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Quantum Geometric Langlands and Gauge Theory

$$\mathcal{N}=4 \text{ SYM} \overset{\text{GL-twist}}{\rightsquigarrow} 4d \ \text{TFT}_{t \in \mathbb{CP}^1}$$

$$Q = Q_l + t Q_r$$

parameters: t, e^2, θ ; $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$

"Relevant" parameter: $\Psi = \frac{\theta}{2\pi} + \frac{t^2-1}{t^2+1} \frac{4\pi i}{e^2}$

"Classical" geometric Langlands:

$${}^LG \quad t=i \quad \xleftrightarrow{S} \quad G, \ t=1, \ \theta=0$$

$(\Psi=\infty)$

$M_4 = C \times \mathbb{R}^2$ $\qquad$ $M_4 = C \times \mathbb{R}^2$

B-branes on $\mathcal{M}_{H,t}({}^LG, C)_J \xleftrightarrow{1}$ A-branes on $\mathcal{M}_{H,t}(G, C)_K$

$\parallel$

$\mathcal{M}_{flat}({}^LG_{\mathbb{C}}, C)$

$\omega_K \sim \int_C \mathrm{Tr}\, \delta A\, \delta\phi$

$\omega = \mathrm{Im}\,\tau \cdot \omega_K$

$\downarrow 2$

$\mathcal{D}$-modules on $\mathrm{Bun}_G\, C$

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(2) Canonical coisotropic A-brane

$$\boxed{B = -\frac{\theta}{2\pi}\omega_I}$$

$$F \sim \operatorname{Im}\tau\, \omega_J \sim \int_C \operatorname{Tr}(\delta A_z\, \delta A_{\bar z} + \delta\phi_z\, \delta\phi_{\bar z})$$

$$(\omega^{-1}F)^2 = -1$$

$$\omega^{-1}F = I$$

$$\mathcal{M}_{H+}(G,C)_I \cong \mathcal{M}_{\substack{\text{Higgs}\\ \text{bundle}}}(G,C)$$

Observables on c.c. brane are holomorphic functions on $\mathcal{M}_{\text{Higgs}}(G,C)$

$$[f,g] = (\operatorname{Im}\tau)^{-1}\{f,g\}_{\Omega_I} + \ldots$$

$$\Omega_I = \omega_K + i\omega_J \sim \int \operatorname{Tr}\, \delta A_{\bar z}\, \delta\phi_z$$

$t = 1$, θ arbitrary

$$\psi = \frac{\theta}{2\pi}, \quad B = -\frac{\theta}{2\pi}\omega_I, \quad (\omega^{-1}(F+B))^2 = -1.$$

$$F = \alpha\,\omega_J \qquad (\text{c.c. A-brane})$$

$$\omega^{-1}(F+B) = \underbrace{\alpha I + \beta J}_{= I_\theta}, \quad \alpha^2 + \beta^2 = 1.$$

$$\beta = \frac{\operatorname{Re}\tau}{\operatorname{Im}\tau}$$

$$\omega = \operatorname{Im}\tau\, \omega_K$$

③

$\mathcal{M}_{Hit}(G,C)_{I_\theta} \cong \underset{\lambda\text{-connections}}{\mathcal{M}_{flat}}(G_{\mathbb{C}},C) =$ "twisted* cotangent bundle" over $Bun_G(C)$

$$\left.\begin{array}{l} \partial: \Gamma(E) \to \Gamma(E\otimes\Omega^1) \\ \partial(f s) = \lambda\partial f\otimes s + f\otimes\partial s \end{array}\right\} \lambda\text{-connections}$$

twisted bundle: $\quad p_i^{(\alpha)} dq^{(\alpha)i} = p_i^{(\beta)} dq^{(\beta)i} + \gamma_{\alpha\beta}$

$$\gamma_{\alpha\beta} \in C^1(\Omega^1_d)$$

$\Rightarrow$ sheaf of diff. operators on $\mathcal{L}_0^{\frac{\theta}{2\pi} - h^\vee}$, $\quad \frac{\theta}{2\pi} - h^\vee = \Psi$

$$S: \Psi \to {}^L\Psi = -\frac{1}{\Psi \cdot n_g} \qquad t \to {}^Lt = e^{i\varphi}t$$

$$n_g = \{1,2,3\}$$

Another approach: Consider $t=0$.

$$Q = Q_\ell \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

$$S = \{Q,V\} + \frac{i\bar{\tau}}{4\pi}\int \mathrm{Tr}\, F\wedge F$$

BPS equations: $(F - \frac{1}{2}[\phi,\phi])^+ = 0, \quad (D\phi)^- = 0, \quad D_\mu\phi^\mu = 0.$

④

$M_4 = C \times \mathbb{R}^2$

A-model with target $\mathcal{M}_{Higgs}(G,C)_I$

$\Downarrow$

A-branes on $\mathcal{M}_{Higgs}(G,C)$ $\longleftrightarrow$ A-branes on $\mathcal{M}_{Higgs}({}^LG,C)$

$B + i\omega = -\bar{\tau}\,\omega_I$

$S: \tau \to {}^L\tau = -\frac{1}{n_g \tau}$ (A, B, A)

Analog of cc branes for $\theta = 0$

$F = \mathrm{Im}\,\tau\, \omega_K$ (d.c. brane)

(A, A, B)

(doesn't solve: $(\omega^{-1}(F+B))^2 = -1$)

$\mathcal{M}_{Hit}(G,C)_J \cong \mathcal{M}_{flat}(G_{\mathbb{C}}, C)$

$\Rightarrow$ algebra of observables on the d.c. brane is the quantization of holomorphic functions on $\mathcal{M}_{flat}(G_{\mathbb{C}}, C) \supset$ twisted cotangent bundle on $\mathrm{Bun}_G(C)$

$\rightsquigarrow$ Sheaf of diff. operators on $\mathcal{L}_0^{-h^\vee - i(\mathrm{Im}\,\tau)^{-1}}$

$(\mathrm{Im}\,\tau)^{-1} = \frac{e^2}{4\pi}$

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Loop operators at $t=0$.

Only 1 exists

1) Wilson operators: not BRST-invariant.

$$\left.\begin{aligned} \delta A_\mu &= i\psi_\mu \\ \delta \phi_\mu &= -i\tilde{\psi}_\mu \end{aligned}\right\} \Rightarrow \text{no BRST-inv. connection.}$$

2) 't Hooft operators

$$F_{ij} = \frac{\varphi}{2}\epsilon_{ijk}\frac{x^k}{r^3}$$

$$F_{4k} = \frac{\rho}{2}\frac{x^k}{r^3}$$

$\Rightarrow$ no 't Hooft op.

3) Reduce 4d TFT on S^2; get a 2d TFT.
Look for branes in this 2d TFT

Surface operators do exist