't Hooft Loops and S-Duality

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KITP, Dualities in Physics and Mathematics

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Motivation

1) Quantum Field Theory

- Provide the path integral definition of all operators in the theory, including order and disorder operators:
  - 't Hooft local operators in $D = 3$
  - vortex loop operators in $D = 3$
  - 't Hooft loop operators in $D = 4$
  - surface operators in $D = 4$
  - ...
- Computation of correlators of renormalized operators
- Understand whether these operators serve as order parameters of novel phases in gauge theory, e.g.

Higgs phase: $< T(C) > \propto \exp(-\tau A(C))$

Confining phase: $< T(C) > \propto \exp(-m P(C))$

2) Duality

- These operators allow us to probe aspects of weak$\leftrightarrow$strong dualities, which are ubiquitous in M-theory and some quantum field theories
- Allows for the exploration of new sectors in holographic correspondences
  - "small" operators $\longleftrightarrow$ bulk D-branes
  - "large" operators $\longleftrightarrow$ topologically rich, asymptotically AdS metrics
- Defining the correlation function of these operators in $\mathcal{N} = 4$ super Yang-Mills allows us to explore the S-duality conjecture for these observables
- Understanding magnetic operators as an intermediate step in deriving the magnetic, dual formulation of $\mathcal{N} = 4$ super Yang-Mills

Duality in Lattice Models

D=3 Ising Model

$$Z_A(K) = \sum_{\sigma} \exp\left(K \sum_{<ij>} \sigma_i \sigma_j \right)$$

D=3 Z_2 Lattice Gauge Theory

$$Z_B({}^L K) = \sum_{U_l} \exp\left({}^L K \sum_{p} U_p \right)$$

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$$\sinh(2K)\sinh(2\,{}^L K) = 1$$

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- The two theories are related by a weak/strong coupling duality
- There is a change of variables in the partition sum

$$\boxed{\sigma_i \longleftrightarrow U_l}$$

Mapping of Observables

	D=3 Ising Model	D=3 Z_2 Lattice Gauge Theory
local:	σ_i	??
non-local:	??	$W(C) = \prod_{l \subset C} U_l$

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non-local:	$\mathcal{O}(C)$: Disorder Operator	$W(C) = \prod_{l \subset C} U_l$

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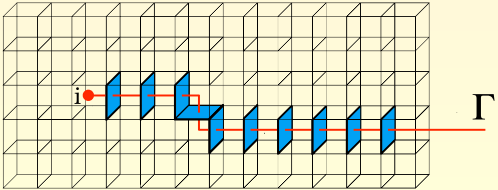
- The 't Hooft operator inserts a monopole at a point. Defined by:

$$< T_i >_B = \frac{\widetilde{Z}_B(^L K)}{Z_B(^L K)}$$

where

$$\widetilde{U_p} = \begin{cases} -U_p \text{ for } p \cap \Gamma \\ U_p \text{ for } p \not\cap \Gamma \end{cases}$$



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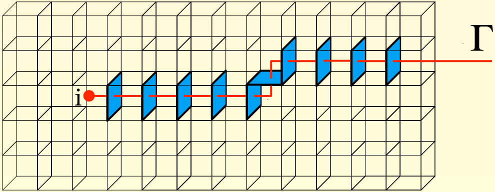

- Dual operators constructed by changing variables in the path integral

$$\text{electric} \longleftrightarrow \text{magnetic}$$

- Duality leads to the study of monopole operators
- These considerations motivate the study of disorder operators supported on various submanifolds in spacetime
- Correlators of observables are mapped into each other by the duality transformation:

$$< \prod_i \sigma_i \prod_{C_a} \mathcal{O}(C_a) >_{A,K} = < \prod_i T_i \prod_{C_a} W(C_a) >_{B,{}^L K}$$

- This theory realizes the picture of confinement as the dual Meissner effect

$$< T_i > \neq 0 \qquad < W(C) > \propto \exp\left(-\tau A(C)\right)$$

't Hoooft Loop Singularity

- Singularity produced by the insertion of a straight line 't Hooft operator

$$F=\frac{B}{2}\mathrm{Vol}(S^2)\,;\quad \phi=\frac{B}{2r}$$

Kapustin

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Comments:

- $B \equiv \sum_i B_i H^i \subset \mathfrak{t}$ characterizes the textcolorscarlet1strength of the

singularity

- $B_i \simeq$ highest weight vector of a representation ${}^L R$ of ${}^L G \Rightarrow T({}^L R)$ GNO
- $T({}^L R)$ topologically non-trivial when ${}^L R$ is charged under $Z({}^L G)$
- $\tilde{r}$ is distance to the circle:

$$\tilde{r}^2 = \frac{(r^2 + x^2 - a^2)^2 + 4a^2 x^2}{4a^2}$$

't Hooft Loop in $AdS_2 \times S^2$

- Map $R^4 \rightarrow AdS_2 \times S^2$ by a Weyl transformation to make the symmetries of the 't Hooft loop $T(^L R)$ manifest

- The choice of AdS_2 depends on the choice of geometry for the loop

straight line	$\Longrightarrow$	AdS_2: upper half-plane
circular loop	$\Longrightarrow$	AdS_2: Poincaré disk

- Field configuration produced by the insertion of a 't Hooft loop $T(^L R)$ in $AdS_2 \times S^2$ when $\theta \neq 0$

$$F = \frac{B}{2}\mathrm{Vol}(S^2) + ig^2\theta\frac{B}{16\pi^2}\mathrm{Vol}(AdS_2)\,; \quad \phi = B\frac{g^2}{4\pi}|\tau|$$

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Comments:

- For $\theta \neq 0 \Longrightarrow$ Witten effect
- 't Hooft loop in $AdS_2 \times S^2$ creates a regular field configuration

Computing the 't Hooft Loop

- Consider the $\mathcal{N}=4$ super Yang-Mills path integral in the presence of a 't Hooft operator $T({}^{L}R)$

- 't Hooft operator specified by a path integral over all fields with a prescribed singularity

$$A = A_0 + \hat{A}$$

$$\phi = \phi_0 + \hat{\phi}$$

Semiclassical Approximation

- The leading order result in the $\hbar$ expansion for the 't Hooft loop is:

$$< T({}^{L}R) > \simeq \exp\left(-S^{(0)}_{\mathcal{N}=4}\right)$$

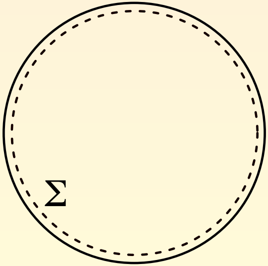
- Evaluate the on-shell action of $\mathcal{N}=4$ super Yang-Mills on $AdS_2 \times S^2$:

$$S^{(0)}_{\mathcal{N}=4} = \frac{1}{g^2}\int \mathrm{Tr}(F\wedge *F) - i\frac{\theta}{8\pi^2}\int \mathrm{Tr}(F\wedge F) = \mathrm{Tr}(B^2)\frac{g^2|\tau|^2}{16\pi}\mathrm{Vol}(AdS_2)$$

- The 't Hooft operator $T({}^{L}R)$ must be renormalized
- Renormalize the operator by adding boundary terms to the $\mathcal{N}=4$ super Yang-Mills action

$$S_{\mathcal{N}=4} \longrightarrow S_{\mathcal{N}=4} + S_{ct}$$

- The boundary terms play the role of counterterms and are part of the path integral definition of the 't Hooft loop operator



- The counterterms associated to the 't Hooft loop operator are:

$$S_{ct} = \frac{1}{g^2} \int_{\Sigma} \mathrm{Tr}\left(F|_{\Sigma} \wedge *_3 F|_{\Sigma} - f \wedge *_3 f\right)$$

• The leading semiclassical result for the 't Hooft operator is:

$$< T(^L R) > \simeq \exp\left(\frac{\mathrm{Tr}(B^2)}{8} g^2 |\tau|^2\right)$$

where

$B \subset \mathfrak{t}$ is the highest weight vector of representation $^L R$ of $^L G$
$\mathrm{Tr}(\ ,\ )$ is the invariant metric on the Lie algebra $\mathfrak{g}$

Quantum 't Hooft loop

• Path integrate over all fields with the prescribed singularity

$$A = A_0 + \hat{A}$$

$$\phi = \phi_0 + \hat{\phi}$$

- Integrate over quantum fluctuations $\hat{A}, \hat{\phi}, \ldots$
- Gauge fix path integral using background field gauge

$$D_0^M \hat{A}_M = 0 \quad \Longrightarrow \quad D_0^\mu \hat{A}_\mu + [\phi_0^I, \hat{\phi}_I] = 0$$

- Add gauge fixing terms and the associated Faddeev-Popov ghosts

$$\mathcal{L}_{gf} = \frac{1}{g^2} \mathrm{Tr} \left(\left(D_0^M \hat{A}_M \right)^2 - \bar{c} D_0^M D_M c \right)$$

- From the gauge fixed path integral can extract Feynman rules and compute the 't Hooft loop correlators in an $\hbar$ expansion

't Hooft Operator at One Loop

- Integrating the fields out at one loop produces a ratio of determinants

$$\frac{\prod \mathsf{det}'_F \cdot \mathsf{det}'_G}{\prod \mathsf{det}'_B} = 1$$

• In order to make the definition of the 't Hooft operator $T({}^L R)$ gauge invariant, we must also integrate over the coadjoint orbit of B

$$\mathcal{O}(B) = \{\mathrm{g} B \mathrm{g}^{-1} \, , \mathrm{g} \subset G\}$$

$\Longrightarrow$

$$< T({}^L R) > \simeq \exp\left(\frac{\mathrm{Tr}(B^2)}{8} g^2 |\tau|^2\right) \cdot \int [d\mu_{\mathcal{O}(B)}]$$

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- The metric on the coadjoint orbit is given by

$$ds^2_{\mathcal{O}(B)} = \frac{g^2|\tau|^2}{4} \sum_{\alpha>0,\, \alpha(B)\neq 0} \alpha(B)^2 \cdot 2\,\mathrm{Tr}(E^{\alpha}, E^{-\alpha}) |d\xi_{\alpha}|^2$$

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where

$$\mathrm{g} = \exp\left(i \sum_i \xi_i H^i + i \sum_\alpha \xi_\alpha E^\alpha\right)$$

$$\sum_{\alpha>0,\, \alpha(B)\neq 0} 2\,\mathrm{Tr}(E^\alpha, E^{-\alpha}) |d\xi_\alpha|^2 = ds^2_{G/H}$$

- Therefore, the t' Hooft operator expectation value is given

$$< T(^L R) > \simeq \exp\left(\frac{\mathrm{Tr}(B^2)}{8} g^2 |\tau|^2\right) \cdot \left(\frac{g^2|\tau|^2}{8\pi}\right)^{\mathsf{dim}(G/H)/2} \cdot \mathsf{Vol}(G/H) \cdot \prod_{\alpha>0,\, \alpha(B)\neq 0} \alpha(B)^2$$

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Comments:

- Valid for arbitrary 't Hooft operator and arbitrary gauge group G
- Non-trivial dependence on the super Yang-Mills coupling constant g from integration over the coadjoint orbit
- Dependence on the stability group $H \subset G$ preserving the singular field configuration characterized by the highest weight vector B of $^L R$
- Once we have the determined measure, can compute the 't Hooft operator to any order in perturbation theory using Feynman diagrams

S-Duality in $\mathcal{N} = 4$ super Yang-Mills

- Theory has a conjectured symmetry group $\Gamma \subset SL(2, R)$
- Γ acts on the operators and coupling constant of the theory

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

- Γ generated by:
  - Classical symmetry T: $\tau \rightarrow \tau + 1$
  - Quantum symmetry S:

$$\begin{aligned} \tau &\rightarrow -1/n_{\mathfrak{g}}\tau \qquad n_{\mathfrak{g}} = 1, 2, 3 \\ G &\rightarrow {}^{L}G \end{aligned}$$

  - S-duality exchanges electric and magnetic charges

$$Z(G) \leftrightarrow \pi_1({}^{L}G)$$

S-Duality in $\mathcal{N} = 4$ super Yang-Mills

- Conjectures about the action of S-duality on a large class of supersymmetric operators exist
- There are two families of loop operators: Wilson and 't Hooft Operators

$$G\colon\ W(R),\ \ T({}^LR)$$

$${}^LG\colon\ W({}^LR),\ T(R)$$

- Under S-duality:

$$W(R) \longleftrightarrow T(R) \qquad T({}^LR) \longleftrightarrow W({}^LR)$$

- S-duality predicts that correlators transform into each other

$$< \ T({}^LR)\prod_i \mathcal{O}_i >_{G,\tau} = < \ W({}^LR)\prod_i {}^L\mathcal{O}_i >_{{}^LG,{}^L\tau}$$

Wilson Operators in $\mathcal{N}=4$ with gauge group ${}^LG$

- Consider the supersymmetric circular Wilson loop operator

$$W({}^LR) = \mathrm{Tr}_{{}^LR} P \exp\left(\oint iA + \phi\right)$$

Maldacena
Rey & Yee

- The expectation value of this Wilson loop captured by a matrix integral

$$< W({}^LR) >_{{}^L\tau} = \frac{1}{\mathcal{Z}} \int_{{}^L\mathfrak{g}} [dM] e^{-\frac{2}{{}^Lg^2}\langle M,M\rangle} \mathrm{Tr}_{{}^LR}\, e^M$$

[ESZ]
Drukker & Gross
Pestun

- Localize the integral to the Cartan subalgebra ${}^L\mathfrak{t}$

$$\frac{\mathsf{Vol}({}^LG/{}^LT)}{|{}^LW|} \int_{{}^L\mathfrak{t}} [dX]\, \Delta(X)^2 e^{-\frac{2}{{}^Lg^2}\langle X,X\rangle} \mathrm{Tr}_{{}^LR}\, e^X$$

where

$$\Delta(X)^2 = \prod_{{}^L\alpha>0} {}^L\alpha(X)^2$$

- Represent Wilson loop as sum over weights v in the representation ${}^LR$

$$\mathrm{Tr}_R\, e^X = \sum_v n_v e^{v(X)}$$

$$\Longrightarrow$$

$$\frac{\mathrm{Vol}({}^LG/{}^LT)}{|{}^LW|} \sum_v n_v e^{\frac{{}^Lg^2}{8}\langle v,v\rangle} \int_{{}^L\mathfrak{t}} [dX] e^{-\frac{2}{{}^Lg^2}\langle X,X\rangle} \prod_{{}^L\alpha>0} \left({}^L\alpha(X) + \frac{{}^Lg^2}{4}\langle {}^L\alpha, v\rangle\right)^2$$

- Interested in the behaviour of Wilson loop for ${}^Lg >> 1$
  - Dominant contribution for v for which $\langle v, v\rangle$ is maximal
    $\Longrightarrow$ $v = w$ highest weight vector in ${}^LR$ up to action of ${}^LW$

$$\frac{\mathrm{Vol}({}^LG/{}^LT)}{|W_{{}^LH}|\mathcal{Z}} e^{\frac{{}^Lg^2}{8}\langle w,w\rangle} \prod_{{}^L\alpha>0,\langle {}^L\alpha,w\rangle\neq 0} \left(\frac{{}^Lg^2}{4}\langle {}^L\alpha, w\rangle\right)^2 \int_{{}^L\mathfrak{t}} [dX] e^{-\frac{2}{{}^Lg^2}\langle X,X\rangle} \prod_{{}^L\alpha>0,\langle {}^L\alpha,w\rangle=0} {}^L\alpha(X)^2$$

- Integration over X yields

$$< W(^L R) > \simeq \exp\left(\frac{\langle w, w\rangle}{8} {}^L g^2\right) \cdot \left(\frac{^L g^2}{8\pi}\right)^{\mathrm{dim}(^L G/^L H)/2} \cdot \mathrm{Vol}\left(^L G/^L H\right) \prod_{^L\alpha>0,\ ^L\alpha(w)\neq 0} \langle ^L\alpha, w\rangle^2$$

$\Longrightarrow$

$$< T(^L R) >_{G,\tau} = < W(^L R) >_{^L G, ^L \tau}$$

- Computations and agreement with S-duality can be extended to the case of correlators with chiral primary operators

$$< T(^L R) \cdot \mathcal{O} >_{G,\tau} = < W(^L R) \cdot {}^L\mathcal{O} >_{^L G, ^L \tau}$$

Conclusions and Outlook

- Given an explicit quantum definition of correlators of 't Hooft operators
- Exhibited S-duality for correlation functions in $\mathcal{N}=4$ super Yang-Mills

$$\text{'t Hooft Operators} \longleftrightarrow \text{Wilson Operators}$$

- Provide the quantum definition of other disorder operators and probe their role in S-duality
- Ultimately find the magnetic description of $\mathcal{N}=4$ super Yang-Mills by changing variables in the path integral