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Towards categorical Langlands correspondence

joint w/ Gaitsgory.

C = compact Riemann surface

G = complex reductive group

$\mathrm{Bun}(G) = \left\{ \begin{array}{c} \text{principal} \\ G\text{-bundles} \\ \text{on } C \end{array} \right\}$

${}^L G$ = its Langlands dual

$\mathrm{LocSys}({}^L G) = \left\{ \begin{array}{c} {}^L G\text{-local systems} \\ \text{principal bundles} \\ \text{with connection} \\ \text{on } C \end{array} \right\}$

Categorical Langlands conjecture?

$$\boxed{\begin{array}{c} \text{(derived categories of)} \\ \text{D-modules on } \mathrm{Bun}(G) \\ \text{(quasi coherent sheaves} \\ \text{w/ flat connection)} \end{array}} \equiv \boxed{\begin{array}{c} \text{quasi-coherent (derived category)} \\ \mathcal{O}\text{-modules} \\ \text{on} \\ \mathrm{LocSys}({}^L G) \end{array}}$$

① $G = GL(2)$

$$\boxed{\mathrm{Bun}(G) \begin{array}{l} \nearrow \text{Cuspidal D-mods} \\ \searrow \text{Eisenstein Series} \end{array}}$$

$$\begin{array}{r} \text{irreducible} \nwarrow \\ \text{reducible} \swarrow \end{array} \mathrm{LocSys}({}^L G)$$

$$^LG = G = \begin{pmatrix} * & * \\ * & * \end{pmatrix} = GL(2)$$
$$\cup$$
$$^LB = B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$^LT = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

$\mathcal{D}$-modules $\qquad\qquad\qquad\qquad$ $\mathcal{O}$-modules

$$\begin{tikzcd}
\mathrm{Bun}(T) = (\mathrm{Pic}(C))^2 \arrow[r, leftrightsquigarrow] & \boxed{\mathrm{LocSys}(^LT)} \\
\mathrm{Bun}(B) \arrow[u] \arrow[d] & \mathrm{LocSys}(^LB) \arrow[u] \arrow[d, "^L\pi"] \\
\mathrm{Bun}(G) \arrow[r, leftrightsquigarrow, "?"] & \mathrm{LocSys}(^LG)
\end{tikzcd}$$

(2) X = algebraic variety (e.g. $X = \mathrm{LocSys}(^LG)$)

$QCoh(X) = \{$quasi coherent $\overset{\text{complexes}}{\mathcal{O}}$-modules on $X\}$
$\cup$
$Coh(X) = \{$coherent $\overset{\text{complexes}}{\mathcal{O}}$-modules on $X\}$

Def $\mathcal{F} \in Coh(X)$ is <u>perfect</u> if it has a finite resolution by vector bundles $(0 \to V_n \to \cdots \to V_1 \to V_0 \to \mathcal{F} \to 0)$

(compact objects in $QCoh(X)$)

Thm If X is smooth, perfect = coherent.

If X is singular, $\mathrm{Perf}(X) \subsetneq Coh(X)$

Problem: $R\pi_*$

$$\begin{array}{c} QCoh(LocSys({}^L B)) \\ \downarrow \\ QCoh(LocSys({}^L G)) \end{array}$$

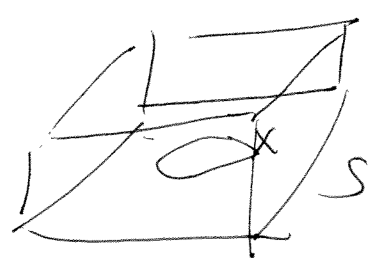
preserves Coh but not Perf.

Def C is *compact* if $Hom(C, \oplus *) = \oplus Hom(C, *)$

(3) $X = \{f_1 = \cdots = f_n = 0\} \subset S =$ smooth

Koszul duality $Coh(X) \xrightarrow{\mathcal{K}} LG$ on $S \times \mathbb{C}^n_{(t_1, \ldots, t_n)}$
for potential $\sum f_i t_i$

F is on $X \hookrightarrow \mathcal{K}(F)$ on S and extra variables



$Coh(X) / Perf(X)$

Given $F \in Coh(X)$ can consider $Supp(\mathcal{K}(F)) \subset S \times \mathbb{C}^n$
(has to be defined)

Fact F is perfect $\Longleftrightarrow$ $Supp(\mathcal{K}(F)) \subset S \times \{0\}$

Now <u>example</u>

$X = \mathrm{LocSys}({}^L G)$ $\quad c \in C$, frame all local systems at c

$\cap$

S = local system on C with a first order pole at $c \in C$

$X = \{ \underbrace{f_1 = \cdots f_n = 0}_{\text{residue} \in \mathfrak{g}} \}$ $\quad (n = \dim {}^L\mathfrak{g})$

$= 0$

$\mathrm{supp}(\mathcal{K}(F)) \subset S \times \mathfrak{g}$

<u>Exercise</u> Actually, $\mathrm{supp}\, \mathcal{K}(F) \subset \{ (E, \varphi) \mid E \in \mathrm{LocSys}({}^L G) = X,\ \varphi = \text{inf'l automorphism} \}$

<u>Definition</u> $\mathcal{N} = \{ (E, \varphi) \mid \varphi \text{ is nilpotent} \}$

$\mathrm{Coh}_{\mathcal{N}}(\mathrm{LocSys}({}^L G)) = \{ F \mid \mathcal{K}(F) \subset \mathcal{N} \}$

<u>"Corrected" version</u>

Compact D-modules on $\mathrm{Bun}(G)$ $\xleftrightarrow{\sim}$ $\mathrm{Coh}_{\mathcal{N}}(\mathrm{LocSys}({}^L G))$

D-modules on Bun_G $\xleftrightarrow{\sim}$ $\mathrm{QCoh}_{\mathcal{N}}(\mathrm{LocSys}({}^L G))$

Thm (for $GL(2)$), $\mathrm{coh}_{\mathcal{N}}$ is generated by $\mathrm{Perf}(\mathrm{LocSys}({}^L G))$ and $\mathrm{Perf}(\mathrm{LocSys}({}^L T))$