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Introduction to Liouville Theory

Plan

(A) Intro; Physically motivated conjectures

(B) Nonperturbative construction via conformal invariance — today

1) Conformal blocks
2) Correlation functions (crossing symmetry ...) — tomorrow

(C) Relation to quantum Teichmüller theory

(A) Classical Liouville theory

C

$$S_L = \int_C \frac{d^2z}{4\pi}\left((\partial_a\phi)^2 + 4\pi\mu e^{2b\phi}\right)$$

!

$$\varphi : C \to \mathbb{R}$$

Extrema (sols. to $\partial\bar\partial\phi = \pi\mu b e^{2b\phi}$)

define metrics of constant negative curvature on C

$(ds^2 = e^{2b\phi}\, dz\, d\bar z)$

Quantization

$$\left\langle \prod_{r=1}^{n} e^{2\alpha\phi(z_r,\bar{z}_r)} \right\rangle_{\mathbb{C}} = \int_{\phi:\mathbb{C}\to\mathbb{R}} [\mathcal{D}\phi]\, e^{-S_L[\phi]} \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r,\bar{z}_r)}$$

Subtle UV-divergences

$\rightsquigarrow$ nonpert. counterterm $\propto \hat{\mu}\, e^{2b^{-1}\phi}$

$\rightsquigarrow$ duality $b \leftrightarrow b^{-1}$

Can. quant. $(b \rightsquigarrow 0)$ $\Rightarrow$ Predictions:

- Liouville theory is conformally inv.

$$\mathcal{H}_L = \int_{\mathbb{R}_+}^{\oplus} dp\; \mathcal{V}_p \otimes \mathcal{V}_p$$

$\nwarrow$ rep of $\overline{\mathrm{Vir}_c}$

$\uparrow$ rep. of Vir_c, $c = 1 + 6Q^2$, $Q = b + b^{-1}$

Df of Vir_c:

generators L_n

algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1)\, \delta_{n+m,0}$$

$$\mathcal{V}_p = V_p \otimes V_p , \qquad L_n v_p = 0,\ n>0$$
$$L_0 v_p = \left(p^2 + \frac{Q^2}{4}\right) v_p$$

$$V_p = \mathrm{Span}\left(L_{n_1} \cdots L_{n_k} v_p\right)$$

Basic observables (primary fields) $\quad V_\alpha(z,\bar z) \sim e^{2\alpha\phi(z,\bar z)}$

$$\left[L_n, V_\alpha(z,\bar z)\right] = z^n\left(z\partial_z + \Delta_\alpha(n+1)\right) V_\alpha(z,\bar z)$$

$\rightsquigarrow$ Can construct corr. fns. as VEV's

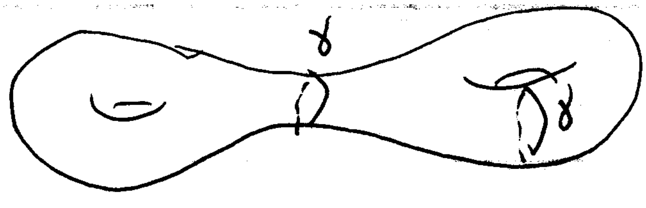
$$\left\langle \prod_{r=1}^{n} e^{2\alpha_r \phi(z_r,\bar z_r)} \right\rangle = \langle 0 | \prod_r V_{\alpha_r}(z_r,\bar z_r) | 0 \rangle$$

$\Rightarrow \ldots \Rightarrow$

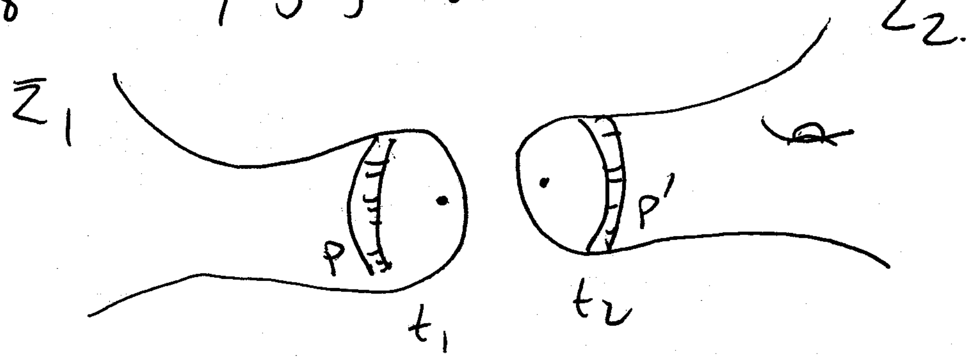
Holomorphic factorization of correlation functions:

$$\left\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r,\bar z_r) \right\rangle_C = \int_{\mathbb{R}^{3g-3+n}} d\mu(P) \left| \mathcal{F}(P, \ell, \alpha, q) \right|^2$$

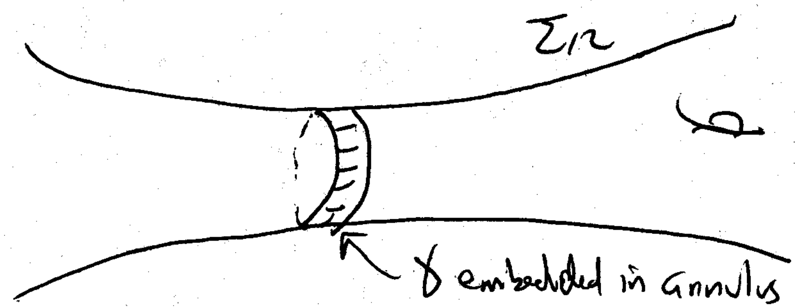
$\gamma \mapsto P_\gamma \qquad (\alpha_1, \ldots, \alpha_n) \qquad \gamma \mapsto q_\gamma$



q_γ defined by gluing constr.



$$t_1(p)\,t_2(p') = q_\gamma$$



Iterate to get $C = C(q)$ from pairs of pants

(B) Function $\mathcal{F}$ is a conformal block, defined entirely by rep theory of the Virasoro algebra.

Given rep. V_r, $r = 1, \ldots, n$ of Vir_c, curve C with n punctures $z_1, \ldots, z_n$

Conf block $\quad F_C : V_{in} \cong \bigotimes_{r=1}^{n} V_r \to \mathbb{C}$, $\quad v_r \in V_r$ $\quad$ s.t. $\quad F_C(T[\chi]v) = 0$ $\quad \forall \chi \in Vir_{out}$ $\quad$ (CWI)

(mero. v.f. on C, poles at z_r.)

$$T[\chi] = \sum_r 1 \otimes \cdots \otimes T[\chi_r] \otimes \cdots \otimes 1$$

$$\sum_r L_r \chi_{k,r} = \int_{\gamma_r} T\chi$$

$$\text{if } \chi(t_r) = \sum_k \chi_{k,r} t_r^{k+1} \partial_{t_r}$$

So $C \rightsquigarrow$ natural action of Vir_{out} on $\mathcal{V}_{[n]}$.

Claim There is (Hilbert-) space of sols. to (CWI), with basis $F_{e,p}$ for each pants decomp of C s.t.

$$\boxed{\mathcal{F}(P,b,\alpha,q) = F_{e(q),p}(v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n})}$$

$v_\alpha = v_p$ if $\alpha = \frac{Q}{2} + iP$.

(CWI) $\rightsquigarrow$ rules to move L_n from one puncture to another.

Example: $C_{0,3} = \mathbb{P}^1 \setminus \{z_1, z_2, z_3\}$, coordinate t

⑥

$\rightsquigarrow t - z_r = t_r$

Consider $\chi = (1 - z_3)^{-\ell(t)} \partial_t$

$\chi_{k,3} = \delta_{k,-\ell}$

$\chi_{k,r} = 0$ if $k > -1, r = 1, 2$

So (CWI) $\Rightarrow$

$$0 = F_e(L_{-\ell} v_3 \otimes v_2 \otimes v_1) + F_e\Big(\sum_{k \geq -1} v_3 \otimes L_k v_2 \otimes 1\Big) \qquad \chi_{k,2}$$

$$+ F_e\Big(\sum_{k \geq 1} v_3 \otimes v_2 \otimes L_k v_1\Big) \qquad \chi_{k,1}$$

creation

(mostly) annihilation

(F_e is linear).

$\rightsquigarrow$ use to compute $F_e\Big(\bigotimes_{r=1}^{3} \mathcal{L}_{\mu_r} v_3\Big)$

$L_{-n_1} \cdots L_{-n_k}$

in terms of $F_e\Big(\bigotimes_{r=1}^{3} L_{-1}^{\ell_r} v_r\Big)$

Holomorphic vector fields

$$X = t^{\ell+1}\partial_t \,, \quad \ell = -1, 0, 1$$

$\rightsquigarrow$ 3 more eqns

$\rightsquigarrow$ Compute $F_e\left(\prod_{r=1}^{3} L_{-1}^{\ell_r} v_r\right)$ in terms of $F_e\left(\bigotimes_{i=1}^{3} v_r\right)$

$$\sum L_x X_{x,r} = \int_{\ell_r} TX$$

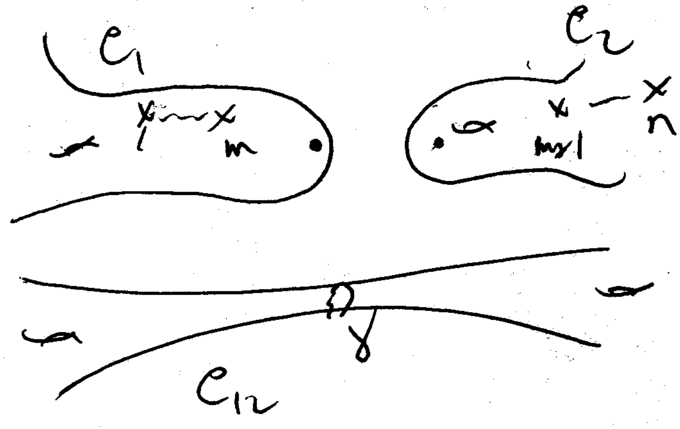
$$\mathcal{F} = F$$

$$\Rightarrow F_{e_{0,3}}\left(\prod_{r=1}^{3} L_{\mu_r} v_{\alpha_r}\right) = N(\alpha_3,\alpha_2,\alpha_1)\, g(\mu_3,\mu_2,\mu_1,\alpha_3,\alpha_2,\alpha_1)$$

$F_e\left(\prod_{r=1}^{3} v_r\right)$

completely determined by (CWI) + $g(0,0,0,\alpha_3,\alpha_2,\alpha_1) \equiv 1$

Conformal blocks for general $\mathcal{C}$ gluing const



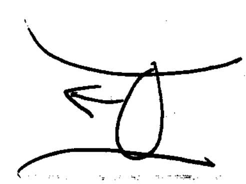
$$F_{\mathcal{C}_{12}, p_\gamma}(v_1 \otimes \cdots \otimes v_n) = \sum_\mu F_{\mathcal{C}_1}(v_1 \otimes \cdots \otimes v_n \otimes q^{L_0} \mathcal{L}_\mu v_{p_\gamma}) \times F_{\mathcal{C}_2}(\mathcal{L}_\mu^\vee v_{p_\gamma} \otimes v_{m+1} \otimes \cdots \otimes v_n) \qquad (G)$$

where $\langle \mathcal{L}_\mu v_p, \mathcal{L}_\mu^\vee v_p \rangle = \delta_{\mu\mu'}$

$$\mathcal{L}_\mu^\vee v_p = \sum_{\mu'} (M^{-1})_{\mu\mu'} \mathcal{L}_{\mu'} v_p, \quad M_{\mu\mu'} \equiv \langle \mathcal{L}_\mu v_p, \mathcal{L}_{\mu'} v_p \rangle$$

<u>Remarks</u>

- $F_{\mathcal{C}_2}$, $F_{\mathcal{C}_1}$ conformal blocks $\Rightarrow$ $F_{\mathcal{C}_{12}, p_\gamma}$ conformal block



- $F_{\mathcal{C}_{12}}$ <u>convergent</u> power series in q_γ

Using (6) recursively

$\Rightarrow$ construction of $\mathcal{F}(P, b, \alpha, q)$

as power series in $q_{\gamma_1}^{k_1}, \ldots, q_{\gamma_{3g-3+n}}^{k_{3g-3+n}}$

Remains to fix normalization factors $N(\alpha_3, \alpha_2, \alpha_1)$.