24 August 2010
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Quantum affine algebras, Yangians, and trigonometric connections

Theme Realise braid group actions arising from quantum groups as monodromy of certain flat connections

Kohno–Drinfeld thm

$\mathfrak{g}$ = complex, simple Lie algebra / $\mathbb{C}$.

$U_q\mathfrak{g}$ = quantum group.

Thm Action of Artin's braid group B_n coming from R-matrix of $U_q\mathfrak{g}$ on $\underbrace{\mathcal{V}\otimes\cdots\otimes\mathcal{V}}_{n}$, $\mathcal{V}\in\mathrm{Rep}(\mathfrak{g})$ is equivalent to the monodromy of the KZ connection on $V=\mathcal{V}/(q-1)\mathcal{V}\in\mathrm{Rep}(\mathfrak{g})$.

Generalized braid group

$\tilde{\mathfrak{g}}$ symm. K-M alg. ($\tilde{\mathfrak{g}}=\mathfrak{g}$ f.d. ; $\tilde{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]$)

$\tilde{I}$ Dynkin diagram

$\tilde{W}$ Weyl group $=\langle s_i\rangle_{i\in\tilde{I}}/s_i^2=1,\ \underbrace{s_is_js_i\cdots}_{m_{ij}}=\underbrace{s_js_is_j\cdots}_{m_{ij}}\ \forall i\neq j$

$$\widetilde{B} = \langle S_i \rangle_{i \in \widetilde{I}} \Big/ S_i S_j S_i = \cdots = S_j S_i S_j \cdots$$

Thm (Lusztig, Kirillov-Reshetikhin, Soibelman)

If $\mathcal{V}$ is an integrable rep. of $U_q \widetilde{\mathfrak{g}}$

there are natural quantum Weyl group operators $S_i^q \in GL(\mathcal{V})$

s.t. (I) $S_i^q S_j^q \cdots = S_j^q S_i^q \cdots \quad i \neq j$

(II) $S_i^q = s_i \mod (q-1)$

$\quad\quad s_i \in \widetilde{W}$

Monodromy interpretation of $\widetilde{B} \circlearrowright \mathcal{V} \in Rep(U_q \widetilde{\mathfrak{g}})$.

$\widetilde{\mathfrak{g}} = \mathfrak{g}$ f.d. simple

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra

$\mathfrak{h}_{reg} = \mathfrak{h} \setminus \bigcup_\alpha \ker(\alpha)$

$$B = \pi_1(\mathfrak{h}_{reg}/W)$$

Thm (TL) The qW action of B on $\mathcal{V} \in Rep(U_q \mathfrak{g})$

is equivalent to the monodromy of the Casimir connection on

$V = \mathcal{V}/(q-1)\mathcal{V}$.

Casimir Connection

$V \in Rep(\mathfrak{g})$

$\mathbb{V} = V \times \mathfrak{h}_{reg} \to \mathfrak{h}_{reg}$

$$\nabla = d - \sum_\alpha \frac{d\alpha}{\alpha} r_\alpha \in End(V)$$

α root $\rightsquigarrow$ $sl_2^\alpha \subseteq \mathfrak{g}$
$\qquad \qquad \qquad \qquad ||$
$\qquad \qquad \qquad \langle e_\alpha, f_\alpha, h_\alpha \rangle$

$$C_\alpha = \text{Casimir of } sl_2^\alpha = \frac{(\alpha,\alpha)}{2}\left(e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{h_\alpha^2}{2}\right)$$

<u>Thm</u> (Millson + TL, DeConcini, FMTV)

The Casimir connection

$$\nabla_C = d - \hbar \sum_\alpha \frac{d\alpha}{\alpha} C_\alpha$$

is flat and W-equivariant for any $\hbar \in \mathbb{C}$ $(\hbar = \log q)$.

<u>Today</u>: affine setting.

$\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$, $\mathfrak{g}$ f.d.

$\tilde{B}$ = affine braid group

$\curvearrowleft qW$

$\mathcal{V} \in \mathrm{Rep}_{fd}(\underbrace{U_q(L\mathfrak{g})}_{\text{quantum Loop algebra}})$

$\widetilde{B} = \pi_1(\cdots)$

$H \subset G$ complex, simply-connected Lie group corresponds to $\mathfrak{g}$

max. torus.

$$H_{reg} = H \setminus \bigcup_\alpha \{e^\alpha = 1\}.$$

e.g. $\mathfrak{g} = \mathfrak{gl}_n$

$$H = \{(z_1 \ldots z_n) \mid z_i \neq 0\}$$

$$H_{reg} = \{(\quad) \mid z_i \neq 0,\ z_i \neq z_j\}$$

$$\widetilde{B} = \pi_1(H_{reg}/W).$$

Connection $\hat{\nabla}$ on H_{reg}?

coefficients of $\hat{\nabla}$ in —

$U_q\mathfrak{g} \longrightarrow$ coeff. in $U\mathfrak{g} = U_q\mathfrak{g}/_{q=1}$.

$U_q(L\mathfrak{g}) \to$ coeff in. $U(L\mathfrak{g}) \cancel{= U_q(L\mathfrak{g})|_{q=1}}$

<u>receptacle</u>

- $Y(\mathfrak{g})$ = Yangian of $\mathfrak{g}$
  = degeneration of $U_q(L\mathfrak{g})$ as $q \to 1$
- $Y(\mathfrak{g})$ has the "same" f.d. rep theory as $U_q(L\mathfrak{g})$

$Y(\mathfrak{g})$ (Hopf) algebra over $\mathbb{C}[\hbar]$

deformation of $U(\mathfrak{g}[s])$

Main features:

- $U\mathfrak{g} \subset Y(\mathfrak{g})$ (constant loops)
- $\{h_{i,r}\}_{\substack{i \in I \\ r \in \mathbb{N}}} \subset Y(\mathfrak{g})$ commutative subalgebra

$h_{i,r} \to h_i \otimes s^n$ as $\hbar \to 0$

Connection

$$\hat{\nabla} = d - \hbar \sum_{\alpha} \frac{d\alpha}{e^{\alpha}-1} C_\alpha - \sum_i du_i A_i$$

- $u_1, \dots, u_n$ basis of $\mathfrak{h}^*$
- du_i translation invariant 1-forms on H

$A_i \in Y(\mathfrak{g})$

If $A_i = 0 \ \forall i$, $\hat{\nabla}$ is neither flat nor W-equivariant.

Thm (TL)

The trigonometric Casimir connection

$$\hat{\nabla} = d - \hbar \sum_{\alpha} \frac{d\alpha}{e^{\alpha}-1} C_\alpha - \sum_i d\alpha_i \left(2h_{i,1} - \frac{h_{i,0}^2}{2}\right)$$

Simple roots

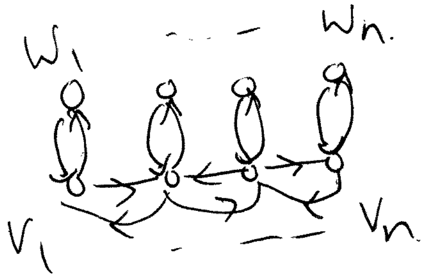
is flat and W-equivariant

$\hat{\nabla} = $ QDE on $QH^*(\mathcal{M})$

Nakajima quiver varieties

$\circ \to \circ \leftarrow \circ \to \circ$ I finite ADE quiver

double the arrows



double the vertices, add dimension vectors

$\mathcal{M}(v,w) =$ HK reduction of the moduli space of reps of the quiver.

$I \to \tilde{I} =$ affine Dynkin diagram.

<u>Example</u>

Then $\mathcal{M}(v,w) =$ moduli space of ASD ~~ins~~tantons on the ALE space $\mathbb{C}^2/\Gamma$ (if v, w well chosen)

$$\Gamma \subset SL_2(\mathbb{C}) \xleftrightarrow[\text{McKay}]{} \tilde{I}$$

$$\mathcal{M}(v,w) \circlearrowleft \mathbb{C}^\times \times GL(w) = \prod_i GL(w_i)$$

<u>Thm</u> (Varagnolo, Nakajima)

$$\bigoplus_{v} H^*_{GL_w \times \mathbb{C}^\times}(\mathcal{M}(v,w))$$

has a $Y(\mathfrak{g})$ action with $\underset{\text{of } Y(\mathfrak{g})}{\hbar} \longleftrightarrow H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

$$\bigoplus QH^*_{GL_w \times \mathbb{C}^\times}(\mathcal{M}(v,w))$$

$$\downarrow$$

$$H^2_{GL_w \times \mathbb{C}^\times}(\mathcal{M}(v,w)) \Big/ \cdots \supseteq H$$

<u>Thm</u> (Braverman-Maulik-Okounkov)

The q-ODE on $\bigoplus_{v} QH^*_{GL_w \times \mathbb{C}^\times}(\mathcal{M}(v,w))$

coincides with the trigonometric Casimir connection

<u>Nekrasov - Shatashvili</u>

Conjecture (TL)

The monodromy of the trigonometric Casimir connection on $V \in \mathrm{Rep}(Y(\mathfrak{g}))$ is equivalent to the qW action of $\widehat{B}$ on the corr. $U_q(L\mathfrak{g})$ module $\mathcal{V}$.

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with Sachin Gautam

$$\mathrm{Rep}_{fd}(Y(\mathfrak{g})) \overset{?}{\longleftrightarrow} \mathrm{Rep}_{fd}(U_q(L\mathfrak{g}))$$

Drinfeld

$$\{\text{fd irred. } U_q(L\mathfrak{g})\text{-mod}\} \longleftrightarrow \left\{\begin{matrix}I\text{-tuples of}\\ \text{monic polys}\\ \text{with roots in } \mathbb{C}^\times\end{matrix}\right\}$$

Ex $U_q(L\mathfrak{sl}_2)$

$$\begin{matrix}\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2\\ a_1 \otimes \cdots \otimes a_N\\ a_i \in \mathbb{C}^\times\end{matrix} \longleftrightarrow p(u) = (u - a_1^{-1}) \cdots (u - a_N^{-1})$$

$$\{\text{f.d irred } Y(\mathfrak{g})\text{-mod}\} \longleftrightarrow \left\{\begin{matrix}I\text{-tuples of monic}\\ \text{polys with roots in } \mathbb{C}\end{matrix}\right\}$$

exp. roots

better (Nakajima)

$$
\begin{tikzcd}
U_q(L\mathfrak{g}) \arrow[r, hook] \arrow[d, dashed, "?"] & \bigoplus_v K_{G_w \times \mathbb{C}^\times}(\mathcal{M}(v,w)) \arrow[d, "ch"] \\
Y(\mathfrak{g}) \arrow[r, hook] & \bigoplus_v H^*_{G_w \times \mathbb{C}^\times}(\mathcal{M}(v,w))
\end{tikzcd}
$$

Theorem (G-TL) explicit

$\exists$ an algebra homomorphism

$$\Phi : U_q(L\mathfrak{g}) \to \widehat{Y(\mathfrak{g})}$$

which becomes an isomorphism after completing $U_q(L\mathfrak{g})$.

$$\widehat{\mathbb{C}[q,q^{-1}]} \qquad \widehat{\mathbb{C}[\hbar]} = \mathbb{C}[[\hbar]]$$

complete at ideal of $q-1$

$$q \longrightarrow e^{\hbar}$$

Functorial monodromy conjecture (G-TL)

The monodromy of $\hat{\nabla}$ (trigonometric Casimir connection) on $V \in \mathrm{Rep}_{fd}(Y(\mathfrak{g}))$ is equivalent to the qW-action on $\Phi^* V \in \mathrm{Rep}_{fd}(U_q(L\mathfrak{g}))$

Thm* (G-TL)
The functional monodromy conjecture is true for $\mathfrak{g} = \mathfrak{sl}_2$.

$$B_{\mathfrak{sl}_2} = \mathbb{Z}$$

$$\widetilde{B}_{\mathfrak{sl}_2} = \text{free group on 2 gens.}$$

$$\begin{tikzcd}
\mathfrak{g}[t,t^{-1}] \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{g}[s] \arrow[d, leftrightarrow] \\
U_q(L\mathfrak{g}) & Y(\mathfrak{g})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, "\exp"'] & \mathfrak{g} \\
\mathbb{C}^\times \arrow[ur] &
\end{tikzcd}$$

$$\widehat{\mathfrak{g}[t,t^{-1}]}_{t=1} \xrightarrow{\sim} \mathfrak{g}[[s]]$$

$$t \to e^s$$

$$\begin{array}{ccc}
\Phi(U_q(\mathfrak{g})) & \not\subset & U\mathfrak{g} \\
\cap & & \cap \\
U_q(L\mathfrak{g}) & & Y(\mathfrak{g})
\end{array}$$