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Geometric Langlands and non-abelian Hodge theory

(joint with Denagi + Simpson)

Geometric Langlands for GL_n

C - smooth compact curve

(smooth Deligne-Mumford stack of dim 1)

Bun = stack of rank r holomorphic vector bundles on C

\underline{Bun} = rigidified stack of rank r vector bundles
 $= Bun / \mathbb{C}^*$

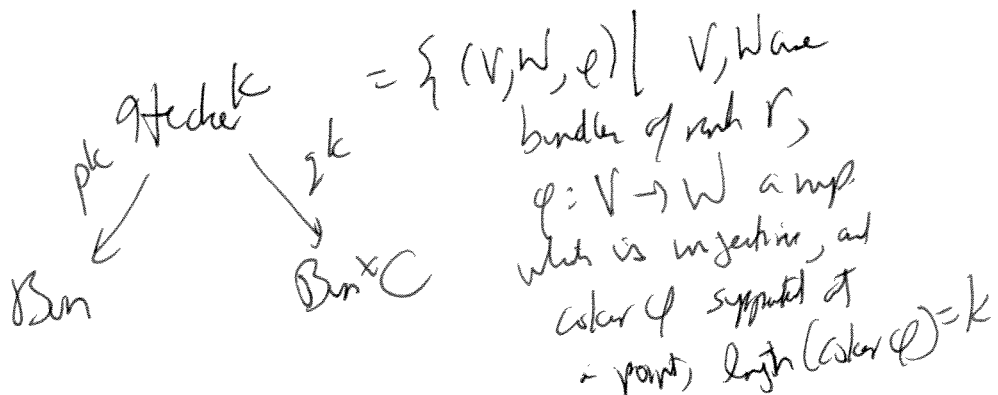
Geometric Langlands Conjecture

If $\mathcal{V} = (V, \nabla)$ is an irreducible rank r local system on C , then $\exists!$ irreducible \mathcal{D} -module $C_{\mathcal{V}}$ on \underline{Bun}

s.t. $H^k(C_{\mathcal{V}}) = C_{\mathcal{V}} \otimes \mathbb{1}^k_{\mathcal{V}}$

Here $\{H^k\}_{k=1}^r$, Hodge operators, are mps: $H^k: D(\underline{Bun}, \mathcal{D}) \rightarrow$

They come from



$$g^k = g_1 \circ p^*$$

Want to construct $C_{\mathbb{W}}$

Idea The assignment $\mathbb{W} \rightarrow C_{\mathbb{W}}$ is a composition

$$\text{quant}_{\text{Bun}} \circ \text{FM} \circ \text{quant}_C^{-1}$$

↑
Fourier-Mukai

of two quantization maps
an Fourier-Mukai
transform (T-duality
along the Hitchin fibres)

$\text{quant}_C =$ quantization procedure for sheaves on T^*C

$\text{quant}_{\text{Bun}} =$ quantization procedure for sheaves on $T^*\text{Bun}$

$\text{quant}_?$ are from non-abelian Hodge theory

Thm (non-abelian Hodge theorem)

$(X, \mathcal{O}_X(1))$ smooth, projective

$\Rightarrow \exists$ equivalence of tensor-dg categories

$$\left(\begin{array}{l} \text{Category of all} \\ \text{finite rank local} \\ \text{systems on } X \end{array} \right) \xrightarrow{\text{ruh}_X} \left(\begin{array}{l} \text{Category of all } \mathcal{O}_X(1)\text{-semistable} \\ \text{finite rank Higgs bundles } (E, \theta) \\ \theta: E \rightarrow E \otimes \Omega^1_X, \theta \wedge \theta = 0 \\ \text{ch}_1 E, \text{ch}_2 E = 0 \end{array} \right)$$

We also need the spectral construction:

If X is a variety,

$$SC_X : \text{Qcoh}(T^*X) \rightarrow \text{Qcoh Higgs}(X)$$

$$\xi \rightarrow (\pi_* \xi, \pi_* (\xi \otimes \lambda))$$

$$\lambda \in \Gamma(T^*X, \pi^* \Omega'_X), \text{ tautological.}$$

In particular, get an identification:

Higgs bundles on C and sheaves on T^*C

If \mathcal{V} is a local system on C ,

$$\text{nah}(\mathcal{V}) = (\text{Semistable Higgs bundle on } C \text{ with } c_1 = 0)$$

$$T^V \text{ Bun}^0 C \stackrel{\cong}{=} T^V \text{ Bun}^{\text{Higgs}} \quad (\text{cotangent stack})$$

Define $\text{quant}_C^{\text{Higgs}}(\mathcal{V}) = \mathcal{O}_{\text{nah}(\mathcal{V})}$

$$\text{Higgs}(E, \theta) \rightarrow \text{Supp}(SC_X^{-1}(E, \theta)) = \{ \det(\pi^* \theta - \lambda \text{id}_E) = 0 \}$$

spectral curve $\bar{C} \rightarrow C$
1:1

$$h \downarrow B = H^0(\Omega'_C) \oplus H^0(\Omega'_C \otimes \omega_C) \oplus \dots \oplus H^0(\Omega'^{\otimes r})$$

The fiber of Higgs over \bar{c} is $\text{Pic}(\bar{c})$.

$$\text{Higgs} \times_B \text{Higgs}$$

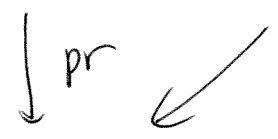
has a Poincaré sheaf \mathcal{P}

FM = Fourier-Mukai for \mathcal{P} .

$$\begin{aligned} \text{FM} \circ \text{quant}_c^{-1}(W) &= \text{FM}(\mathcal{O}_{\text{nah}(W)}) \\ &= L_{\bar{c}} \otimes L_W \end{aligned}$$

L_W is a line bundle on $\text{Pic}(\bar{c})$

$$\text{Higgs} = T^V \underline{\text{Bun}} \rightarrow \underline{\text{Pic}}(\bar{c})$$



$$L_{\bar{c}} \otimes L_W \in \text{Coh}(T^V \underline{\text{Bun}})$$

$\text{SC}_{\underline{\text{Bun}}}(\mathcal{F}, \theta)$ Higgs sheaf on $\underline{\text{Bun}}$

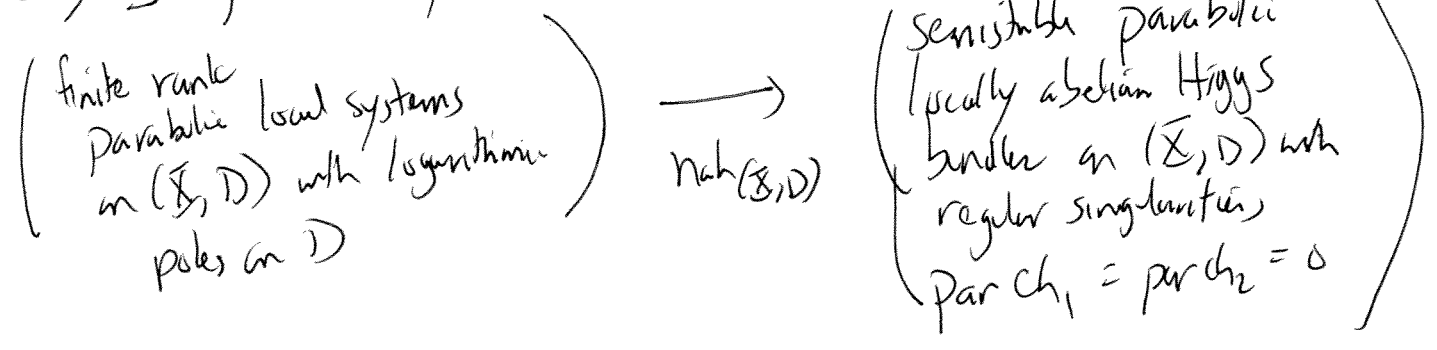
"quant $\underline{\text{Bun}}$ " should be $\text{nah}_{\underline{\text{Bun}}}$, but it does not exist

Can fix this by using nah with ramification

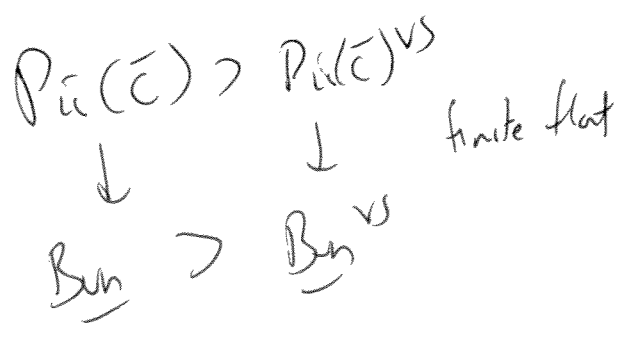
Idea: $\text{Bun} \supset \text{Bun}^s =$ stack of stable bundles,
a smooth quasi projective variety

Thm (Mochizuki) If $(\bar{X}, \mathcal{O}_{\bar{X}}(1))$ projective,
 $D \subset \bar{X}$ effective divisor, s.t. $\exists Z \subset \bar{X}$ of codim ≥ 3
so that $\bar{X} \setminus Z$ smooth, $D \rightarrow Z$ normal crossing.

$\Rightarrow \exists$ equivalence of tensor dg categories



$\text{Bun} \supset \text{Bun}^{vs} =$ very stable bundles.



Can convert $(F, \theta) / \underline{Bun}^{VS}$ to a local system

if we can find

\overline{Bun}^{VS} satisfying Mochizuki's conditions.

So that

(F, θ) extends to a parabolic, locally a Selmer, semistable with vanishing Chern classes.

Compatibility by $\underline{Bun}^{VS} \subset \underline{Bun}^{SS}$, and blowup to get normal crossings.

Thm (Mochizuki) Let $\overline{X}, \overline{Y}$ be projective varieties



$\overline{X} - U = \text{normal crossings}$

$\overline{Y} = \text{anything}$

$$\mathcal{N}_{X!} := \left(\begin{array}{l} \text{logarithmic local} \\ \text{systems on } \overline{X}, \\ \text{Smooth on } U \end{array} \right) \xrightarrow{\sim} \left(\begin{array}{l} \mathcal{D}\text{-module on } \overline{Y}, \\ \text{Smooth on } U \end{array} \right)$$

Example \mathbb{P}^1 with 5 pts, $GL(2)$.