20 March 2009

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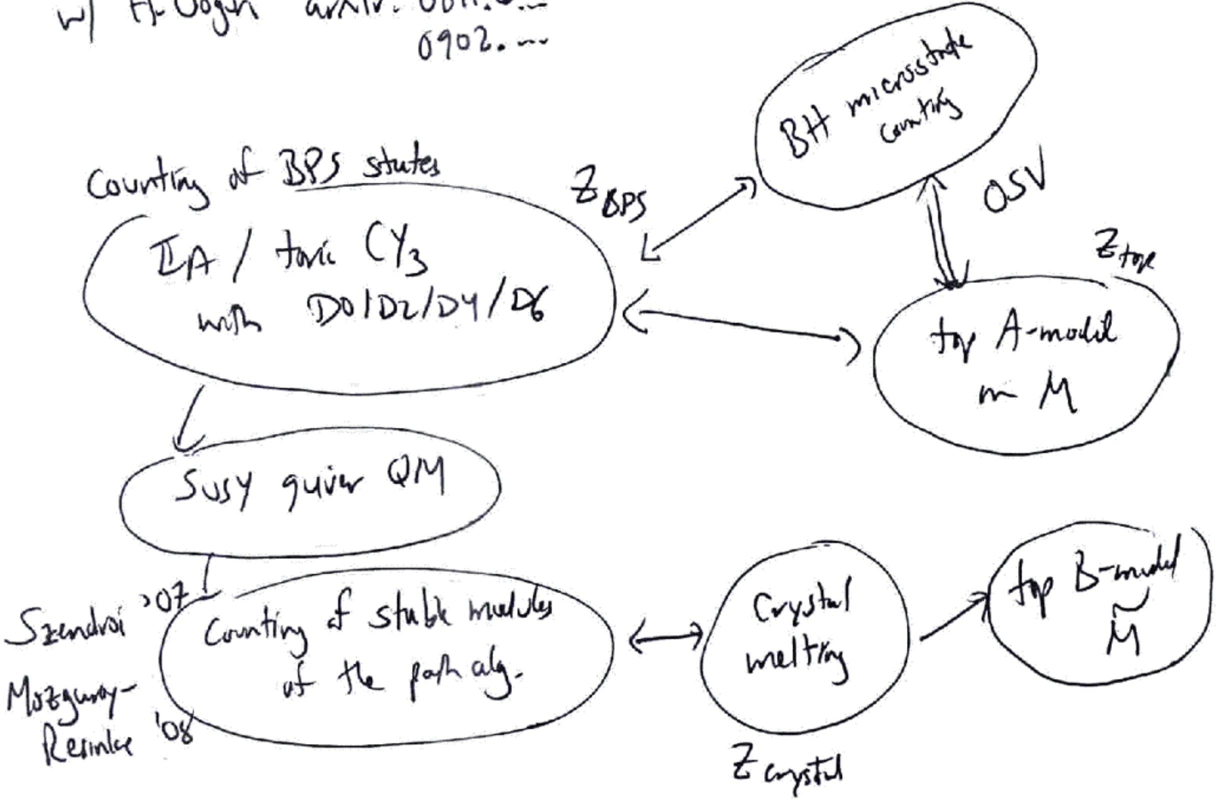
Crystal Melting and Non-Commutative Donaldson-Thomas theory

w/ H. Ooguri arXiv: 0811.…
0902.…



※ wall crossing

$$Z_{BPS}(t'_\infty) = Z_{crystal}$$

$$Z_{BPS}(t''_\infty) = Z_{top}$$

$Z_{crystal} \sim Z_{top}$ (up to wall crossings)

Today type IIA string
M = conifold

※ · our story applies to arbitrary toric CY_3

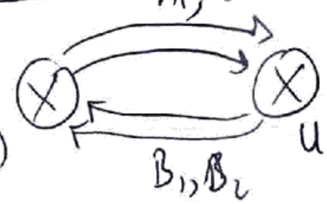
with N_0 D0-brane, N_2 D2-brane, one D6-brane
(physical string, not topological string)

Quiver QM

K-W

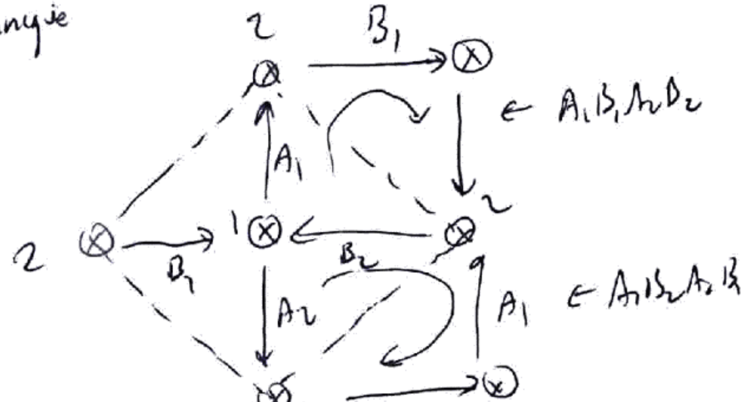
$\mathbb{C}^{N_0 - N_2} = \mathbb{C}^{n_2}$

$\mathbb{C}^{n_1} = \mathbb{C}^{N_0}$



$W = \mathrm{tr}\, A_1 B_1 A_2 B_2$
$\quad - \mathrm{tr}\, A_1 B_2 A_2 B_1$

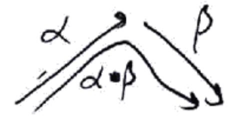
brane tiling technique



periodic quiver

3

Hanany-Vegh : similar construction for any toric CY_3



noncommutative algebra

NCDT : path algebra

$$\mathcal{A} = \underset{\substack{\uparrow \\ \text{trivial path}}}{\mathbb{C}[e_1,e_2]}\langle a_1, b_1, a_2, b_2 \rangle \Big/ \left(\begin{array}{l} a_1 b_i a_2 = a_2 b_i a_1 \\ b_1 a_i b_2 = b_2 a_i b_1 \end{array} \right)$$

$$\frac{\partial W}{\partial a_i} = \frac{\partial W}{\partial b_i} = 0$$

(F-term constraints)

Counting of BPS states
= Counting of θ-stable representations M of $\mathcal{A}$.
(module)

D-term constraints $\Rightarrow$ θ-stability.

$\theta = \theta_1, \theta_2 \longleftrightarrow$ nodes of diagram.

$$M = M_1 \oplus M_2, \quad \dim M_1 = n_1, \quad \dim M_2 = n_2$$

BPS degeneracy $\Omega(\underset{\substack{\uparrow \\ D_0}}{N_0}, \underset{\substack{\uparrow \\ D_2}}{N_2}) = \chi(\mathcal{M}_{n_1,n_2})$

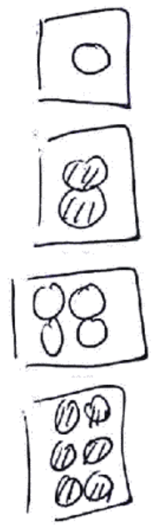
localization theorem

$$= \chi(\underset{\substack{\uparrow \\ \text{torus fixed points}}}{\mathcal{M}^T_{n_1,n_2}})$$

(4)

<u>Thm</u> (Szendrői, Mozgovoy – Reineke, Ooguri – Y.)

element of $\mathcal{M}^{J}_{n_1, n_2}$ $\xleftrightarrow{1:1}$ molten crystal



pyramid

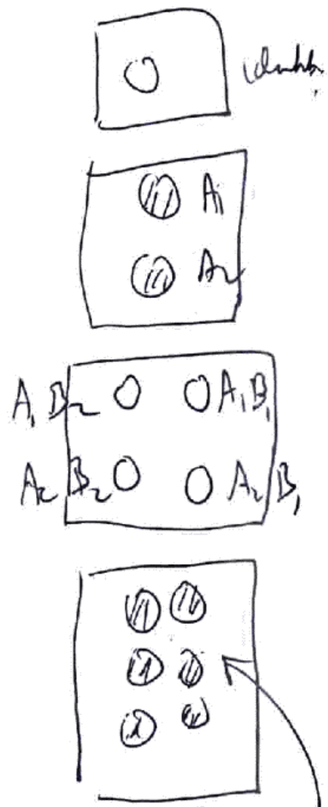
lift of periodic quiver

$$\begin{tikzcd}
\overset{1}{\otimes} & \overset{2}{\otimes} \arrow[l, "B_2"'] \arrow[r, "B_1"] & \overset{1}{\otimes} \arrow[d, "A_2"] \\
2\ \otimes \arrow[u, leftarrow, "A_2"] \arrow[r, "B_1"'] & \overset{1}{\otimes} \arrow[u, "A_1"'] \arrow[d, "A_2"] & \otimes\ 2 \arrow[l, "B_2"'] & \cdots \\
\underset{1}{\otimes} \arrow[u, "A_1"] & \underset{2}{\otimes} \arrow[l, "B_2"] \arrow[r, "B_1"'] & \underset{1}{\otimes} \arrow[u, "A_1"']
\end{tikzcd}$$

an atom of crystal = a path starting from vertex 1,
modulo F-term relations

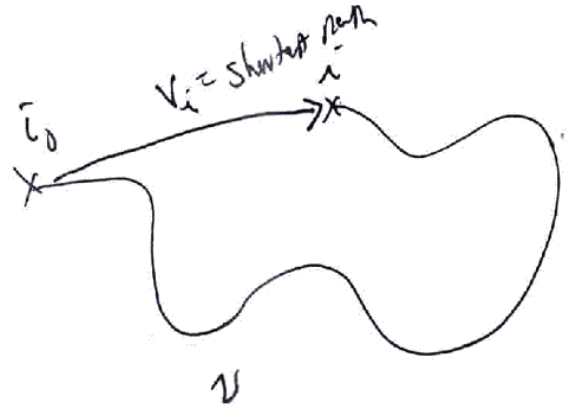
ABABAB...

5



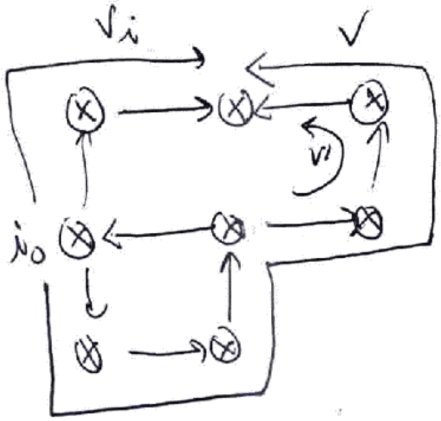
$A_i B_j A_k$ 8 choices
but there are F-term relations

$A_1 B_1 A_2 = A_2 B_1 A_1$

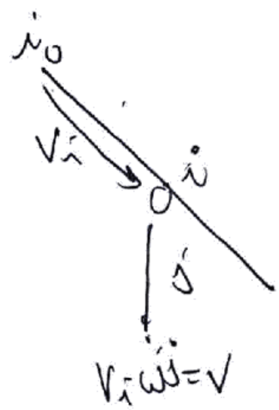


Prop $v \sim v_i \omega^j$ $j \in \mathbb{Z}_{\geq 0}$, ω = a path around a face
mod F-term

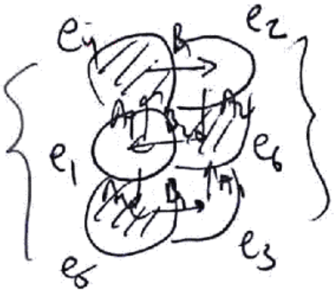
⑥



$V \sim V' \sim V_i \omega^1$



Excited state = removing finite # of atoms Ω from crystal.

e.g. $\Omega = \Big\{$  $\Big\}$

melting rule: 

$a\alpha \in \Omega \Rightarrow \alpha \in \Omega$

$\Omega \to$ module

• For each atom, prepare a one-dim vector space $\mathbb{C}e_i$

$$M = \bigoplus_{i=1}^{6} \mathbb{C}e_i = \underbrace{\left(\bigoplus_{i=1}^{3} \mathbb{C}e_i\right)}_{\bigcirc} \oplus \underbrace{\bigoplus_{i=4}^{6} \mathbb{C}e_i}_{\bullet}$$

$$= M_1 \oplus M_2$$

$$M_1 \otimes \rightleftarrows M_2$$

action of path alg

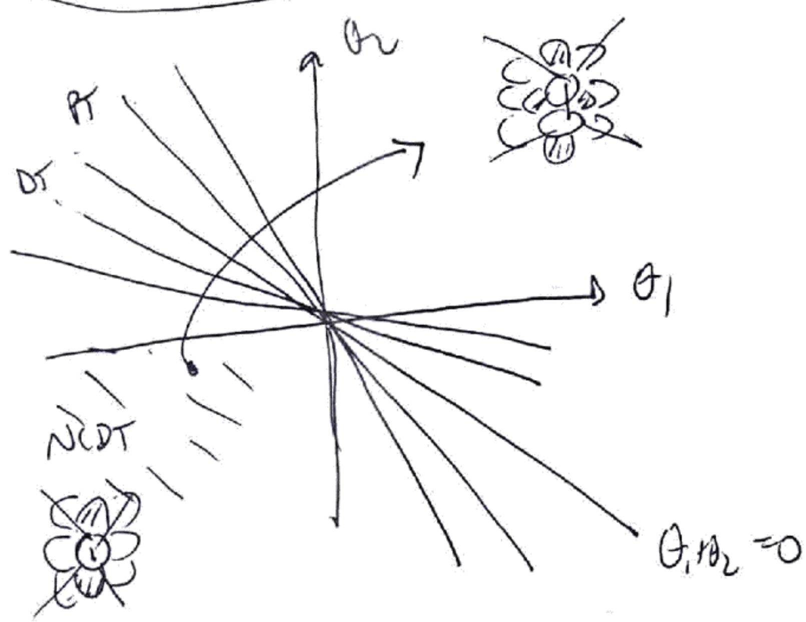
e.g. $A_1 : \begin{array}{l} e_1 \to e_4 \\ e_2 \to 0 \\ e_3 \to e_6 \end{array}$

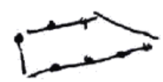
$e_\alpha \xrightarrow{a} a e_\alpha$, $a \in \Omega$.

$$a(e_\alpha) = \begin{cases} a \cdot e_\alpha & \text{if } a e_\alpha \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

$A_1, B_1, A_2 : e_1 \to e_6$
$A_2, B_1, A_1 : e_1 \to e_6$

Counting of BPS state $\longleftrightarrow$ Counting of molten crystal





Nakajima-Nagao, Nagao
Jafferis-Chuang

- top. vertex for NCDT ?
- crystal: $Z_{BPS} = \cancel{Z_{crystal}}\ Z_{top}$ $\to$ $Z_{top} \sim |Z_{top}|^2$ up to WC

OSV: $Z'_{BPS} = |Z_{top}|^2$