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O. Ganor

From S-duality to Chern-Simons via minimal strings

Yong-Pyo Hong & O.G, 0904.????

$$G = {}^LG = U(n), \quad n<6$$

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$

Quest: $S: \mathcal{H}(\tau, U(n)) \to \mathcal{H}(-1/\tau, U(n))$

spacetime: $\mathbb{R}^{3,1}$ or $\mathbb{R}^3 \times S^1$ or $S^1 \times \mathbb{R}^{2,1}$

$\uparrow$ time

Favorite gauge $A_0=0$

for $U(1)$

conn on $\mathbb{R}^3$

$$\tilde{\Psi}(\tilde{A}) = \int [\mathcal{D}A]\, S_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)}(\tilde{A}, A)\, \Psi(A)$$

$$E_i = -2\pi i \frac{\delta}{\delta A_i}$$

$$|\tilde{\Psi}\rangle = \hat{S}_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} |\Psi\rangle$$

generalization: $S_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)} : \mathcal{H}(\tau, U(n)) \longrightarrow \mathcal{H}\left(\frac{a\tau+b}{c\tau+d}, U(n)\right)$

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$SL(2,\mathbb{Z})$ $\qquad \tau \to \frac{a\tau+b}{c\tau+d}$

$$S_{(\ )}(\tilde{A},A) = e^{\frac{i}{4\pi c}\int\left[\underline{d}\, A\wedge dA - 2\tilde{A}\wedge dA + a\, A\wedge d\tilde{A}\right]}$$

$$\tilde{E}_x \hat{S}_{(\ )} = \hat{S}_{(\ )}(aE_x + bB_x)$$

$$\tilde{B}_x \hat{S}_{(\ )} = \hat{S}_{(\ )}(cE_x + \underline{d}B_x)$$

Lozano '94

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$e^{\int A\wedge d\tilde{A}} \qquad e^{i\int A\wedge dA}\ \delta(A)$$

- topological * (gravitational CS)
- $A = \tilde{A} \Rightarrow S(A,A) = \exp\left\{\frac{ik}{4\pi}\int A\wedge dA\right\}$

$$k = \frac{a+d-2}{c} \qquad U(n)?$$

<u>Physical</u>

$$\tau = \frac{a\tau+b}{c\tau+d} \Rightarrow \quad \tau = i \quad \pm\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tau = e^{i\pi/3} \quad \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ $|k|=2$ $\quad$ $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $|k|=1$

$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ $|k|=3$

S-duality twist

$U(1)$ on $S^1 \times \mathbb{R}^{3,1}$

$$\mathrm{tr}\left(S(\dots)e^{-2\pi R H}\right) \to \mathrm{tr}(S \dots)$$ $U(1)$ C.S. level 1, 2, 3

IR, $R \to 0$ $\quad$ $U(n)$?

$$\tau = \frac{a\tau + b}{c\tau + d} \Rightarrow (\mathrm{Im}\,\tau > 0) \quad c\tau + d = e^{i\theta}$$

3 cases

$$k = \frac{a+d-2}{c}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	k	τ
$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$	2	$\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$
$\begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}$	1	$\mathbb{C}/\mathbb{Z}+e^{i\pi/3}\mathbb{Z}$
$\begin{pmatrix} -1 & 1 \\ -1 & \end{pmatrix}$	3	$\mathbb{C}/\mathbb{Z}+e^{i\pi/3}\mathbb{Z}$

(4)

(1) Identify Hilbert Space $\mathcal{H}$ of States on T^2

(2) Identify Some operators on $\mathcal{H}$

Susy.

$N=4$ SYM fields

$$A_\mu,\ \psi^a_\alpha,\ \bar\psi_{\dot\alpha a},\ \Phi^I_{\ I\in\{1\to 6\}}$$

$a\in\{1,2,3,4\}$ R-symm $SU(4)$

$$z^j=\phi^j+i\phi^{j+3}\qquad j=1,2,3$$

$Q_{\alpha a}$

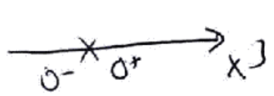
S-duality $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$

$$Q_{\alpha a}\to \underbrace{\left(\frac{c\tau+d}{|c\tau+d|}\right)^{1/2}}_{e^{i\theta/2}\in\{e^{i\pi/2},\,e^{i\pi/3},\,e^{i2\pi/3}\}} Q_{\alpha a}\qquad \theta=\frac{\pi}{2},\frac{2\pi}{3},\frac{4\pi}{3},$$

Add R-sym twist

$\gamma\in SU(4)$.

$$\gamma=\begin{pmatrix} e^{i\varphi_1} &&& \\ & e^{i\varphi_2} && \\ && e^{i\varphi_3} & \\ &&& e^{i\varphi_4}\end{pmatrix}\to\begin{pmatrix} e^{i\theta/2} &&& \\ & e^{i\theta/2} && \\ && e^{i\theta/2} & \\ &&& e^{-3i\theta/2}\end{pmatrix}$$



~~$Z^3(0^-)=$~~

$$\psi^a_\alpha(0^-)=\gamma\psi^a_\alpha(0^+)$$

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This leads to $\mathcal{N}=6$ in 3D

$3D = \mathbb{R}^{2,1}$ $(0,1,2)$ directions. What is the IR limit?

Is there a Coulomb branch?

$\langle Z^{\bar{j}}(0^-)\rangle = e^{i\theta}\langle Z^{\bar{j}}(0^+)\rangle$

$\Lambda^{-1}\langle Z^{\bar{j}}(0^-)\rangle\Lambda = e^{i\theta}\langle Z^{\bar{j}}(0^+)\rangle$, $\Lambda \in U(n)$

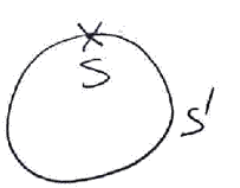
$$Z^{\bar{j}} = \begin{pmatrix} Z_1^{\bar{j}} & & \\ & Z_2^{\bar{j}} & \\ & & Z_n^{\bar{j}} \end{pmatrix} \subset U(n)$$

$\Lambda \in \mathrm{Weyl}(U(n)) = S_n \subseteq U(n)$

$\Lambda^{-1}\langle Z^{\bar{j}}\rangle\Lambda = e^{i\theta}\langle Z^{\bar{j}}\rangle$

Λ must have order $\frac{2\pi}{\theta} \in \{4, 3, 6\}$

$U(n)$, $\mathcal{N}=4$, $n<4$, $\tau = i$.



S^1 (direction 3) $\times M_3$ (dirs 0,1,2)

$M_3 \to T^2$ (dirs 1,2) $\times \mathbb{R}$ (time)

STRING THEORY ...

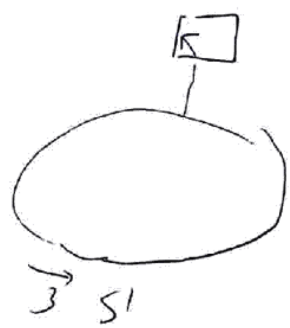
S-duality ⟶ geometric realization

$\tau \to -1/\tau$ $\qquad$ $\tilde{T}^2(\tau)$

	brane	0	1	2	3	...	10	
IIB	D3	—	—	—	✗		~	T-duality on 1
IIA	D2	—		—	✗		~	to M
M	M2	—		—	✗		•	reduce to IIA on 2
IIA	F1	—	•	~	✗		•	

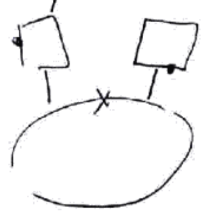
$$\begin{array}{c} \tilde{T}^2(\tau) \longrightarrow X \quad \text{3dim} \\ \text{dirs } (1,10) \qquad \downarrow \qquad \\ \qquad S^1 \ (\text{dir-}3) \end{array}$$

$\mathbb{R}/\mathbb{Z}$ dir 2



How many ground states ⟶ how many minimal length strings?

$\underline{n=1}$ Γ - section of fiber bundle



minimal length : Γ = constant.

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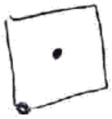


$\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$

$\underline{n=1}$

$N_s = 2$

$\tau = e^{i\pi/3}$

$N_s = 1$ $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$\tau = e^{i\pi/3}$

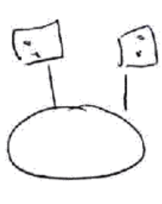
$N_s = 3$ $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$U(1)$ CS/T^2 at level k has $N_s = k$.

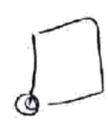
$$W_1 = e^{i\oint_1 A}, \quad W_2 = e^{i\oint_2 A}$$

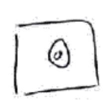
$$W_1 W_2 W_1^{-1} W_2^{-1} = e^{2\pi i/k}$$

$\underline{n=2}$



$\tau = i$





2 zero-particle states, winding #1
or
1 two-particle state, winding #2

$N_s = 1$ $N_s = 1$

$N_s = 2$ $N_s = 2$

total: $N_s = 6$

⑧

$SL(2,\mathbb{Z})$	"k"	τ	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$	2	i	2	3	5	10	15
$\begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}$	1	$e^{\pi i/3}$	1	6	12		
$\begin{pmatrix} -1 & 1 \\ -1 & \end{pmatrix}$	3	$e^{\pi i/3}$	3	9			

$$U(n)_k \text{ C.S.} = (U(1)_{kn} \times SU(n)_k)/\mathbb{Z}_n$$

$$\underset{U(1)}{A} \rightarrow \underbrace{\begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix}}_{n} \in U(n)$$

$$\overset{U(1)}{\frac{n}{4g^2}} \int F \wedge {*F} \rightarrow \frac{g^2}{4n} (\tilde{F} \wedge {*\tilde{F}}) = \frac{g^2 n}{4} \int \tilde{\tilde{F}} \wedge {*\tilde{\tilde{F}}}$$

$$\tilde{A}/n = \tilde{\tilde{A}} \times A \qquad \frac{ik}{4\pi} \int \tilde{\tilde{A}} \wedge dA$$

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$SU(2)_k$ on T^2 $\quad N_S = k+1$

$$\frac{(k-2)(k+1)}{2^2}$$

operators on $\mathcal{H}$

$n=1 \quad \tau = i \quad k=2$

$$T^2 \to X$$
$$\downarrow$$
$$S^1$$

$\mathbb{Z}_k$-momentum

$$W_1 = e^{\pi i P} |\square\rangle = |\boxdot\rangle$$

$$e^{\pi i P}(|\boxdot\rangle) = |\square\rangle$$

$e^{\pi i W}$ $\mathbb{Z}_2$-winding

$H_1(X) \quad \underset{\alpha}{\mathbb{Z}} \oplus \underset{\beta}{\mathbb{Z}/2\mathbb{Z}} \quad n\alpha + w\beta$

$\uparrow$ around S^1

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$b = a = -b, \quad 2b \equiv 0.$$

$$W_2 = e^{i\pi W} |\square\rangle = |\square\rangle, \quad e^{\pi i W}|\boxdot\rangle = -|\boxdot\rangle$$

$SL(2, \mathbb{Z})$

$\hookrightarrow$ T-duality

G = 2 copies of $U(1)_2$

If $n > 5$, get moduli, or $n=4$ $k=1$

moduli space is

$(\mathbb{R}^6 \times T^2)/\mathbb{Z}_4 \to \mathbb{R}^8/\mathbb{Z}_4$

the ABJM space!