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The work of Argyres and Seiberg

$\mathcal{N}=4$ SUSY Yang-Mills

S-duality Montonen-Olive (1977)

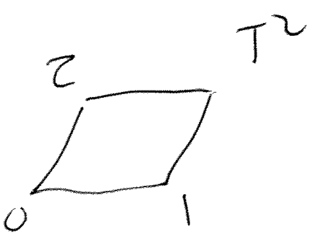
G at coupling τ

$\updownarrow$

^{L}G at coupling $-\frac{1}{n_G \tau}$

$n_G \leftarrow$ ratio $\dfrac{|\text{long root}|^2}{|\text{short root}|^2}$

6d theory $\mathfrak{g}$ A_n, D_n, E_n. Vafa (1999)



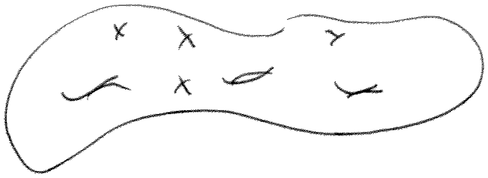
$\mathcal{N}=2$ SUSY theory

$\mathcal{N}=2$ SU(2) 4 flavors — self dual (1994)

Argyres-Seiberg (2007)
Argyres-Wittig

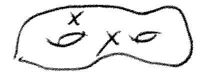
$\mathcal{N}=2$, SU(3) with 6 flavors
$\updownarrow$
very strange dual theory

6d theory $\mathfrak{g} = A, D, E$



Gaiotto (2009)

6d theory $\mathfrak{g} = A, D, E$

${}^{4d}SCFT_{\mathfrak{g}}$:  $\longrightarrow$ ${}^{4d}SCFT_{\mathfrak{g}}$ []

(an associated 4d SCFT)

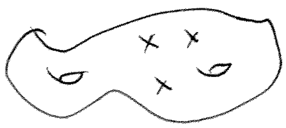
$SCFT_{\mathfrak{g}}$ [] = $SCFT_{\mathfrak{g}}$ [] + $SCFT_{\mathfrak{g}}$ []

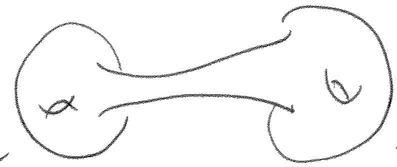
coupled to G at $\tau = \log q$

q = "length" e.g. 4-punctured sphere

$\mathrm{Im}\,\tau \gg 0$ weakly coupled

2d TQFT
CFT

Z :  $\longrightarrow$ $\mathbb{C}$

$Z\Big[$  $\Big] = \sum_a Z\Big[$  $\Big]\, Z\Big[$  $\Big]$

analogy:

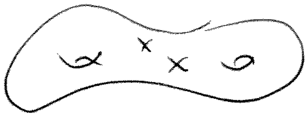
X = Riemannian manifold

$X \to$ vol, $H^*(X)$, $\chi(X)$

$J \to Z_{S^4}$, prepotential Nekrasov part., n_h, n_v, Coulomb, Higgs, Superconformal index

Z_{S^4}, prepotential Nekrasov part.: depends on τ

n_h, n_v, Coulomb, Higgs, Superconformal index: indep't of τ

 $\xrightarrow{\text{SCFT } T_g} J \xrightarrow{Z_{S^4}} \mathbb{C}$

$J \xrightarrow{\text{superconformal index}} \mathbb{C}$

$J \xrightarrow{\text{Higgs}}$ HK manifold

morally satisfies axioms of 2d CFT

$\mathrm{Higgs}(T_1) \times \mathrm{Higgs}(T_2) /\!/\!/ G$

- Lagrangian vs. non-Lagrangian SCFT

Lagrangian $\downarrow$ has a path integral description

non-Lagrangian $\to$ does not have a path integral

$\mathcal{N}=2$ Lagrangian theory

G = compact Lie group

$\tilde{\rho}$ = pseudoreal rep. of G.

$$\mathbb{R}^4 \qquad \gamma^\mu(\partial_\mu + \tilde{\rho}(A_\mu))\chi = 0$$

$$\chi \in 2 \otimes \tilde{\rho}$$

2: $\mathrm{Spin}(4) \simeq SU(2) \otimes SU(2)$.

$$C\chi = \chi.$$

$$\left.\begin{array}{l} 2 \text{ pseudo real} \\ \tilde{\rho} \text{ pseudo real} \end{array}\right\} \text{strictly real.}$$

conj. linear map $C: \chi \to \chi$, $C^2 = 1$.

special case: $\tilde{\rho} = \rho \oplus \rho^*$ $\qquad \gamma^\mu(\partial_\mu + \rho(A_\mu))\psi = 0$

$k_G(\rho \oplus \rho^*) = 2 \times \dim_{\mathbb{C}}$ Index of $D_{A,\rho}$

when A is an 1-instanton configuration

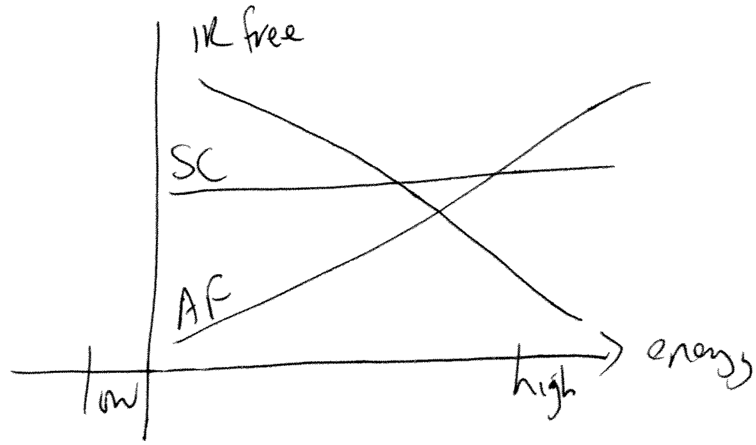
$$k_G(\mathrm{adj} \oplus \mathrm{adj}) = 4h^\vee(G)$$

$$k_{SU(N)}(N \oplus \bar{N}) = 2$$

$k_G(\tilde{\rho}) > 4h^\vee(G)$ infrared free

$= 4h^\vee(G)$ superconformal

$< 4h^\vee(G)$ asymptotically free



G, $\tilde{\rho} = \text{adj} \oplus \text{adj}$

$\mathcal{N}=4$ SYM.

$G = SU(N)$, $\tilde{\rho} = N \otimes \overline{2N} \oplus \overline{N} \otimes 2N$

of $SU(N) \times U(2N)$ ← F = flavor symmetry

$\mathcal{N}=2$ $SU(N)$ with $2N$ flavors

$n_v(\tilde{\rho}/G) = \dim G$

$n_h(\tilde{\rho}/G) = \dim_{\mathbb{H}} \tilde{\rho} = \dim_{\mathbb{C}} \rho$ when $\tilde{\rho} = \rho \oplus \rho^*$.

$k_F(\tilde{\rho}/G) = k_F(\tilde{\rho})$ (normalized quadratic Casimir)

$\text{Coulomb}(\tilde{\rho}/G) \underset{\text{Spec}}{=} \mathbb{C}[\phi]^G$ ← ϕ: adj. of G.

$\underset{\text{Spec}}{=} \mathbb{C}[u_1, u_2, \ldots u_r]$ $r = \text{rank}\, G$

$d(u_i) = e_i + 1$ ← e_i: exponent of G.

e.g. $SU(N)$, $u_i = \operatorname{tr} \phi^{i+1}$ $i = 1, \ldots N-1$

$\mathrm{Higgs}(\tilde{\rho}/G) = \tilde{\rho} /\!/\!/\!/ G.$

Things depending on τ:

prepotential / Nekrasov partition function / Z_{S^4} / SW curves — to discuss next week

$\mathcal{N}=4$ SYM

$$\begin{array}{cc} G & \tau \\ \updownarrow & \updownarrow \\ {}^L G & -1/\tau \end{array}$$

e.g. $G = USp(2n) \longleftrightarrow {}^L G = SO(2n+1)$

$\mathcal{N}=2$ $SU(2)$ 4 flavors

$$\tilde{\rho} = 2 \otimes \bar{4} \oplus \bar{2} \otimes 4$$

$$= 2 \otimes_{\mathbb{R}} \mathbb{R}^8 \qquad SU(2) \times SO(8)$$

$SU(2)$

$$\tilde{\rho} = 2 \otimes_{\mathbb{R}} \mathbb{R}_V \qquad \tau$$

$$2 \otimes_{\mathbb{R}} \mathbb{R}_S^8 \qquad -1/\tau$$

$$2 \otimes_{\mathbb{R}} \mathbb{R}_C^8 \qquad 1/(1-\tau)$$

$\tilde{\mathcal{S}} /\!/\!/ SU(2)$ is 1-instanton moduli of $SO(8)$
(depends on the adjoint of $SO(8)$)

non-Lagrangian theories

Minahan + Nemeschansky in 1996:

$MN(E_{6,7,8})$

	n_v	n_h	Coulomb	Flavor Symmetry
$MN(E_6)$	5	16	$\mathbb{C}[u]$ $d(u)=3$	E_6
$MN(E_7)$	7	24	$\mathbb{C}[u]$ $d(u)=4$	E_7
$MN(E_8)$	11	40	$\mathbb{C}[u]$ $d(u)=6$	E_8

Higgs branch	$\dim_{\mathbb{H}}$ higgs	k_E	family of SW curve
centered, framed 1-inst. moduli space of E_6	11	6	Kodaira
of E_7	17	8	
of E_8	29	12	

Given two $\mathcal{N}=2$ SCFT A, B

(1) theory $A+B$.

$$\left.\begin{array}{l} n_v \\ n_h \end{array}\right\} \text{additive}$$

$$\left.\begin{array}{l} \text{Coulomb} \\ \text{Higgs} \end{array}\right\} \text{direct product}$$

If a theory A has a flavor symmetry $F \supset G$ s.t.

$$k_G(A) = 4 h^\vee(G)$$

then "theory A/G at coupling τ" satisfies

$$n_v(A/G) = n_v(A) + \dim G$$

$$n_h(A/G) = n_h(A)$$

$$\text{Coulomb}(A/G) = \text{Coulomb}(A) \otimes \mathbb{C}[\phi]^G$$

$\deg \phi = 1$ (adjoint of G)

$$\text{Flavor Symm}(A/G) = \text{commutant of } G \text{ in } \text{Flavor}(A)$$

$$\text{Higgs}(A/G) = \text{Higgs}(A) /\!\!/\!\!/ G$$

$\tilde{\rho}$ = free theory

$\tilde{\rho}/G$ (Lagrangian theory) in a special case.

Argyres - Seiberg example

$$G = SU(3) \qquad \tilde{\rho} = \mathbf{3} \otimes \bar{\mathbf{6}} \oplus \bar{\mathbf{3}} \otimes \mathbf{6}$$

$$\curvearrowleft SU(3) \times U(6)$$

dual

$$k_{SU(2)}(\tilde{\rho}) = 2$$

$$G = SU(2), \qquad \tilde{\rho} = \mathbf{2} \otimes_{\mathbb{R}} \mathbb{R}^2 \leftarrow SU(2) \times SU(3)$$

$$+ MN(E_6) \leftarrow E_6 \supset SU(2) \times SU(6)$$

$$k_{SU(2)}(MN(E_6)) = k_{E_6}(MN(E_6)) = 6.$$

$$\text{total } k_{SU(2)} = 8 = 4h^\vee(SU(2)).$$

So we can couple via $SU(2)$.

Higgs branch:

$$\mathbb{B}\otimes\overline{\mathbb{G}} \oplus \overline{\mathbb{B}}\otimes\mathbb{G} /\!/\!/ SU(3)$$ dim $18-8=10$

$$\mathbb{H}\otimes_{\mathbb{R}}\mathbb{R}^2 \times \mathcal{N}_{min}(E_6) /\!/\!/ SU(2)$$ dim $2+11-3=10$

Gaiotto - Neitzke - Tachikawa: defining eqns agree