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Geometric Langlands and non-abelian Hodge theory

(joint with Donagi + Simpson)

Geometric Langlands for GL_n

C – smooth compact curve
(smooth Deligne-Mumford stack of dim 1)

$\mathbb{B}\mathrm{un}$ = stack of rank r holo vector bundles on C

$\underline{\mathbb{B}\mathrm{un}}$ = rigidified stack of rank r vector bundles
$= \mathbb{B}\mathrm{un} /\!\!/ \mathbb{C}^\times$

Geometric Langlands Conjecture

If $\mathbb{V} = (V, \nabla)$ is an irreducible rank r local system on C, then $\exists!$ irreducible $\mathcal{D}$-module $c_{\mathbb{V}}$ on $\underline{\mathbb{B}\mathrm{un}}$ s.t. $H^k(c_{\mathbb{V}}) = c_{\mathbb{V}} \boxtimes \wedge^k \mathbb{V}$

Here $\{H^k\}_{k=1}^{r}$, Hecke operators, are maps: $H^k : D(\underline{\mathbb{B}\mathrm{un}}, \mathcal{D}) \circlearrowleft$

They come from

$$\begin{tikzcd} & \mathcal{H}ecke^k \arrow[dl, "p^k"'] \arrow[dr, "q^k"] & \\ \mathbb{B}\mathrm{un} & & \mathbb{B}\mathrm{un} \times C \end{tikzcd}$$

$\mathcal{H}ecke^k = \{(V, W, \varphi) \mid V, W$ are bundles of rank r, $\varphi : V \to W$ a map which is injective, and coker φ supported at a point, length(coker φ) $= k\}$

$\mathcal{H}^k = q_! \circ p^*$.

Want to construct $C_{\mathbb{V}}$

Idea The assignment $\mathbb{V} \to C_{\mathbb{V}}$ is a composition

$\mathrm{quant}_{\underline{\mathrm{Bun}}} \circ FM \circ \mathrm{quant}_C^{-1}$ of two quantization maps and Fourier-Mukai transform (T-duality along the Hitchin fibers)

Fourier-Mukai

quant_C = quantization procedure for sheaves on T^*C

$\mathrm{quant}_{\underline{\mathrm{Bun}}}$ = quantization procedure for sheaves on $T^*\underline{\mathrm{Bun}}$.

$\mathrm{quant}_?$ are from non-abelian Hodge theory

Thm (non-abelian Hodge theorem)

$(X, \mathcal{O}_X(1))$ smooth, projective

$\Rightarrow \exists$ equivalence of tensor-dg categories

$$\begin{pmatrix}\text{Category of all}\\ \text{finite rank local}\\ \text{systems on } X\end{pmatrix} \xrightarrow[\mathrm{nah}_X]{\sim} \begin{pmatrix}\text{Category of all } \mathcal{O}_X(1)\text{-semistable}\\ \text{finite rank Higgs bundles } (E, \theta)\\ \theta: E \to E \otimes \Omega^1_X,\ \theta \wedge \theta = 0\\ ch_1 E,\ ch_2 E = 0\end{pmatrix}$$

We also need the spectral construction:

If X is a variety,

$$SC_X : QCoh(T^*X) \to QCoh\,Higgs(X)$$

$$\xi \mapsto (\pi_* \xi, \pi_*(\otimes \lambda))$$

$\lambda \in \Gamma(T^*X, \pi^*\Omega^1_X)$, tautological.

In particular, get an identification

Higgs bundles on C and sheaves on T^*C

If $\mathbb{V}$ is a local system on C,

$$nah(\mathbb{V}) = (\text{semistable Higgs bundle on } C \text{ with } c_1 = 0)$$

$$\begin{array}{c} \cong \\ T^\vee Bun^0 \subset \underset{\mathcal{H}iggs}{\underset{\|}{T^\vee Bun}} \end{array}$$

(cotangent stack)

Define $quant_C^{-1}(\mathbb{V}) = \mathcal{O}_{nah(\mathbb{V})}$

$$\mathcal{H}iggs \ni (E,\theta) \mapsto Supp(SC_X^{-1}(E,\theta)) = \{\det(\pi^*\theta - \lambda\, id_E) = 0\}$$

spectral cover $\bar{C} \xrightarrow[r:1]{} C$

$$h \downarrow \quad B = H^0(\Omega^1_C) \oplus H^0(\Omega_C^{1\otimes 2}) \oplus \cdots \oplus H^0(\Omega^{1\otimes r})$$

The fiber of Higgs over $\bar{C}$ is $\mathrm{Pic}(\bar{C})$.

$$\mathcal{H}iggs \times_{B} \mathcal{H}iggs$$

has a Poincaré sheaf $\mathcal{P}$

FM = Fourier–Mukai for $\mathcal{P}$.

$$FM \circ \mathrm{quant}_C^{-1}(\mathbb{V}) = FM(\mathcal{O}_{\mathrm{nah}(\mathbb{V})})$$

$$= \iota_{\bar{C}*} \mathcal{L}_{\mathbb{V}}$$

$\mathcal{L}_{\mathbb{V}}$ is a line bundle on $\mathrm{Pic}(\bar{C})$

$$\begin{tikzcd}
\mathcal{H}iggs = T^\vee \underline{\mathrm{Bun}} \arrow[d, "\mathrm{pr}"] & \supset \mathrm{Pic}(\bar{C}) \arrow[dl] \\
\underline{\mathrm{Bun}} &
\end{tikzcd}$$

$$\iota_{\bar{C}*} \mathcal{L}_{\mathbb{V}} \in \mathrm{Coh}(T^\vee \underline{\mathrm{Bun}})$$

$SC_{\underline{\mathrm{Bun}}}$ $\quad (\mathcal{F}, \theta)$ Higgs sheaf on $\underline{\mathrm{Bun}}$

"$\mathrm{quant}_{\underline{\mathrm{Bun}}}$" should be $\mathrm{nah}_{\underline{\mathrm{Bun}}}$, but it does not exist

Can fix this by using nah with ramification

Idea: $\underline{Bun} \supset \underline{Bun}^s$ = stack of stable bundles, a smooth quasi projective variety

Thm (Mochizuki) If $(\bar{X}, \mathcal{O}_{\bar{X}}(1))$ projective, $D \subset \bar{X}$ effective divisor, s.t. $\exists\ Z \subset \bar{X}$ of codim ≥ 3 so that $\bar{X} \setminus Z$ smooth, $D - Z$ normal crossing.

$\Rightarrow \exists$ equivalence of tensor dg categories

$$\begin{pmatrix} \text{finite rank} \\ \text{parabolic local systems} \\ \text{on } (\bar{X}, D) \text{ with logarithmic} \\ \text{poles on } D \end{pmatrix} \xrightarrow[\mathrm{nah}_{(\bar{X},D)}]{} \begin{pmatrix} \text{semistable parabolic} \\ \text{locally abelian Higgs} \\ \text{bundles on } (\bar{X}, D) \text{ with} \\ \text{regular singularities} \\ \mathrm{par}\,ch_1 = \mathrm{par}\,ch_2 = 0 \end{pmatrix}$$

$\underline{Bun} \supset \underline{Bun}^{vs}$ = very stable bundles.

$$\begin{array}{ccc} \mathrm{Pic}(\bar{C}) & \supset & \mathrm{Pic}(\bar{C})^{vs} \\ \downarrow & & \downarrow \ \text{finite flat} \\ \underline{Bun} & \supset & \underline{Bun}^{vs} \end{array}$$

Can convert $(\mathcal{F},\theta)|_{\underline{Bun}^{vs}}$ to a local system

if we can find

$$\overline{\underline{Bun}}^{vs}$$ satisfying Mochizuki's conditions.

So that

$(\mathcal{F},\theta)$ extends to a parabolic, locally abelian, semistable with vanishing Chern classes.

Compactify by $\underline{Bun}^{vs} \subset \underline{Bun}^{ss}$, and blowup to get normal crossings.

$\underline{\text{Thm}}$ (Mochizuki) Let $\overline{X}, \overline{Y}$ be projective varieties

$$\begin{tikzcd} \overline{X} & & \overline{Y} \\ & U \arrow[ul, hook, "\varphi"] \arrow[ur, hook, "\psi"'] & \end{tikzcd}$$

$\overline{X} - U$ = normal crossings

$\overline{Y}$ = anything

$$\psi_{*!} : \begin{pmatrix} \text{logarithmic local} \\ \text{systems on } \overline{X}, \\ \text{smooth on } U \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \mathcal{D}\text{-modules on } \overline{Y}, \\ \text{smooth on } U \end{pmatrix}$$

$\underline{\text{Example}}$ $\mathbb{P}^1$ with 5 pts, $GL(2)$.