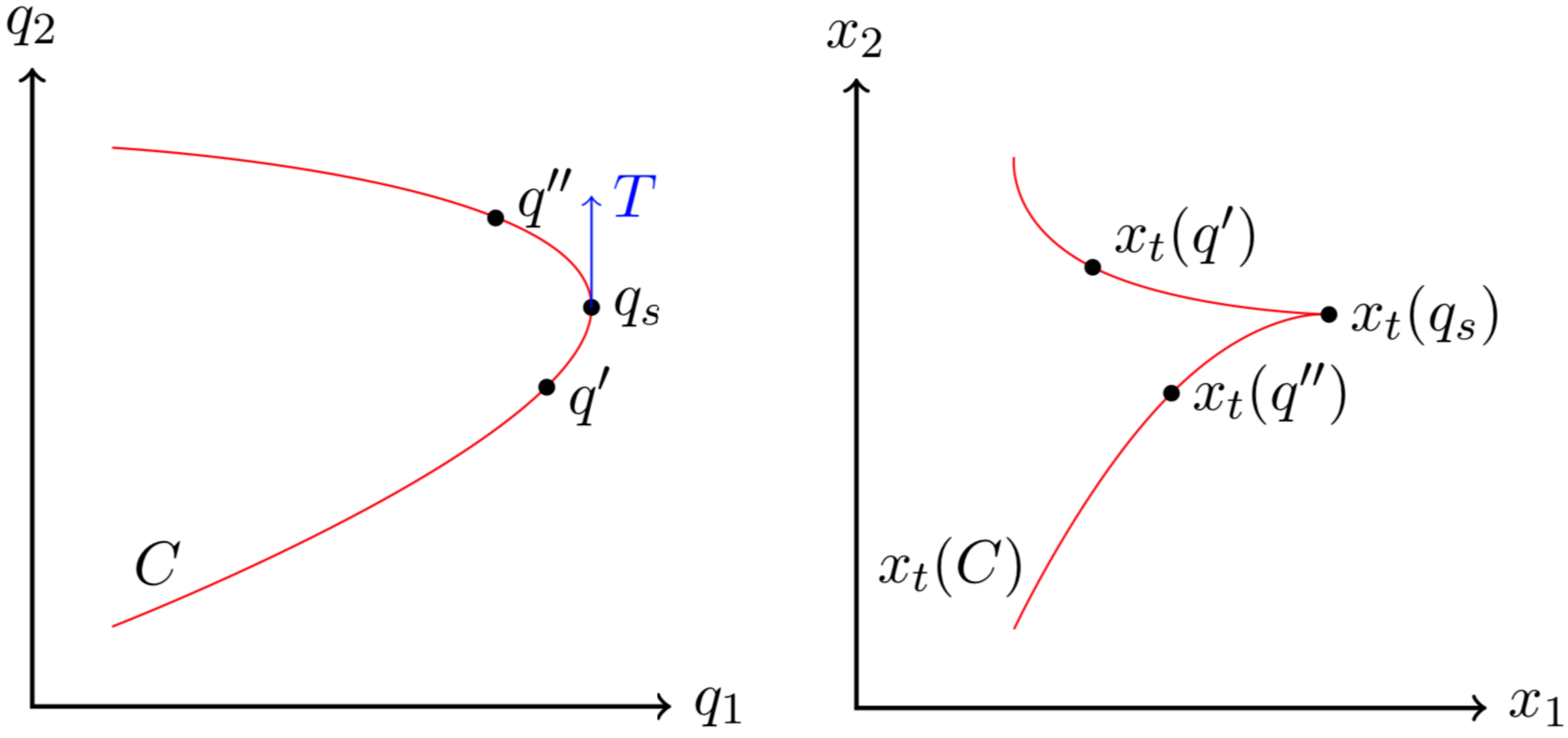
$$\rho_t(\mathbf{x}') = \sum_{\mathbf{q} \in \mathbf{x}_t^{-1}(\mathbf{x}')} \frac{\bar{\rho}}{|\det \nabla \mathbf{x}_t(\mathbf{q})|} = \sum_{\mathbf{q} \in \mathbf{x}_t^{-1}(\mathbf{x}')} \frac{\bar{\rho}}{|1 + \mu_1(\mathbf{q})||1 + \mu_2(\mathbf{q})||1 + \mu_3(\mathbf{q})|}$$

with the eigenvalues of the deformation tensor $\nabla \mathbf{s}_t(\mathbf{q}) \mathbf{v}_i(\mathbf{q}) = \mu_i(\mathbf{q}) \mathbf{v}_i(\mathbf{q})$

Caustic conditions

Up to recently, we only knew how to evaluate the fold caustic in 2D. Now we can evaluate any caustic in any dimension

$$\frac{\Delta x}{|\Delta q|} = \frac{\|x_t(q') - x_t(q'')\|}{\|q' - q''\|} \to 0 \qquad q', q'' \to q_s$$



Theorem: *A manifold $M \subset L$ forms a singularity under the mapping x_t in the point $x_t(q_s) \in x_t(M) \subset E$ at time t, meaning that $x_t(M)$ is not smooth in $x_t(q_s)$, if and only if there exists at least one nonzero tangent vector $T \in T_{q_s}M$ satisfying*

$$(1 + \mu_{it}(q_s))v_{it}^*(q_s) \cdot T = 0$$

for all $i = 1, 2, \ldots, dim(L)$.

Feldbrugge et al (2019)

Caustic conditions

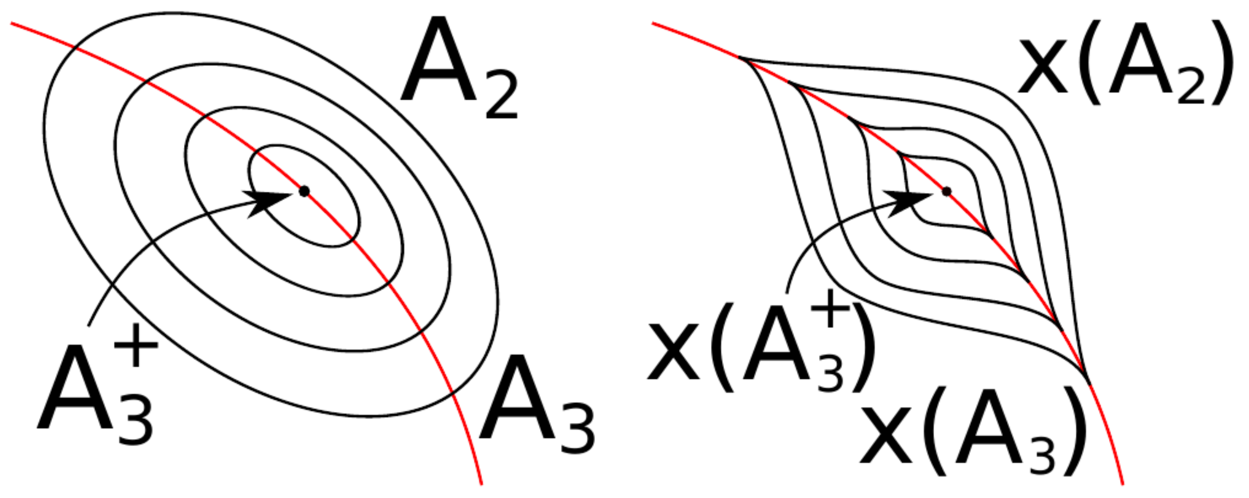
When applying the caustic condition to a Lagrangian space, we obtain the fold condition

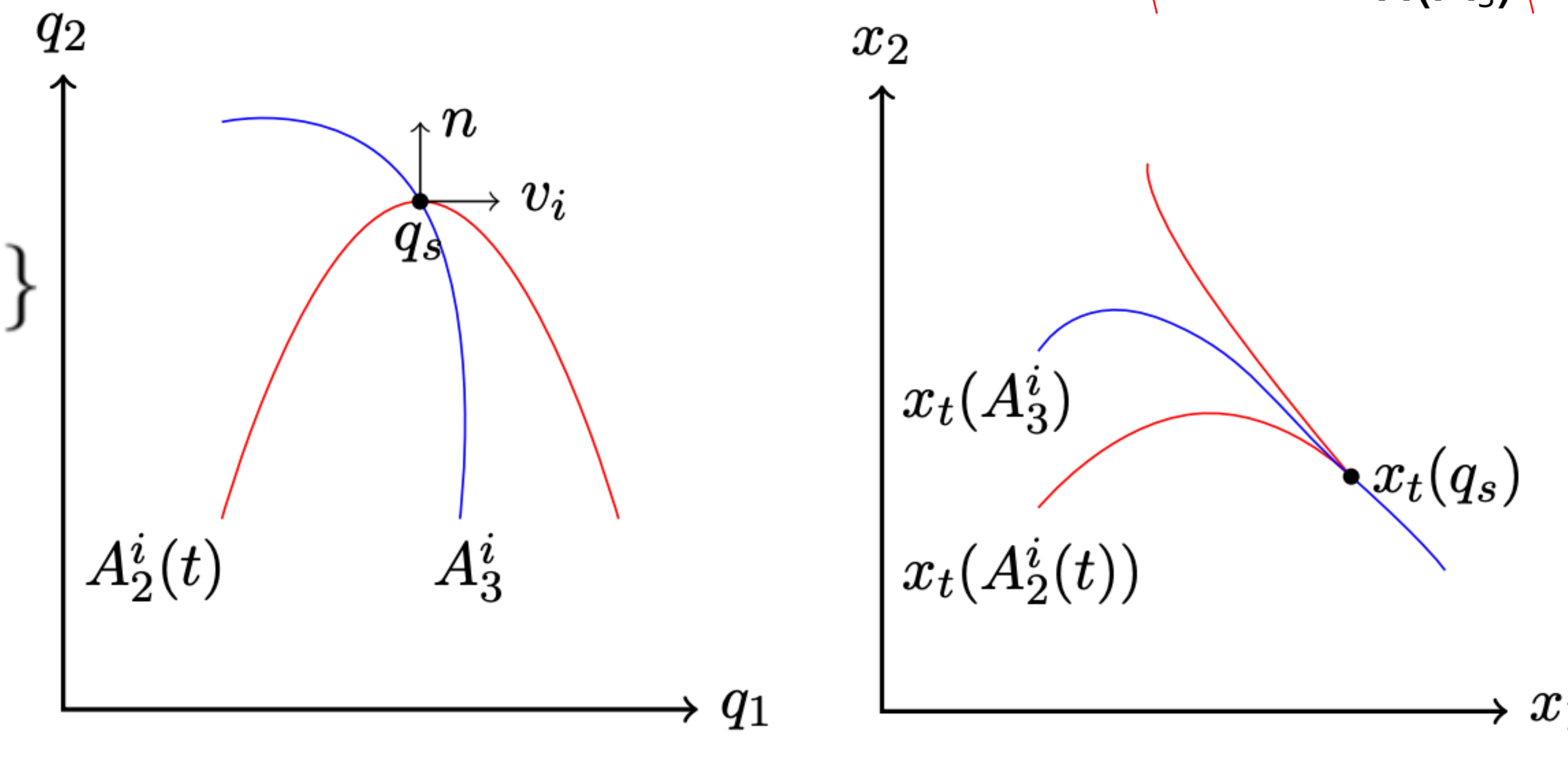
$$A_2^i(t) = \{\mathbf{q} \in L \mid 1 + \mu_{it}(\mathbf{q}) = 0\}$$

Applying the caustic condition a second time, we obtain the cusp caustic in terms of both the eigenvalue and eigenvector fields

$$A_3^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_2^i(t), \mathbf{v}_i \cdot \nabla \mu_{it} = 0\}$$

$$(1 + \mu_{it}(q_s))v_{it}^*(q_s) \cdot T = 0$$

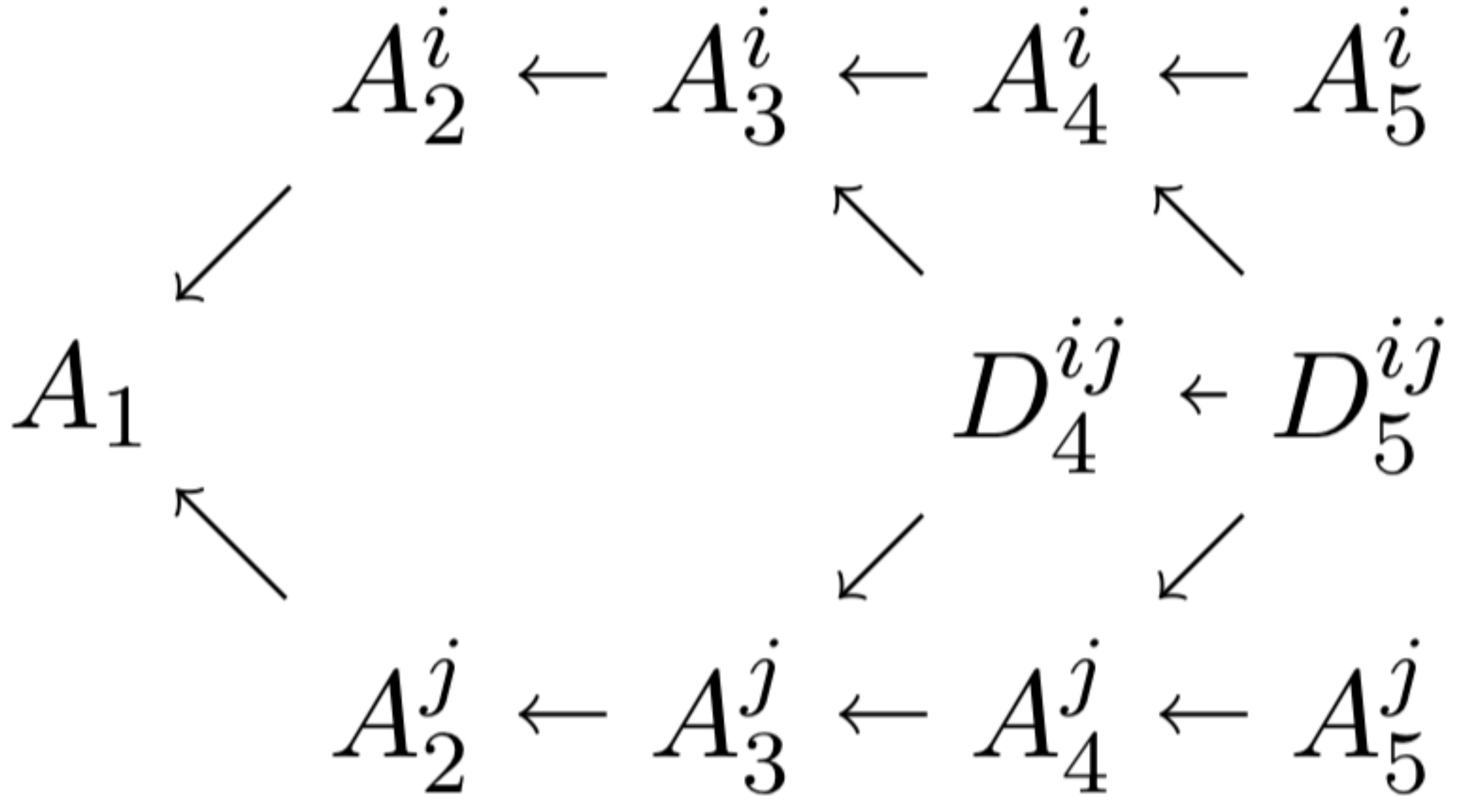




(a) Lagrangian space (b) Eulerian space

Feldbrugge et al (2019)

Caustic conditions



Iterative application of the shell-crossing condition

$$(1+\mu_{it}(q_s))v_{it}^*(q_s)\cdot T = 0$$

leads to the caustic conditions on both the eigenvalue and eigenvector fields:

Fold:	$A_2^i(t) = \{\mathbf{q} \in L \mid 1+\mu_{it}(\mathbf{q}) = 0\}$
Cusp:	$A_3^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_2^i(t), \mathbf{v}_i \cdot \nabla \mu_{it} = 0\}$
Swallowtail:	$A_4^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_3^i(t), \mathbf{v}_i \cdot \nabla(\mathbf{v}_i \cdot \nabla \mu_{it}) = 0\}$
Butterfly:	$A_5^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_4^i(t), \mathbf{v}_i \cdot \nabla(\mathbf{v}_i \cdot \nabla(\mathbf{v}_i \cdot \nabla \mu_{it})) = 0\}$
Umbilic:	$D_4^{ij}(t) = \{\mathbf{q} \in L \mid 1+\mu_{it}(\mathbf{q}) = 1+\mu_{jt}(\mathbf{q}) = 0\}$
Parabolic:	$D_5^{ij}(t) = \{\mathbf{q} \in L \mid q \in D_4^{ij}(t), \mathbf{v}_i \cdot \nabla \mu_i = \mathbf{v}_j \cdot \nabla \mu_j = 0\}$

Morse-Smale theory of full deformation tensor field. No free parameters!

Feldbrugge et al (2019)

Caustic conditions

Singularity class	Singularity name	Feature in the 2D cosmic web	Feature in the 3D cosmic web
A_2	fold	collapsed region	collapsed region
A_3	cusp	filament	wall or membrane
A_4	swallowtail	cluster or knot	filament
A_5	butterfly	not stable	cluster or knot
D_4	hyperbolic/elliptic	cluster or knot	filament
D_5	parabolic	not stable	cluster or knot

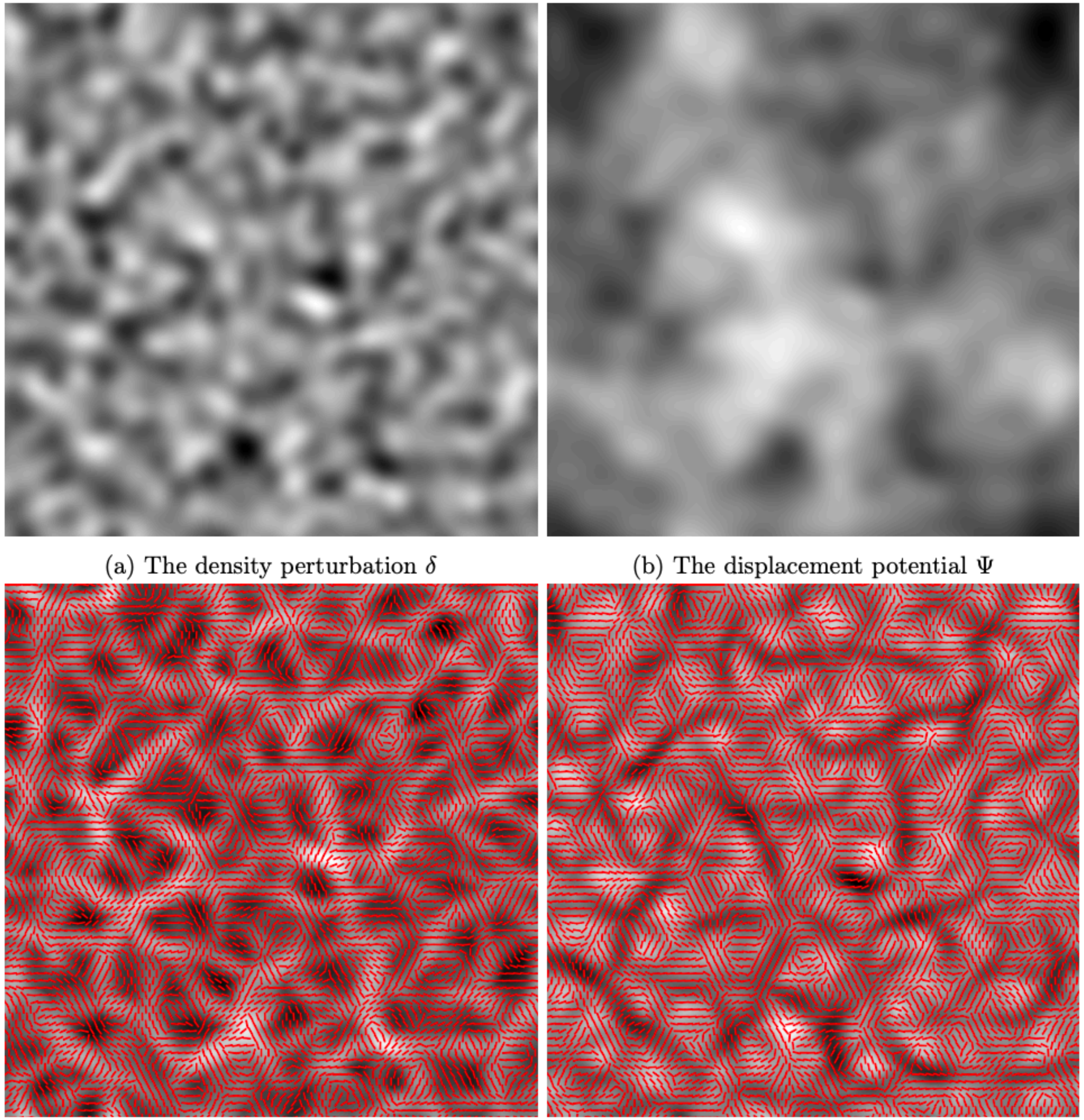
The identification of the different caustics in the 2- and 3-dimensional cosmic web

Feldbrugge et al (2019)

Caustic conditions

Note that:

- The eigenvalue and eigenvector fields are non-linear transformations of the density perturbations
- The web-like nature is embedded in the distribution of the eigenvalue *and* eigenvector fields



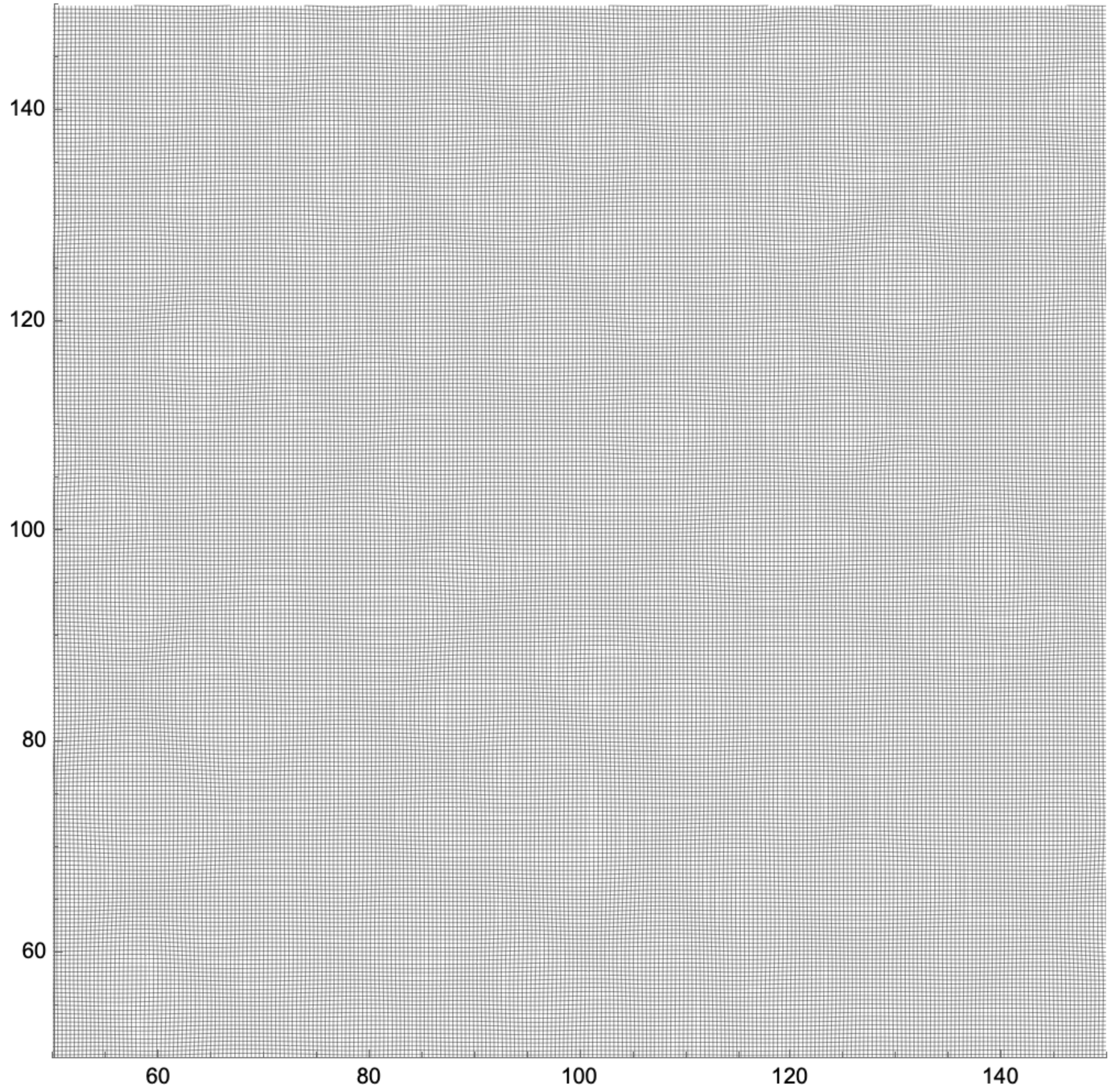
(a) The density perturbation δ

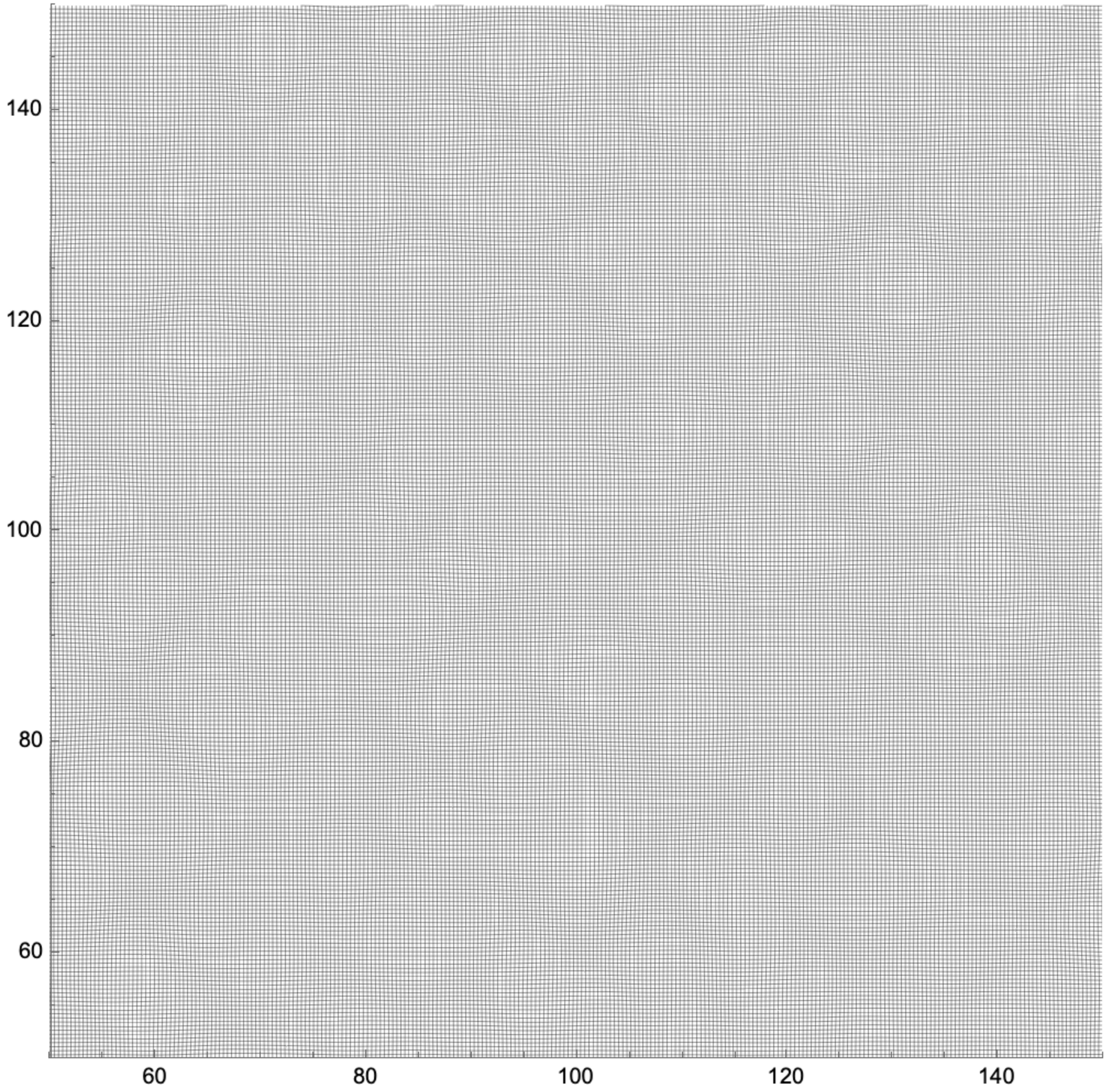
(b) The displacement potential Ψ

(c) The first eigenvalue and eigenvector fields λ_1, and $\boldsymbol{v}_1$

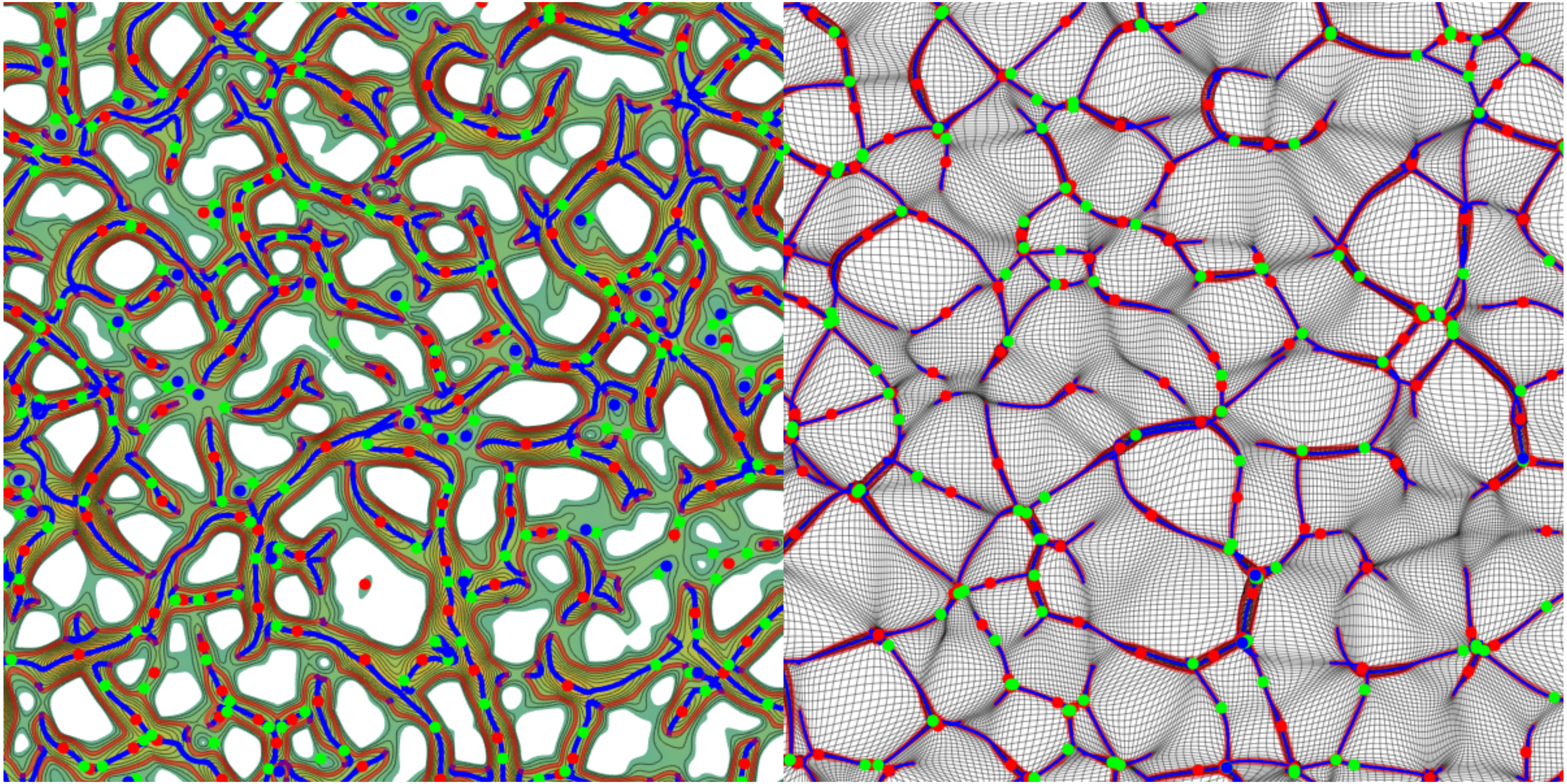
(d) The second eigenvalue and eigenvector fields λ_2, and $\boldsymbol{v}_2$

Caustic conditions

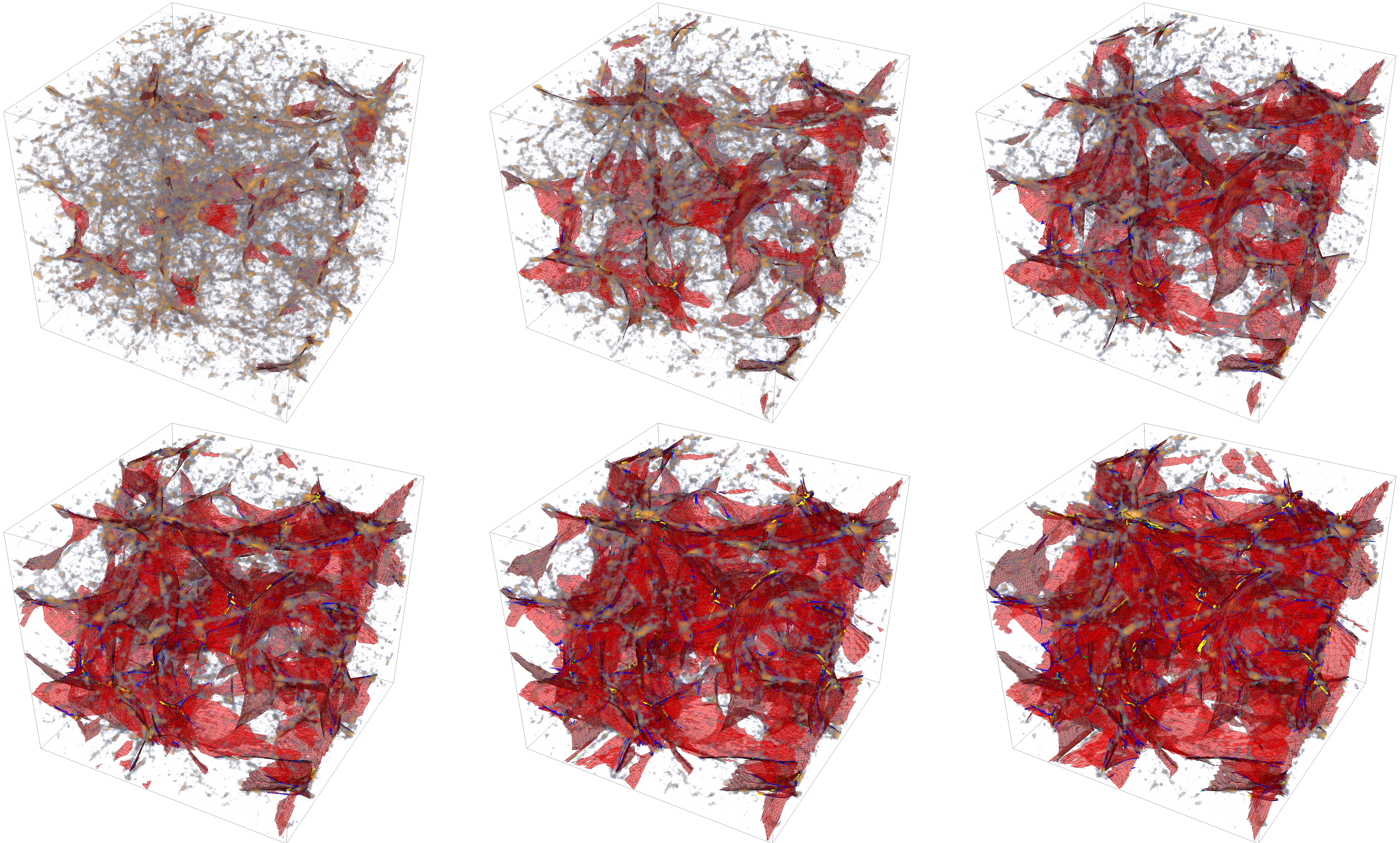




Caustic conditions



Caustic conditions



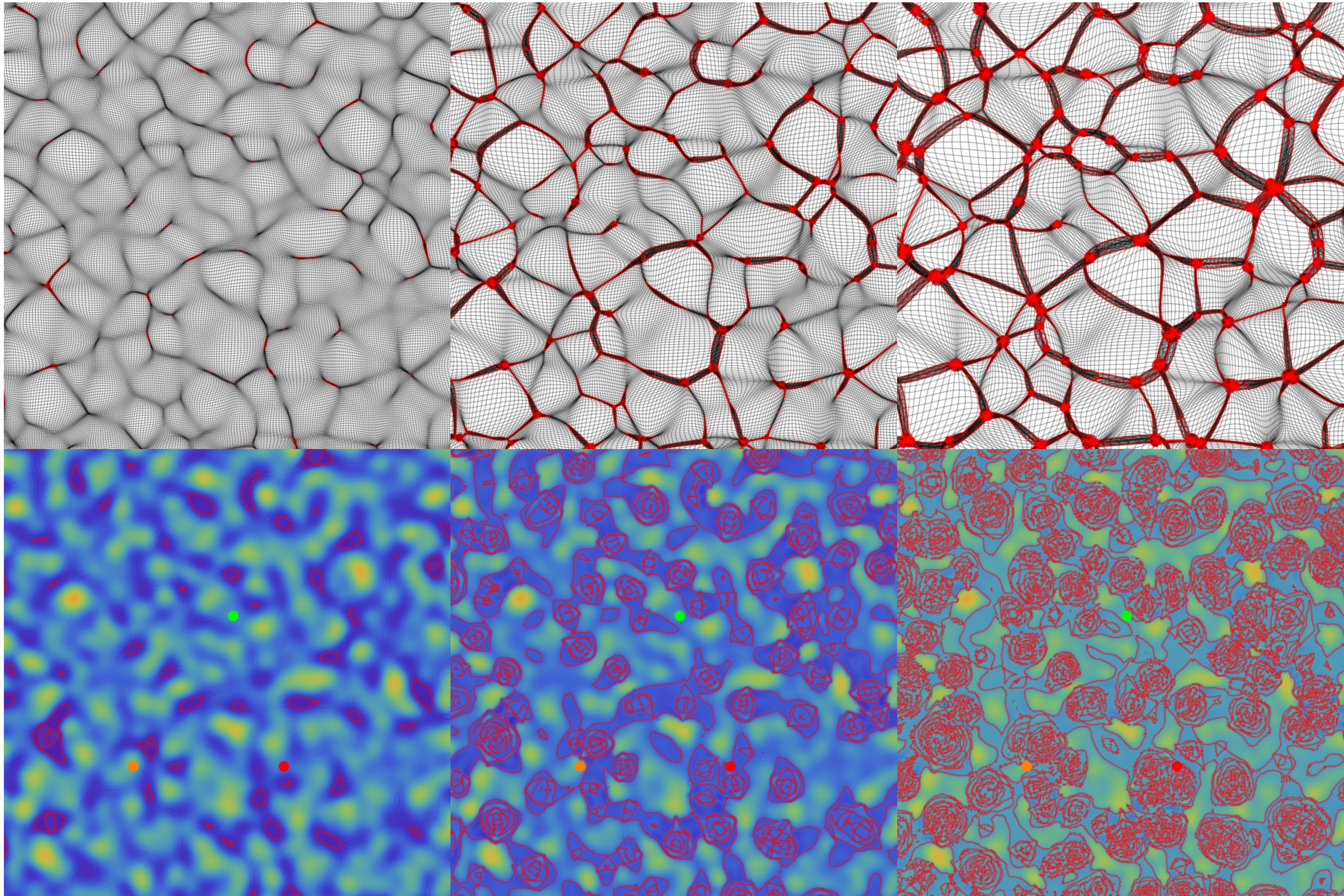
Caustic conditions

Topology of the cosmic web

Topology

The Lagrangian map forms a fold caustic when the deformation tensor becomes singular

$$\det \nabla \mathbf{x}_t(\mathbf{q}) = 0$$

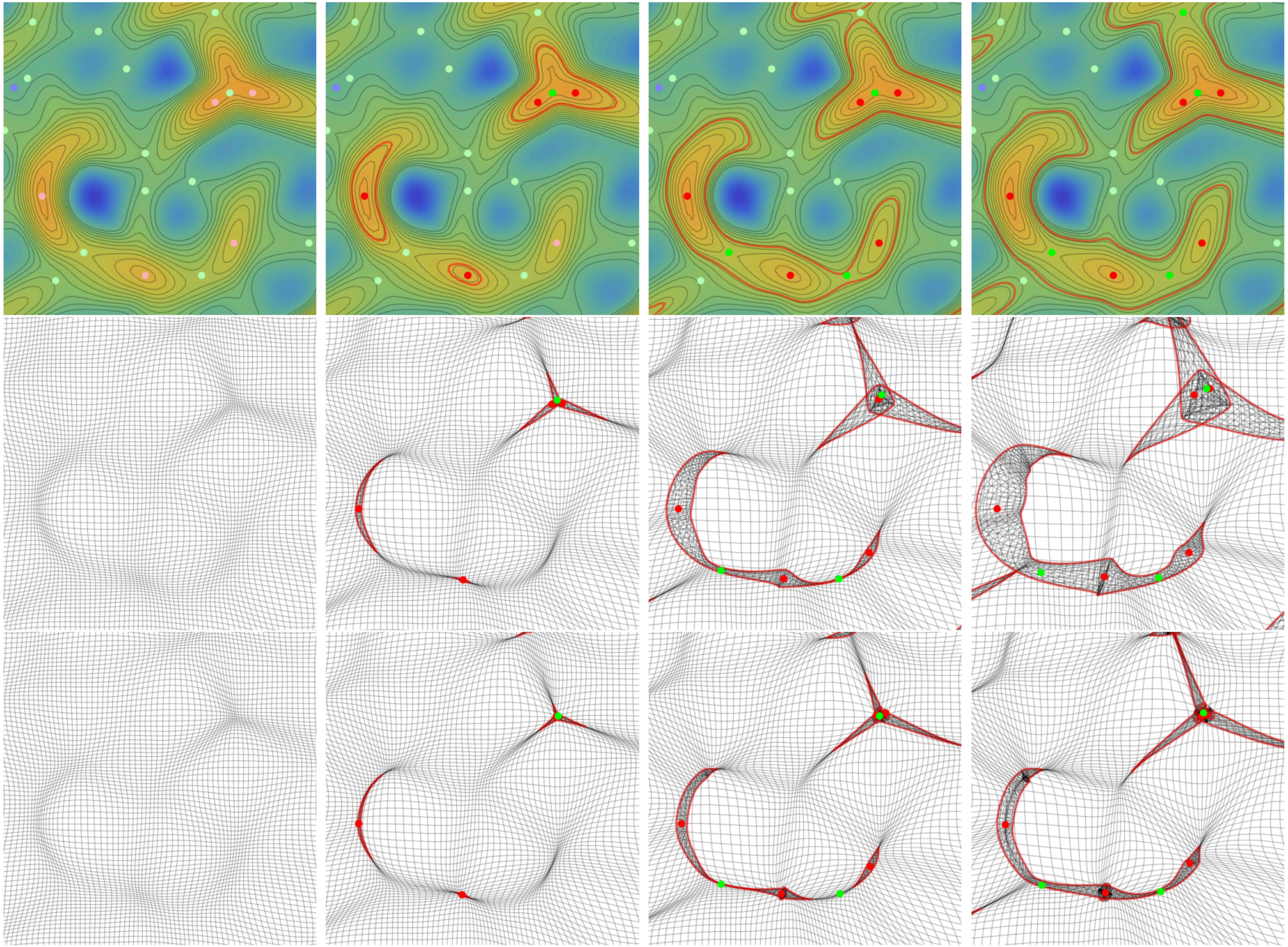


Topology

Critical points of the eigenvalue fields

Zel'dovich approximation

N-body simulation

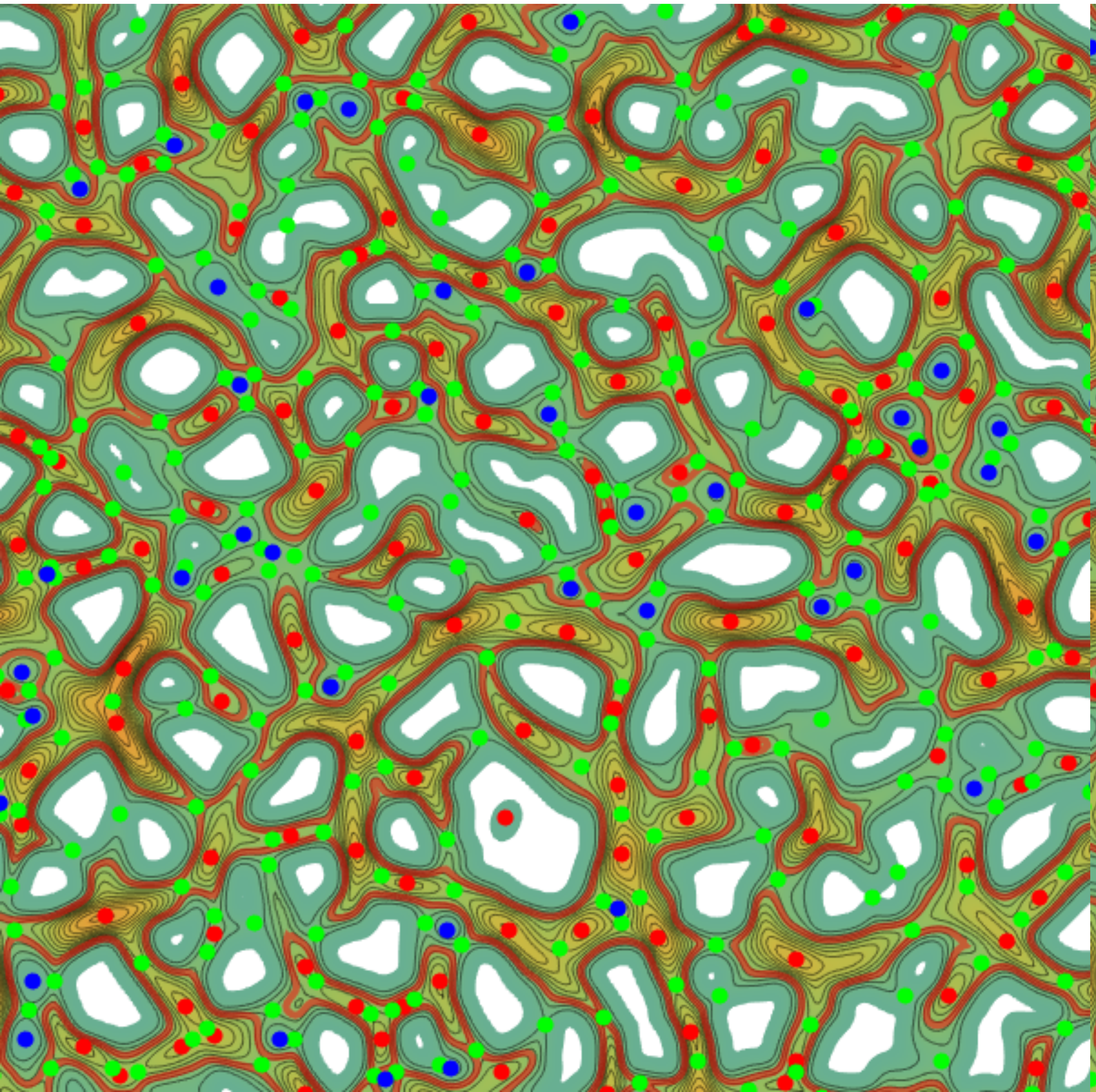


Topology

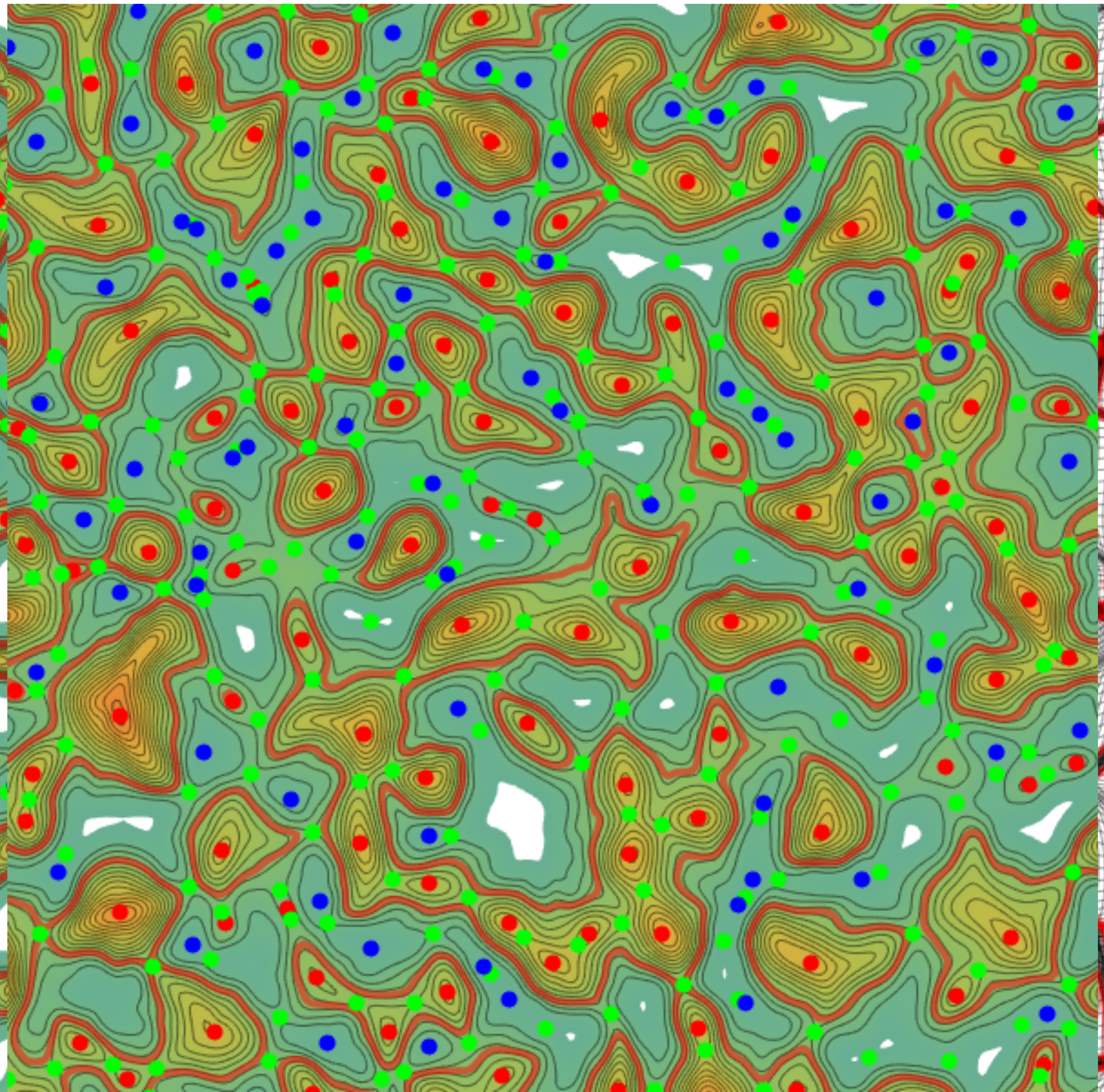
Comparison of skeleton of the Zel'dovich approximation and the N-body simulation

Feldbrugge, Yan, Weygaert (2023)
Wilding, Feldbrugge, Weygaert (in prep)
Sklansky, Feldbrugge, Weygaert (in prep)

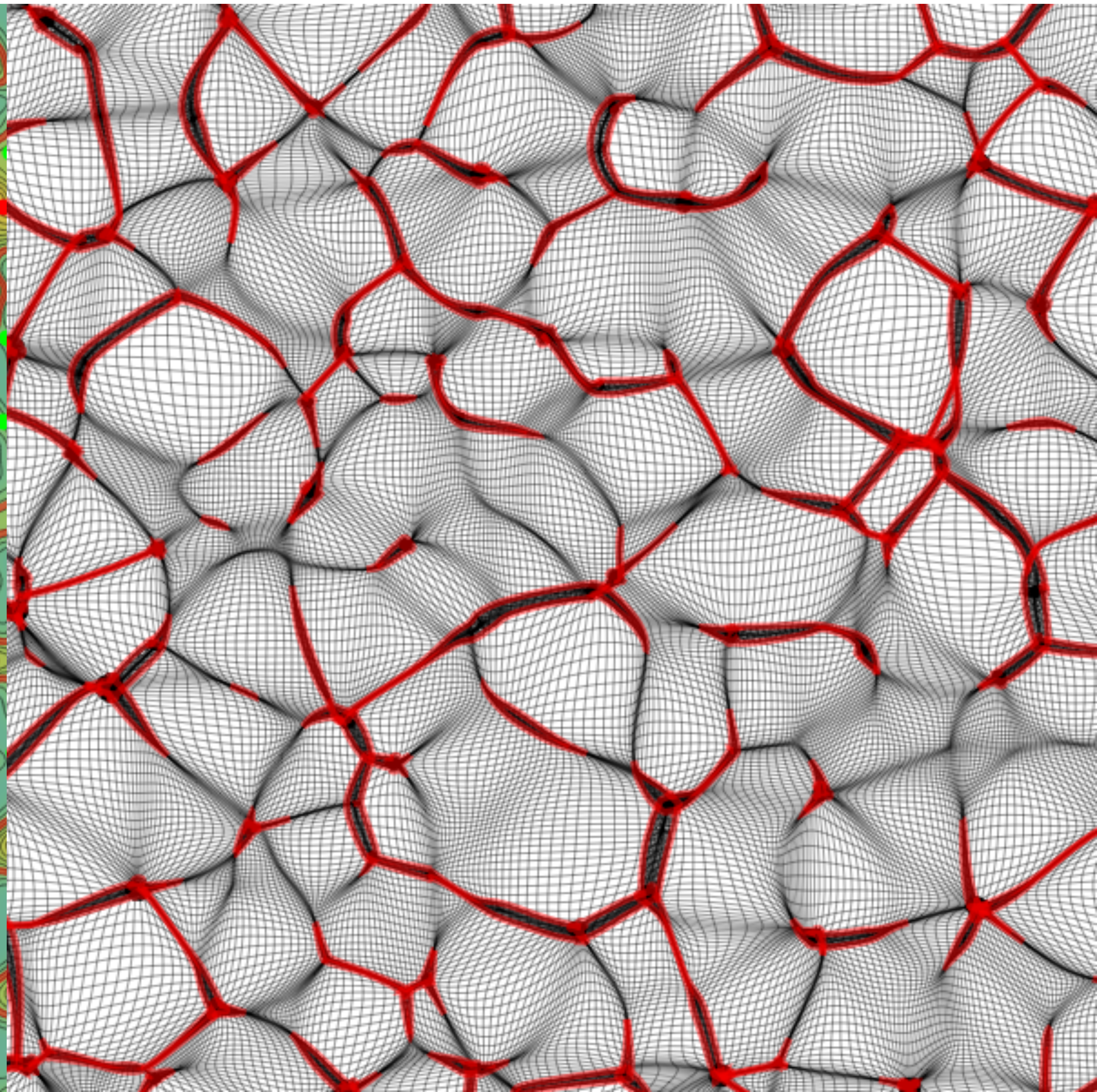
Zel'dovich approximation



N-body simulation



N-body simulation

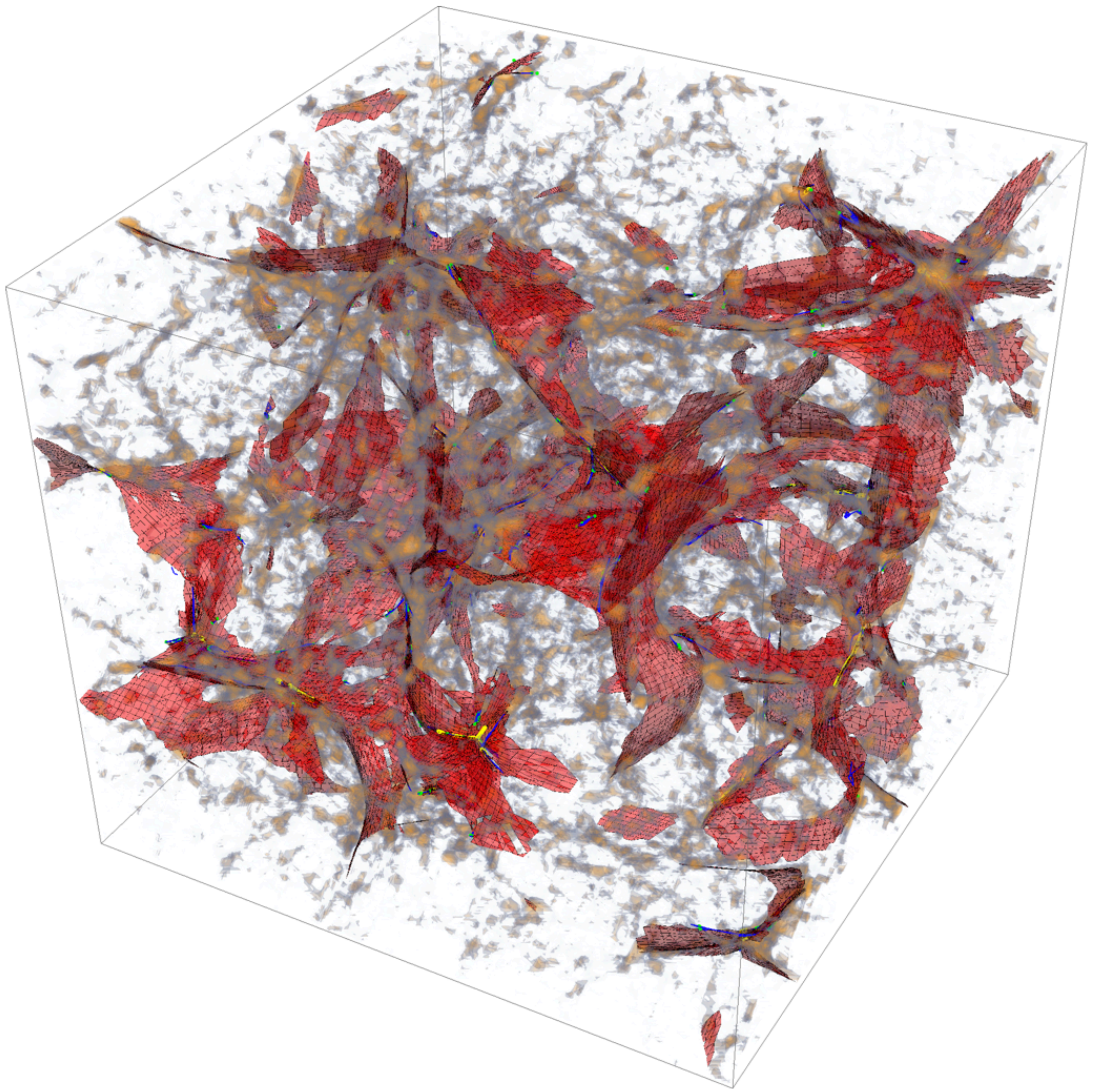


Non-linear constrained GRFs

Dressing the caustic skeleton

Assessing the mass distribution in and around the caustic spine of the cosmic web:

- What are the properties of walls/ filaments/clusters?
- What is their mass distribution?
- How do they form and relate to the initial conditions?
- Detailed merging history and hierarchical evolution of the filamentary network?



Dressing the caustic skeleton

Can we construct dedicated initial conditions for the systematic study of the different elements of the cosmic web forming at specified times?

Constrained GRF theory

- Goal: Generate Gaussian random fields subject to constraints.
- Linear constraints: Hoffman-Ribak algorithm:

Bertschinger 1987
Hoffman, Ribak 1991
van de Weygaert & Bertschinger 1996

  - The statistical properties of the residue of a cGRF with respect to the mean field is independent of the values of the constraints
  - Generate GRF ➔ Measure constraint values ➔ Evaluate corresponding mean field ➔ Evaluate residue ➔ Add residue to mean field with target constraint values
- Drawback: Algorithm only works for **linear** constraints on a Gaussian field

Log-normal: Ravi Sheth 1997

Constrained GRF theory

Given the constraints: $\Gamma = \{C_i[f; \boldsymbol{q}_i] = c_i, \;\; i = 1, \ldots, M\} \, ,$

Bertschinger 1987
Hoffman, Ribak 1991
Bertschinger and Weygaert 1996

we obtain the mean field:

$$\bar{f}_{\boldsymbol{c}}(\boldsymbol{q}) = \langle f(\boldsymbol{q}) | \Gamma \rangle = \bar{f}(\boldsymbol{q}) + \sum_{i,j=1}^{M} \xi_i(\boldsymbol{q}) \xi_{ij}^{-1} (c_j - \bar{C}_j)$$

and the variance of the residue:

$$\langle \delta f(\boldsymbol{q})^2 | \Gamma \rangle = \sigma_0^2 - \sum_{i,j=1}^{M} \xi_i(\boldsymbol{q}) \xi_{ij}^{-1} \xi_j(\boldsymbol{q})$$

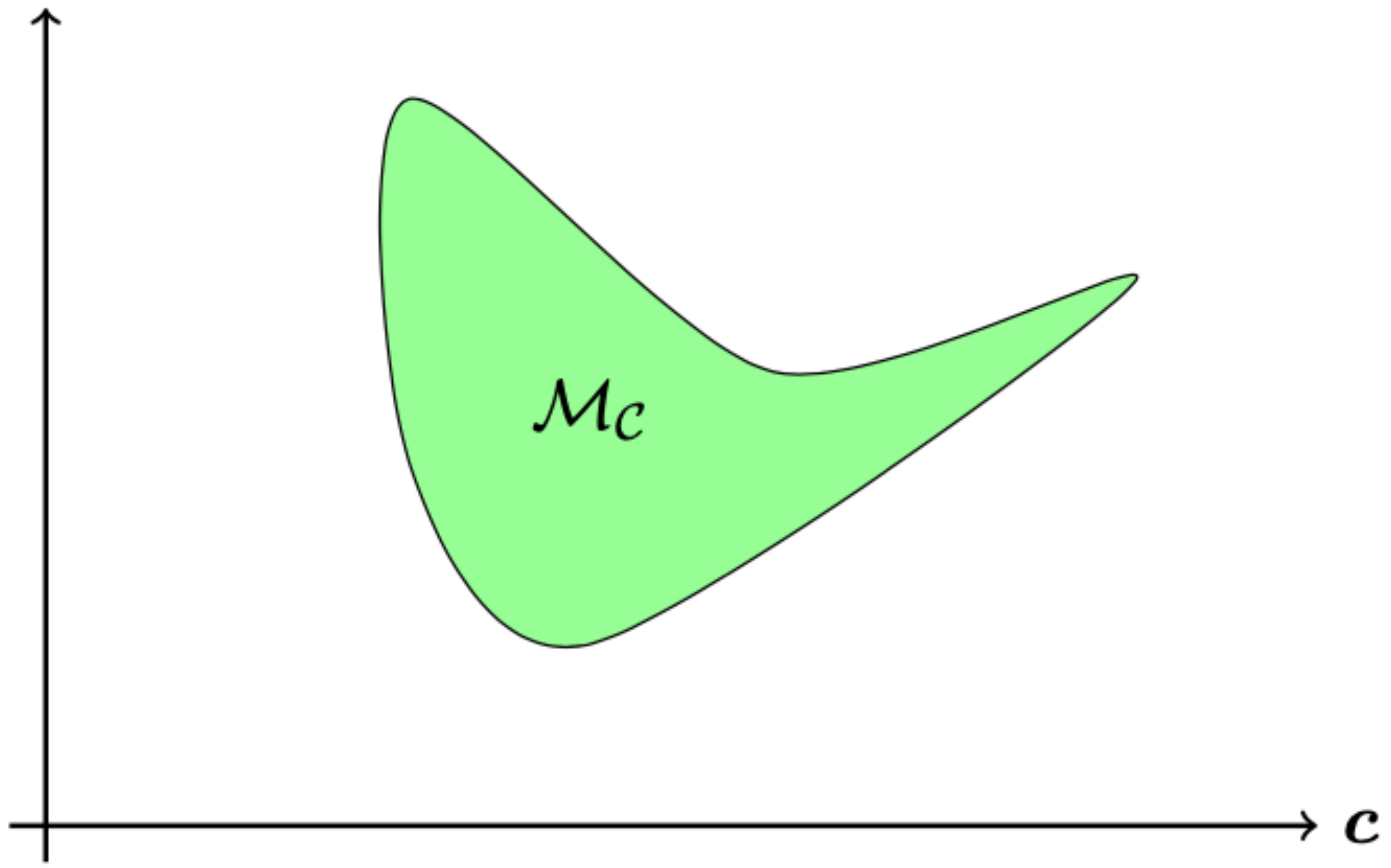
To generate realizations, we generate an unconstrained GRF, find the c_i's, find the residue and add the residue to the mean field of the target constraint values

Log-normal: Ravi Sheth 1997

Non-linear constraints

- The eigenvalue and eigenvector fields are not Gaussian, and the caustic conditions are non-linear. For this reason, we develop non-linear constraint Gaussian random field theory

$$\mathcal{M}_{\mathcal{C}} = \{\boldsymbol{c}\,|\,\mathcal{C}_i(\boldsymbol{c}) = 0 \text{ for all } i = 1, \ldots, N\}.$$



- On this constraint manifold, we find the induced probability density

$$p(\boldsymbol{c}\,|\,\boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}) = \frac{p(\boldsymbol{c})}{\int_{\mathcal{M}_{\mathcal{C}}} p(\boldsymbol{c})\,\mathrm{d}\boldsymbol{c}}\,.$$

Feldbrugge and Weygaert (2022)

Non-linear constraints

- Using the properties of the constraint manifold we can leverage the Hoffman-Ribak principle for non-linear constraints:
  - The mean field: $$\bar{f}_{\mathcal{C}}(\boldsymbol{q}) = \bar{f}_{\bar{\boldsymbol{c}}}(\boldsymbol{q})\,, \quad \bar{\boldsymbol{c}} \equiv \langle \boldsymbol{c} \,|\, \Gamma \rangle = \int_{\boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}} \boldsymbol{c}\, p(\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}) \mathrm{d}\boldsymbol{c}\,,$$
  - The variance:

$$\left\langle \delta f(\boldsymbol{q})^2 \,|\, \Gamma \right\rangle = \sigma_0^2 - \sum_{i,j=1}^{M} \xi_i(\boldsymbol{q}) \zeta_{ij}^{-1} \xi_j(\boldsymbol{q})\,, \quad \zeta_{ij}^{-1} = \xi_{ij}^{-1} - \sum_{k,l=1}^{M} \xi_{ik}^{-1} \mathrm{cov}(c_k, c_l \,|\, \boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}) \xi_{lj}^{-1}$$

- To generate realizations, we first sample the constraint manifold. Given the constraint values, we use the Hoffman-Ribak algorithm

- Very efficient and does not require expensive MCMC techniques

Feldbrugge and Weygaert (2022)

Generating the cusps/filaments

Generating the swallowtails/clusters

Generating the umbilics/clusters

Caustic condition

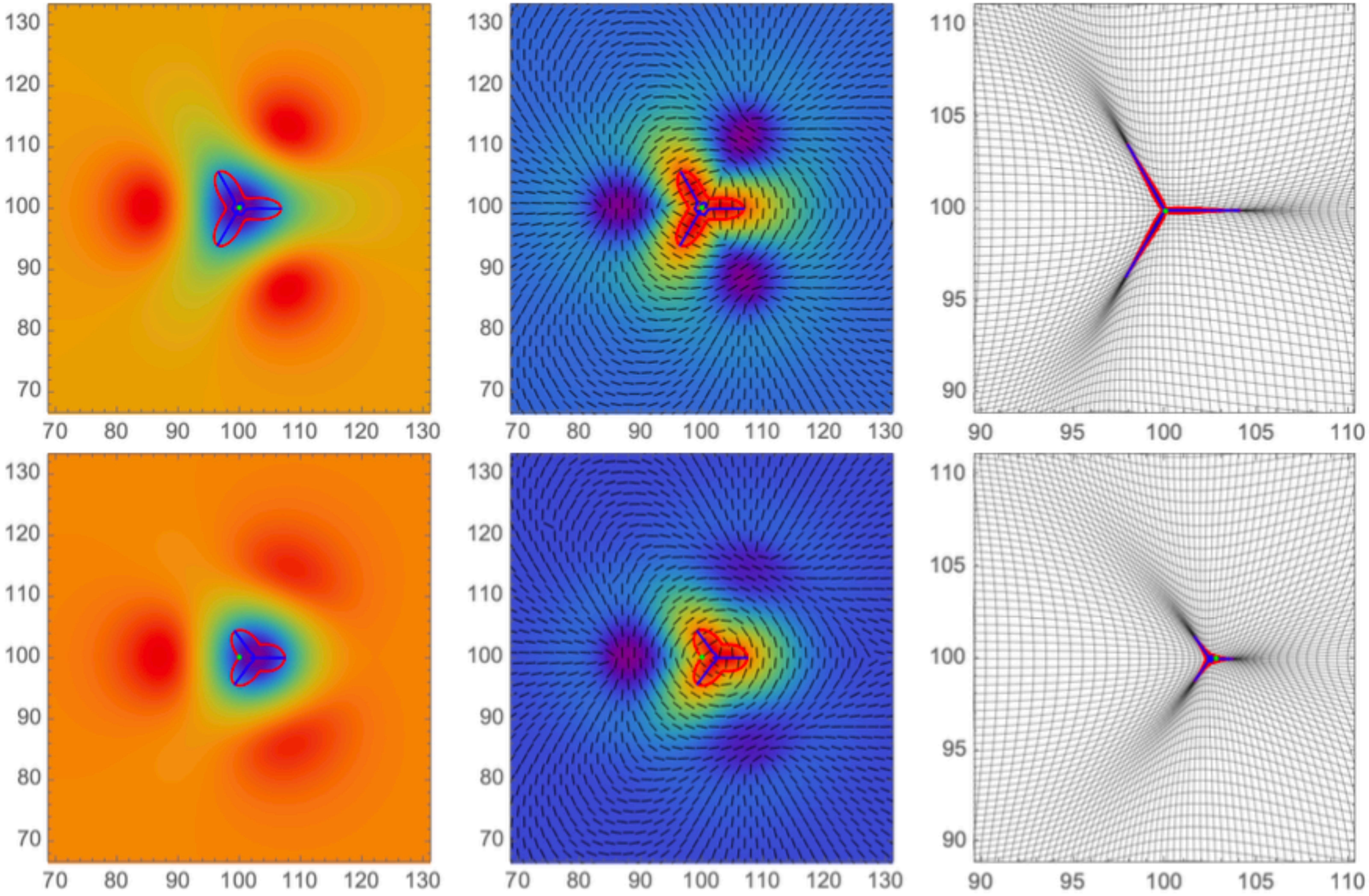
$$b_+(t)\lambda_1(\boldsymbol{q}_c) = b_+(t)\lambda_2(\boldsymbol{q}_c) = 1\,.$$

Elliptic umbilic:

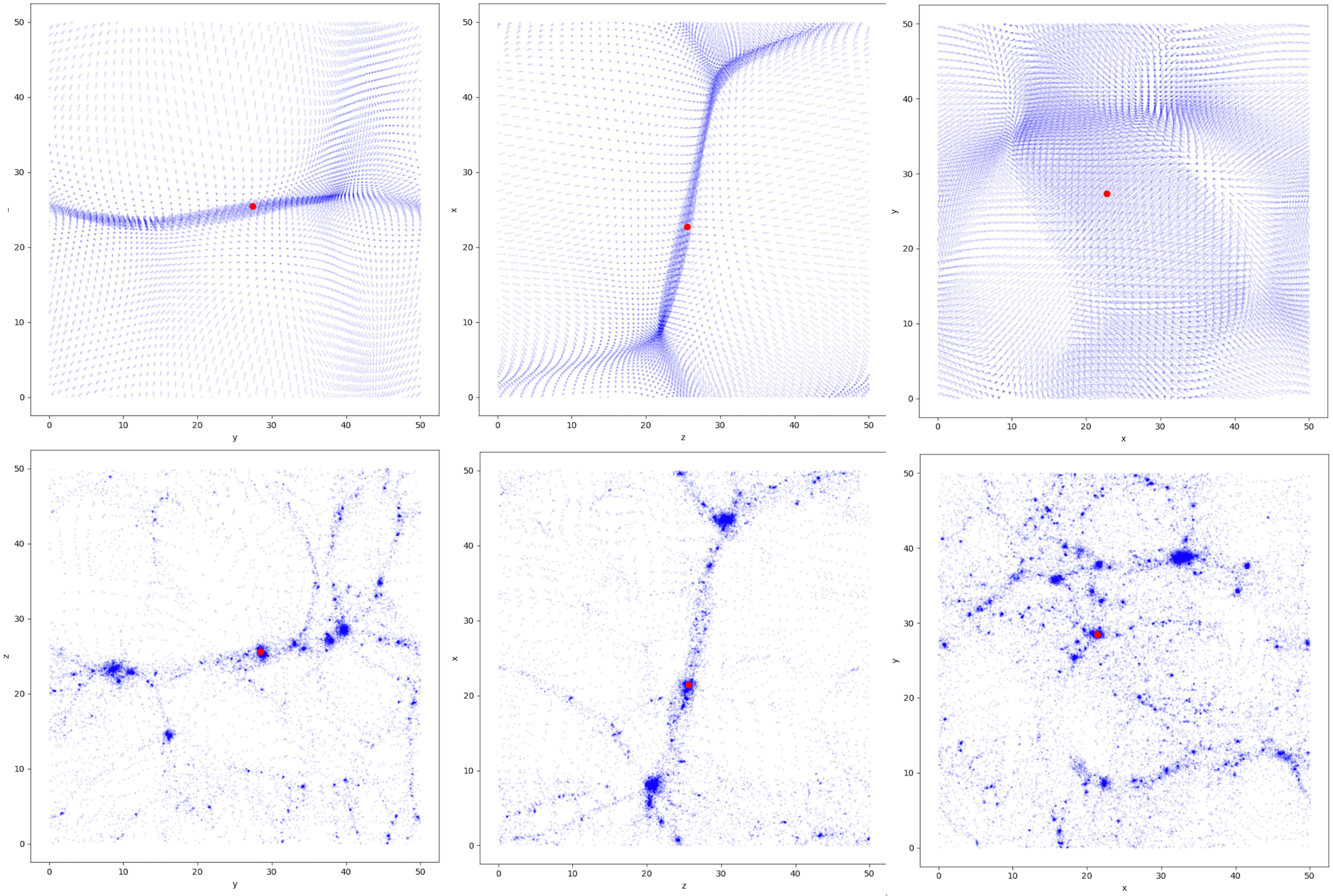
$$\det(\mathcal{H}\left[\det(I - b_+\boldsymbol{\psi})\right]) > 0\,.$$

Hyperbolic umbilic:

$$\det(\mathcal{H}\left[\det(I - b_+\boldsymbol{\psi})\right]) < 0\,.$$



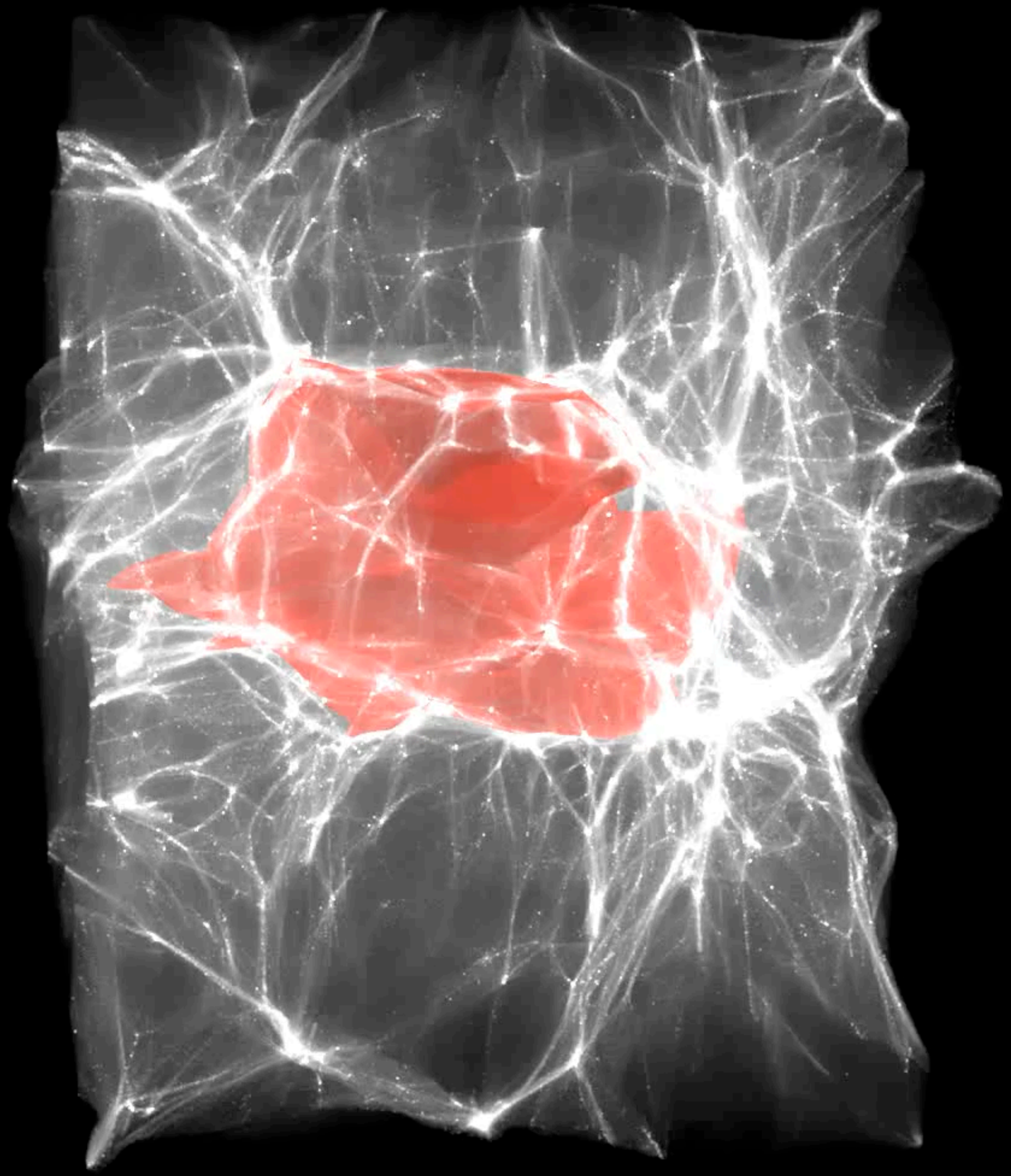
Realizations



Constrained Gaussian Random field theory

By generating customized initial conditions, using non-linear constrained Gaussian random field theory, we can systematically study the different elements of the cosmic web

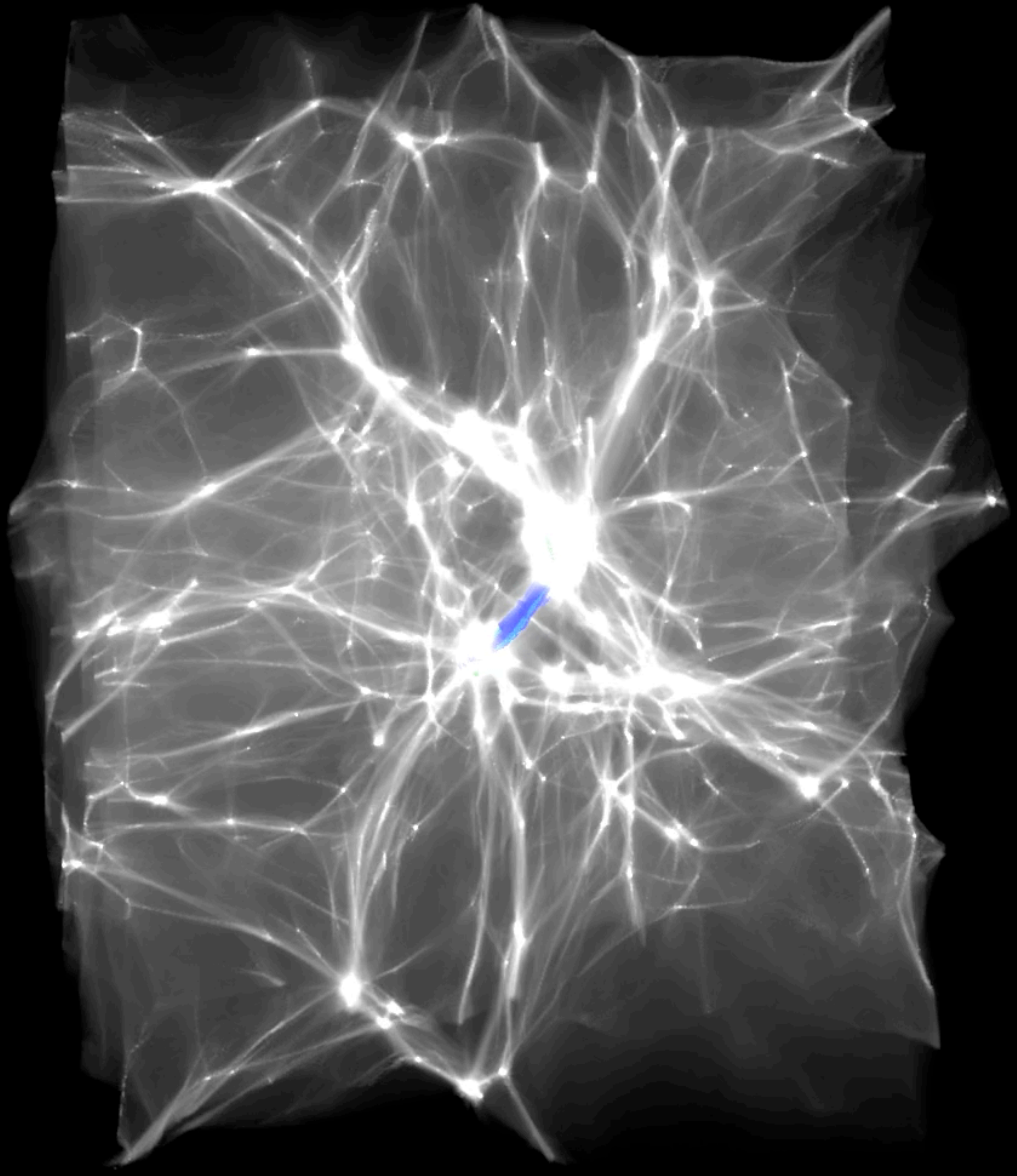
$$A_3^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_2^i(t), \mathbf{v}_i \cdot \nabla \mu_{it} = 0\}$$



Constrained Gaussian Random field theory

By generating customized initial conditions, using non-linear constrained Gaussian random field theory, we can systematically study the different elements of the cosmic web

$$D_4^{ij}(t) = \{\mathbf{q} \in L \mid 1 + \mu_{it}(\mathbf{q}) = 1 + \mu_{jt}(\mathbf{q}) = 0\}$$



Conclusion

- Shell-crossing condition enables us to derive caustic conditions in 3D
- Caustic skeleton of cosmic web depends on the eigenvalue *and* eigenvector fields
- Filaments and walls do not require multiple shell-crossings
- This formalism enables a systematic analysis of the topology of the cosmic web
- We extend constrained Gaussian random field theory to non-linear constraints
- We generate constrained initial conditions tied to the dynamics of structure formation