16 August 2010

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3 dimensions

Chern-Simons gauge theory

$\sim$ 2000 Khovanov + . .

Khovanov homology, $\underset{4}{\overset{3+1}{\shortparallel}}$ dim'l theory

2004 Gukov, Schwarz, Vafa

Start with quantum path integral

$$\underset{\text{momenta}}{p_i} \quad \underset{\text{coordinates}}{q^i}$$

$$\int \mathcal{D}p_i(t)\,\mathcal{D}q^i(t)\ \exp i\int\left(p_i\frac{dq^i}{dt} - H(p,q)\right)dt$$

M = classical phase space

Functions on M $\longrightarrow$ operators on a Hilbert space

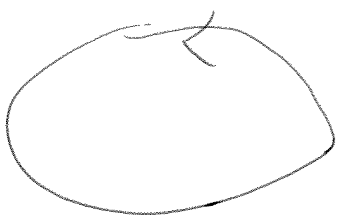
$\mathcal{R}$ = ring of observables act on $\mathcal{H}$

$u_1(p(t_1), q(t_1))$ $\qquad$ $u_1(t_1), \dots, u_k(t_k)$

$u_1, \dots, u_k$ fns on M

Today: traces.

$$\mathrm{Tr}\; u_1(t_1)\, u_2(t_2) \cdots u_k(t_k)$$

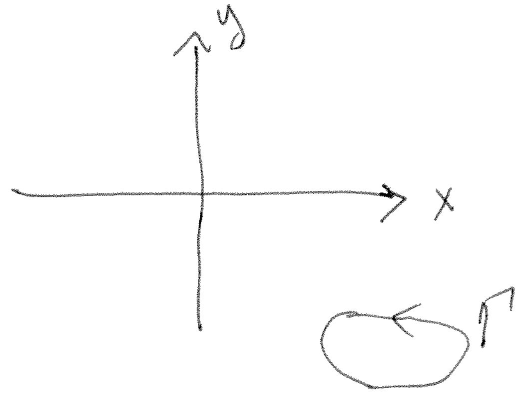


$p_i(t)$ $\quad q^i(t)$ $\quad$ fn on S^1

$$\int_{-\infty}^{\infty} dx \, \exp(i f(x))$$

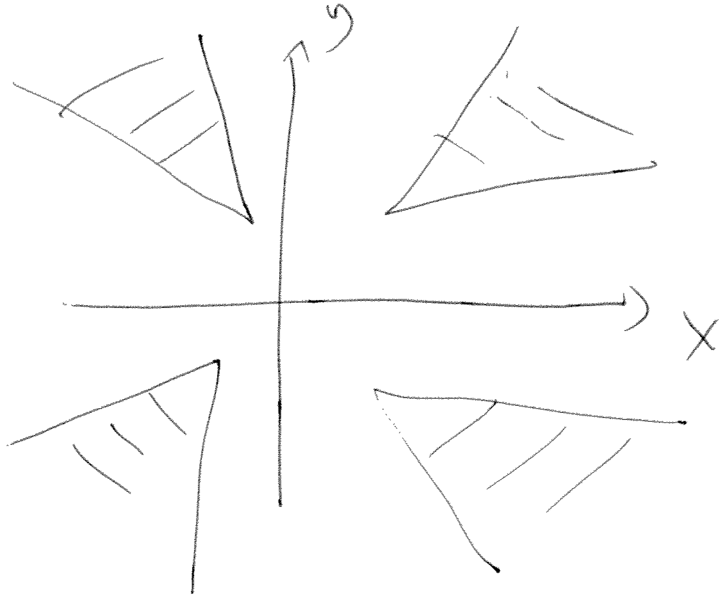
$$dx \, \exp i(x^4 + ax)$$

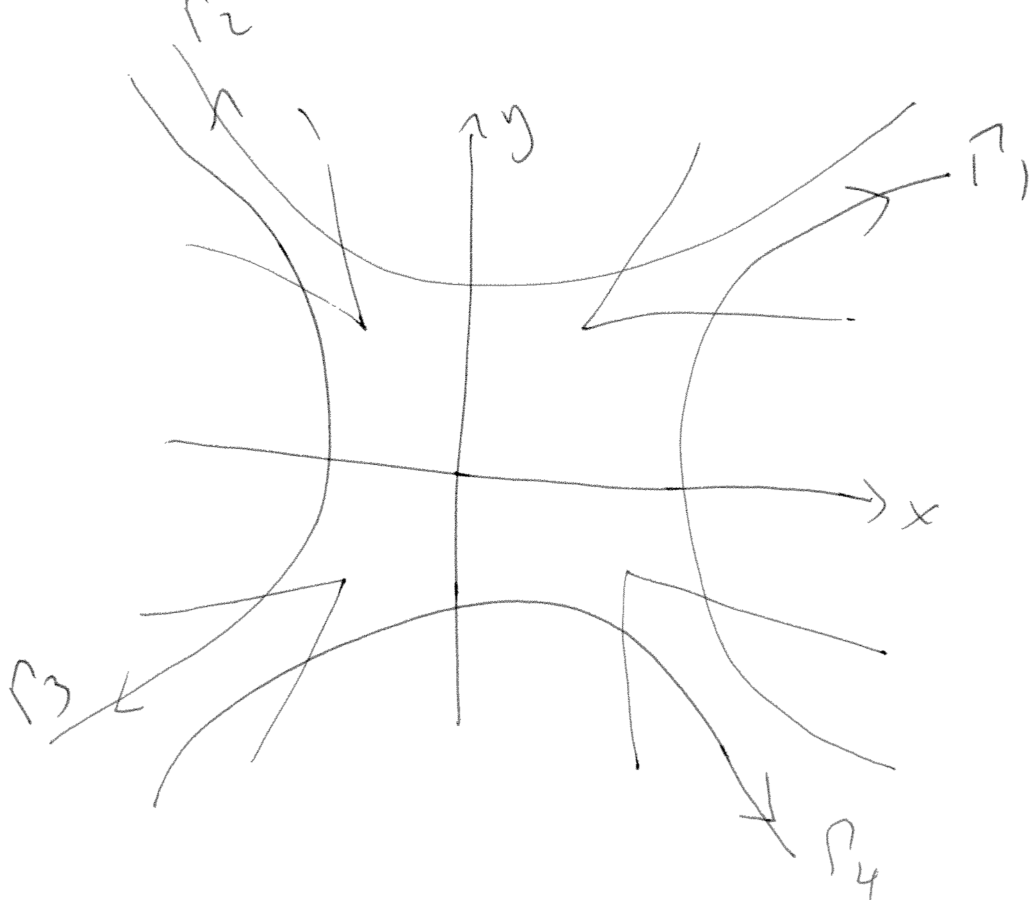
$$z = x + iy$$



$$\int_\Gamma dz \, \exp i(z^4 + az)$$

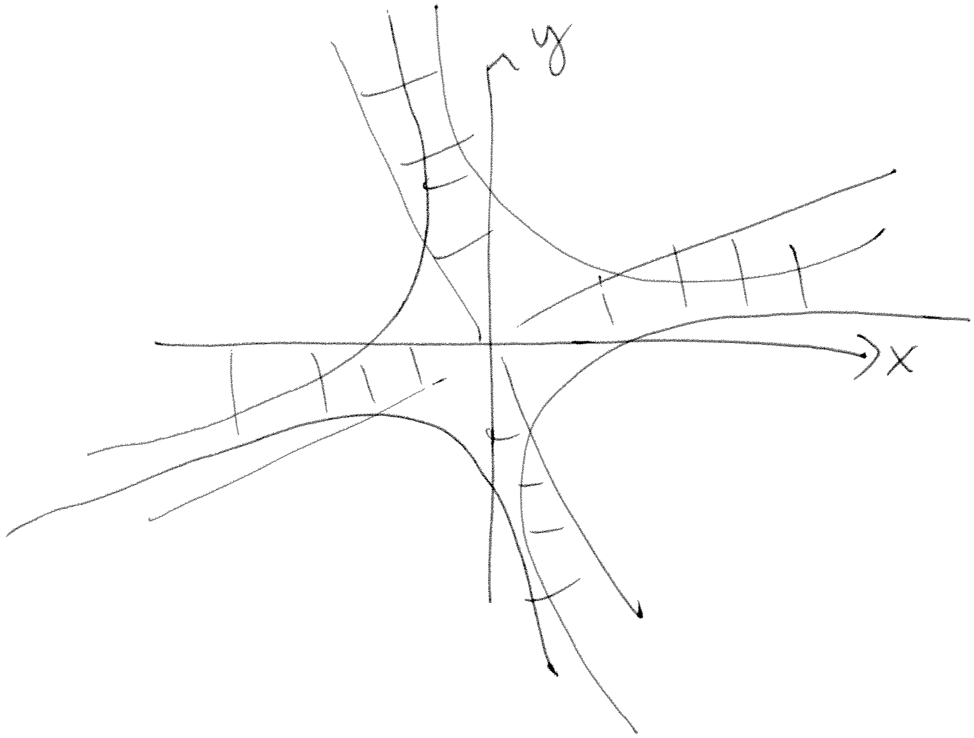
$$\mathrm{Re}\; i(z^4 + az) \ll 0$$

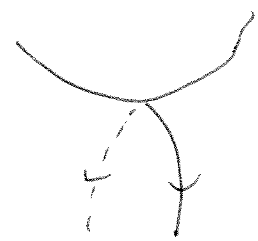




$\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 = 0$

3 good contours





$h = \mathrm{Re}\,(i\,(z^4 + az))$

Critical points or saddle points of h.

are critical pts of $F(z) = i(z^4 + az)$

$F'(z) = 0, \quad F'(z) = 4iz^3 + a.$

Integration cycle for each critical point.

Pick a metric $ds^2 = dz \otimes d\bar{z} = dx^2 + dy^2 \qquad z = x + iy$

Write the "flow-eqn"

$$\frac{dx^i}{ds} = -g^{ij}\frac{\partial h}{\partial x^j}$$

$\int dx_1 \cdots dx_n \exp i F(x_1, \ldots, x_n)$

$\int_\Gamma dz_1 \cdots dz_n \exp i F(z_1, \ldots, z_n)$

$$h = \mathrm{Re}(iF)$$

a) Start with a critical point of h

b) flow down $\quad \dfrac{du^i}{ds} = -g^{ij} \dfrac{\partial h}{\partial u^j} \qquad u = x \text{ or } y$

$z_i = x_i + i y_i$

$\tilde{N} =$ crit set of $h \subset \tilde{M}$

$n = \dim M = \dim_{\mathbb{C}} \tilde{M}$

$\tilde{M} =$ complexification of M

$\dim_{\mathbb{C}} \tilde{N} = k.$

$n-k$ directions for downward flow

Pick any k-dim'l cycle $C \subset \tilde{N}$; flow from there.

$$F = a + 0 \cdot z_i + a_{ij} z_i z_j + \cdots$$

$$+ \sum_{i=1}^{n} z_i^2$$

$$z = x + iy \,, \quad \pm z^2 = \pm(x^2 - y^2)$$

$$dx_1 \cdots dx_n$$

$$dz_1 \cdots dz_n$$

$$H(p,q) = 0 .$$

$$\int \mathcal{D}p_i(t)\, \mathcal{D}q^j(t) \, \exp i \int \left(p_i \frac{dq^i}{dt} \right) dt$$

$$\mathrm{Tr}_{\mathcal{H}} \; u_1(t_1) \cdots u_n(t_n)$$

Take P, Q to be complex valued, t still real.

$$\int_{\Gamma} \mathcal{D}P_i(t)\, \mathcal{D}Q^j(t) \, \exp i \int P_j \, dQ^j$$

Γ = mid dim. cycle in $\mathrm{Maps}(S^1, \tilde{M})$

Ex $M = S^2$

$x_1^2 + x_2^2 + x_3^2 = j^2$ in $\mathbb{R}^3$.

$X_1^2 + X_2^2 + X_3^2 = j^2$ in $\mathbb{C}^3$ $\qquad \widehat{M}$

$$f = \frac{1}{j^2} \varepsilon_{ijk}\, x_i\, dx_j\, dx_k = \frac{dx_2\, dx_3}{x_1} \qquad f = f_{ab}\, dx^a dx^b$$

$$\Omega = i \frac{1}{j^2} \varepsilon_{abc}\, X^a dX^b dX^c$$

$$\{u, v\} = \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} f^{ij}$$

$i = \sqrt{-1}$ in formula for Ω.

$$\Omega = \cancel{d\Lambda}\; d\Lambda$$

$$\oint p_i\, dq^i \to \oint a_i\, dx^i$$

$$i \oint P_i\, dQ^i \to \oint \Lambda_a\, dx^a$$

$$\int_\Gamma dX^a(t) \left(\exp \oint \Lambda_a dX^a\right) \prod_{i=1}^{n} u_i(t_i)$$

$h = \mathrm{Re} \oint \Lambda_a dX^a$

$u^I = (\mathrm{Re}\, X^a, \mathrm{Im}\, X^a)$

$u^I(t) \to u^I(t,s)$

G = metric on loop space of $\hat{m}$

$$\frac{\partial u^I}{\partial s} = -G^{IJ} \frac{\delta h}{\delta u^J}$$

Pick an ordinary metric g on $\hat{m}$

Define G on $\mathcal{L}\hat{m}$ by

$$|\delta u|^2 = \oint dt\, g_{IJ}\, \delta u^I(t)\, \delta u^J(t)$$

$$= \ldots$$

$h = \mathrm{Re} \oint \Lambda_a du^a$

$\delta h = \mathrm{Re} \oint (\partial_A \Lambda_B - \partial_B \Lambda_B)\, \delta u^A$

$$\frac{\partial u^I}{\partial s} \cancel{\delta u} = -g^{IJ} (\mathrm{Re}\, \Omega_{JK}) \frac{du^K}{dt}$$

$$\cancel{\mathrm{Re}\left(\Omega_{AB}\frac{du^A}{dt}\right) = -\partial_B H}$$

$$\delta\int\left(p\frac{dq}{dt}\right) = \int\left(\delta p\cdot q - \delta q\cdot p\right)dt$$

$$\Omega = \omega + i f.$$

$$(*) \quad \frac{\partial u^I}{\partial s} = -J^I{}_K \frac{\partial u^K}{\partial t} \qquad J = g^{-1}\omega$$

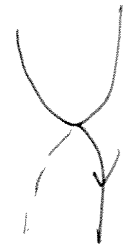
Can always pick g so that $J^2 = -1$.

e.g. $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\hat{M}$ already a complex manifold.
but now we've given it an almost complex structure J.

(*) becomes invariant to conformal mappings of $w = s + it$.

$$\frac{\partial a}{\partial s} = \frac{\partial b}{\partial t}, \quad \frac{\partial b}{\partial s} = -\frac{\partial a}{\partial t} \qquad \text{Cauchy-Riemann.}$$



$h = \frac{x^2 - y^2}{2}$

$\frac{dx}{ds} = -x, \quad x = Ce^{-s}$

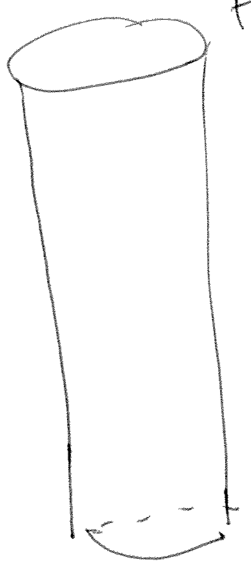
$\frac{dy}{ds} = y, \quad y = \tilde{C}e^{s}$

$s \in (-\infty, 0]$

Start at crit pt

anywhere.

Start at $s = -\infty$, flow to somewhere at $s = 0$.



t ← any $u^I(t, s=0)$

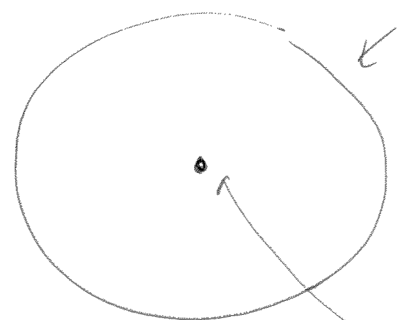
s

← start at crit pt

$u(t, s=\infty)$ indept of t.

Conformal mapping:

$z = \exp(w)$



← any $u^I|_{|z|=1}$

$s = -\infty$

Continuous at origin

$u(z)$ well-defined for $|z| \leq 1$, even at $z=0$.

Critical pts are $\widehat{m} \subset \mathcal{L}\widehat{m}$.

Pick $V \subset \widehat{m}$ a middle dim'l cycle.

Impose: $u(z=0) \in V$.

<u>Example</u> $V = m \subset \widehat{m}$

Eqn (*) is the eqn for an $\mathcal{I}$-pseudoholomorphic map.

<u>Example</u>

$$m = \quad x_1^2 + x_2^2 + x_3^2 = j^2$$

$$\widehat{m} = \quad X_1^2 + X_2^2 + X_3^2 = j^2$$

$\widehat{m}$ is a complex manifold s.t. X_1, X_2, X_3 hol. function.

Call that complex structure J.

Can we have $\mathcal{I} = J$?

No, because Ω is type $(2,0)$ for J.

$\Omega = \omega + i f$; ω is type $(2,0)+(0,2)$ for J

ω is type $(1,1)$ for $\mathcal{I}$

Best: $dJ = -J\,d$. Let $K = dJ$. Almost hyperkähler structure.

In example, hyperkähler using Eguchi-Hansen

$$\int \mathcal{D}u^I(z)\,\delta\!\left(\frac{\partial u^I}{\partial s} + J\frac{\partial u^I}{\partial t} = 0\right)\delta\big(u(s=-\infty)\in V\big)$$

$$\cdot \prod \mu_i\left(\tau_i = \exp i\,t_i\right) \cdot \exp \oint \Lambda_A du^A$$

$|z|\leq 1$

$$\delta(x) \longrightarrow \int \frac{d\lambda}{2\pi}\exp i\lambda x$$

$$\mathcal{D}(u) = \frac{\partial u^I}{\partial s} + J\frac{du}{\partial t}$$

$$S_V = \delta\big(u(s=-\infty)\in V\big)$$

$$\int du^I(s,t)\,dT(s,t)\,\exp i\int T\,\mathcal{D}(u)\,\exp\oint\Lambda_A du^A \prod_i \mu_i(\tau_i)$$

$\delta(\mathcal{D}(u))\ldots$ $\qquad \delta(f(x)) = f'(a)\,\delta(x-a)$ if $f(a)=0$

$$\frac{1}{\det \mathcal{D}_L}$$

add fermions χ, ψ.

$$\int \mathcal{D}\chi\, \mathcal{D}\psi \int du^I(s,t)\, dT(s,t)\, \exp i \int \left(T\, \mathcal{D}(u) + \chi \mathcal{D}_L \psi \right)$$

$$\cdot \exp \oint \Lambda_A du^A \prod_i u_i(\tau_i) \cdot \delta_V$$

ψ lies in tangent space to u

Fermionic symmetry

$$\delta u = \psi, \quad \delta \psi = 0$$

$$\delta \chi = T, \quad \delta T = 0 \qquad \delta^2 = 0$$

$$T\mathcal{D}(u) + \chi \mathcal{D}_L \psi = \delta(\chi \mathcal{D}(u))$$

$\delta = Q$ (BRST operator)

perturb $\exp\left(i \int \delta(\chi\mathcal{D}(u)) - \underbrace{\varepsilon T^2}_{\delta(\varepsilon \chi T)} \right)$

do the path integral over T

$$\mathcal{D}u = \frac{du}{ds} + J\frac{du}{dt}$$

$$\int (\mathcal{D}u)^2 = \int \left(\frac{du}{ds}\right)^2 + \left(\frac{du}{dt}\right)^2 + \int_{\partial(\text{Disc})} (\ldots)$$

$$\int \mathcal{D}u\, \mathcal{D}\chi\, \mathcal{D}\psi \exp\left(-\frac{1}{\varepsilon}\int (\mathcal{D}(u))^2 + \int \chi \mathcal{D}_L \psi\right) \prod_i U_i(t_i) \cdot S_V \cdot \exp(\Lambda_A du^A) \cdot \exp\left(\frac{1}{\varepsilon}(\ldots)\right)$$

We've arrived at the 2d A-model with symplectic structure Ω and target $\widehat{M}$.

S_V is a standard A-model observable.

<u>at $\varepsilon = 1$,</u> $\exp(\Lambda_A du^A)\exp\left(\frac{1}{\varepsilon}(\ldots)\right) = \underbrace{\exp \operatorname{Im}\oint \Lambda_A du^A}_{\exp i\int p\,dq}$

$$\exp\int (\mathcal{D}u)^2 = \exp\left(-\int \left(\frac{du}{ds}\right)^2 + \left(\frac{du}{dt}\right)^2\right) \frac{1}{\varepsilon}\exp\left(-\oint \operatorname{Re}\Lambda_A du^A\right)$$

Kapustin - Orlov