17 August 2010
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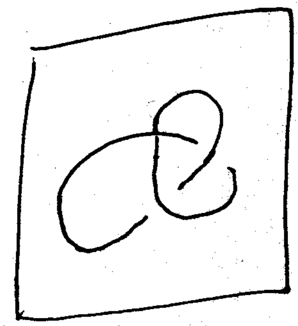
Fivebranes and Knots

Chern-Simons in $d=3$

$$I = \frac{k}{4\pi}\int \mathrm{Tr}\left(A\,dA + \frac{2}{3}AAA\right)$$



M_3

$M_3 = \mathbb{R}^3.$

$$J_R(q) = \int \mathcal{D}A\; e^{iI}\; \mathrm{Tr}_R\, P\exp\oint A$$

When you compute, one finds $\quad J_R(q;C) = \sum_{n=-m}^{m'} a_n q^n$

$$q = \exp\frac{2\pi i}{k+h} \qquad a_n \in \mathbb{Z}$$

I. Frenkel, Khovanov

There is a "Hilbert" space $\mathcal{H}$, $\mathbb{Z}\times\mathbb{Z}$ graded

(two conserved quantities H, F, $u(1)\times u(1)$)

Hamiltonian   fermion #

$$\mathrm{Tr}_{\mathcal{H}}\, q^H (-1)^F = J_R(q;C). \qquad \mathcal{H} = \mathcal{H}(R;C).$$

$\mathrm{Tr}_{\mathcal{H}}\, q^H y^F$

Expect $M_3 \longrightarrow M_3 \times \mathbb{R}$ or $\underbrace{M_4 = M_3 \times S^1}$

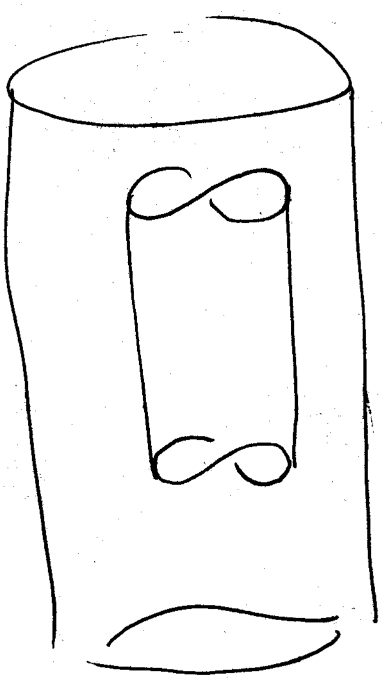
time

path integral $\Rightarrow$ number

Path integral $\Rightarrow$ number

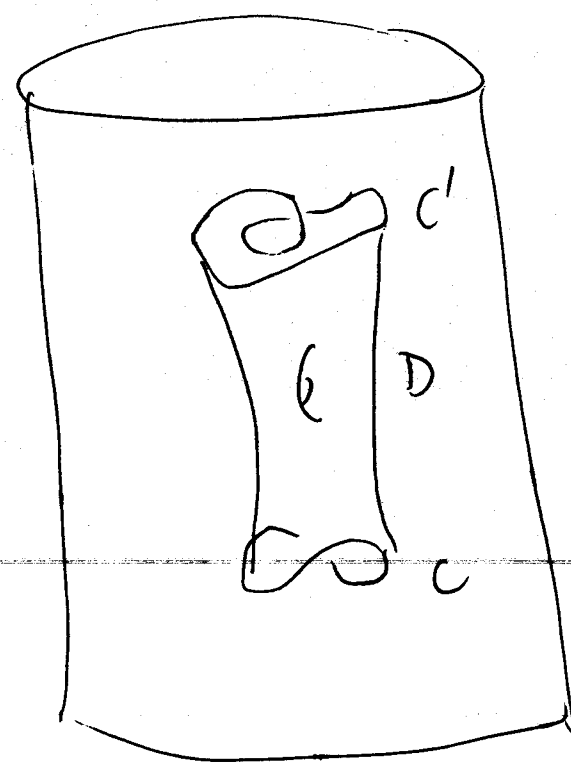
Space of physical states

$\mathbb{Z}_2$ graded trace
Supertrace in $\mathcal{H}$



World volume

= support of a surface operator



$\Phi_D : \mathcal{H}_C \to \mathcal{H}_{C'}$

2004: Gukov-Vafa-Schwarz

⋮

1998: Ooguri-Vafa

$$\int \mathcal{D}A \, \exp \frac{ik}{4\pi} \int \mathrm{Tr}(A\,dA + \tfrac{2}{3}A^3) \quad \mathrm{Tr}_R \, P\exp \oint_C A$$

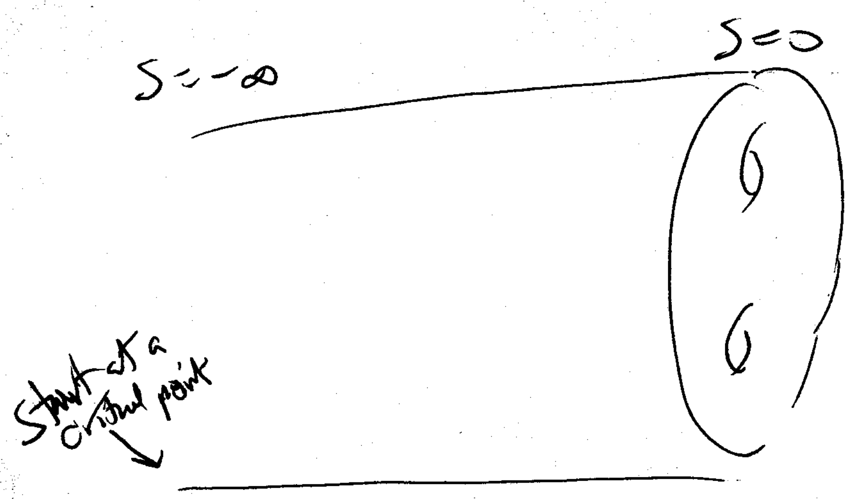
$$A \to a$$

$$\int_\Gamma \mathcal{D}a \exp \underbrace{\frac{ik}{4\pi} \int \mathrm{Tr}(a\,da + \tfrac{2}{3}aaa)}_{h = \text{Real Part}(\cdots)} \mathrm{Tr}_R \, P\exp \oint_C a$$

$$ds^2 = \int_{M_3} \mathrm{Tr} \, \delta a \wedge * \delta \bar{a}$$

$$= \int_{M_3} d^3x \sqrt{g} \, g \, \mathrm{Tr} \, \delta a_i \, \delta \bar{a}_j$$

$$\frac{da}{ds} = - \frac{\delta h}{\delta a}$$



$M_3 \times (-\infty, 0]$
$\quad\quad\quad\; s$

Cycle Γ consists of all values at $s=0$ of solns of flow eqn on $\mathbb{R}_+ \times M_3$

If $M_3 = \mathbb{R}^3$, the only critical point is $a=0$ (up to gauge)

$$A = \sum c_\lambda A_{(\lambda)}$$

$$\int dc_\lambda \exp i \sum m_\lambda c_\lambda^2$$

$\exp(i \int a da + \cdots)$

$\exp(h + i\Phi)$

Floer:

$$\frac{dA_i}{ds} = -\frac{\delta}{\delta A_i}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$$

$$\frac{dA_i}{ds} = \frac{1}{2}\underbrace{\varepsilon_{ijk} F_{jk}}_{\vec{B}_i} \qquad F^+ = 0$$

$$A_s = 0 \qquad \vec{E}_i = \vec{B}_i$$

$$\frac{da}{ds} = -g^{a\bar{a}}\frac{\delta h}{\delta \bar{a}^a}$$

A real-valued connection

$$\phi \in \Omega^1(M_3 \times \mathbb{R}_+, \mathrm{ad}(P))$$

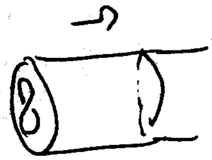
Flow equations are

$$F - \phi \wedge \phi = * D\phi$$

$$D * \phi = 0$$

↑ Kapustin-Witten parameter t

$$\mathcal{A} = A + i\Phi \qquad (t = i)$$



The theory on $M_3 \times \mathbb{R}_+$ is $\mathcal{N}=4$ SYM with the twist related to geometric Langlands (GL twist

Consider the case $G = U(n)$.

$$\delta(F - \varphi\wedge\varphi = *D\varphi)$$

$\mathcal{N}=4$ SYM theory
$M \times \mathbb{R}_+$

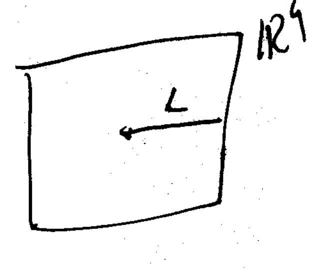
n parallel D3-branes
Here, the D3's end on an NS5
Bulk theory has $\theta \neq 0$, in fact $\theta = 2\pi k$.

Geometry: 10 dim'l spacetime is $T^*M_3 \times \mathbb{R}^4$

NS5 supported on
$T^*M_3 \times \{0\}$

D3 supported on
$M_3 \times L$
$\cap$
T^*M_3

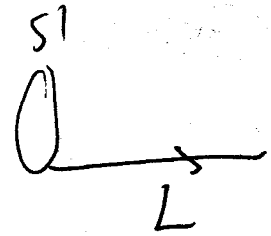
L = half line from $\{0\}$ to ∞



$\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$

$T^*M_3 \times \mathbb{R}^3 \times S^1$

$L \subset \mathbb{R}^3 \times \overset{\substack{S^1 \\ \cup}}{\{p\}}$



II 3

T-duality on S^1:

$D3 \to D4$

$NS5 \to (\ldots)$

$T^*M_3 \times \mathbb{R}^3 \times S^1$ asymptotically

$D4$ on $M_3 \times L \times S^1$

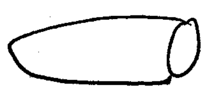
Taub-Nut

<u>Exact</u>

$T^*M_3 \times TN$

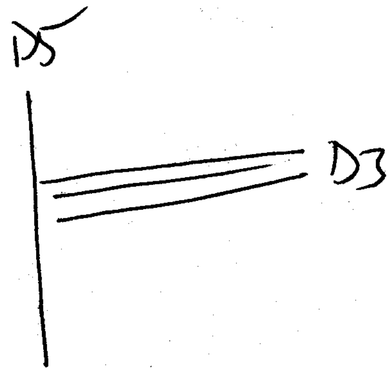
TN hyperkähler

$$\begin{array}{c} TN \to \mathbb{R}^3 \\ \text{generic} \uparrow \\ S^1 \end{array}$$

Actually, $D4$ on $M_3 \times$ 

Take the S-dual (before T-dual)

IIB: NS5 $\to$ D5



T dual is a a D4-D6 system.

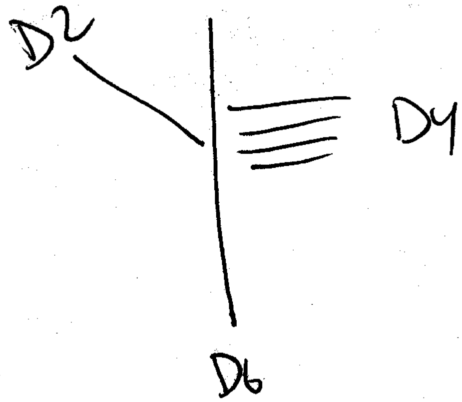
D6 on $T^*M_3 \times \underset{\mathbb{R}^3}{\underset{\cap}{\{0\}}} \times S^1$

D4 on $M_3 \times L \times S^1$

D2

$\cancel{C \times S^1 \times L}$ IF

$T^*C \times S^1$



$C \subset M_3 \subset T^*M_3$

$T^*C \subset T^*M_3$

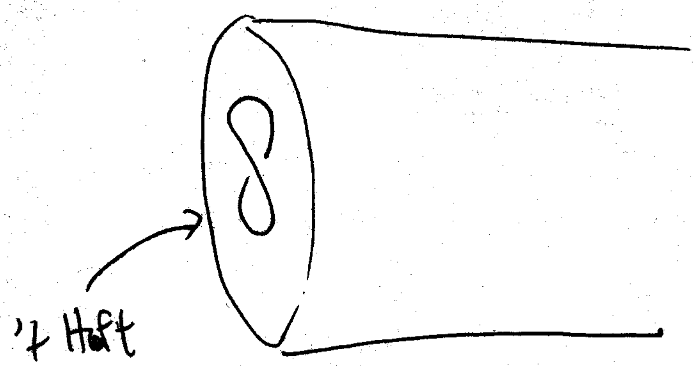
U(n) gauge theory on D4's.

$$M_3 \times \mathbb{R}_+ \times \underbrace{\mathbb{R}}_{\text{time}}$$

D2 end on $C \times \{s=0\} \times \mathbb{R}$.

$$\begin{array}{ccc} C & & \{s=0\} \\ \cap & & \cap \\ M_3 & & \mathbb{R}_+ \end{array}$$

$$M_3 = \partial(M_3 \times \mathbb{R}_+)$$



In general, could be on $M_4 \times \mathbb{R}_+$

$$M_4 = M_3 \times S^1$$

$$M_3 \times \mathbb{R}$$

$$M_4 \times \mathbb{R}_+$$

D = 2-manifold

$$\begin{array}{ccc} D = D' & \times & \{0\} \\ \cap & & \cap \\ M^4 & & \mathbb{R}_+ \end{array}$$

G = Chern-Simons

Fields are

A = gauge field
= connection on $G^\vee$ bundle $E \to M_4 \times \mathbb{R}_+$

$$B \in \Gamma(M_4 \times \mathbb{R}_+, \pi^* \Omega^{2,+}(M_4) \otimes \mathrm{ad}(E$$

$F^+ + B\times B = D_s B$

$\Lambda^2(\Omega^{3+}) \cong \Lambda^{3+}$

$*_{(4)} DB + \iota_{(\frac{\partial}{\partial s})} F = 0$

$\|$

$\iota_{(\frac{\partial}{\partial s})}(*DB + F)$

instanton # $= H = \int_{M_3\times\mathbb{R}_+} \left(\frac{\mathrm{Tr}\, F\wedge F}{8\pi^2} - \frac{\mathrm{Tr}(R\wedge R)}{8\pi^2}\right)$

a_n = "number" of solns for given $H=n$.

$Z(M_4\times\mathbb{R}_+) = \sum a_n q^n$.

$M_4 = M_3\times\mathbb{R}$ (time)

1) chain groups
   classical approx to ground states.
   time-indept solutions

2) define a differential by "counting" time dependent solutions

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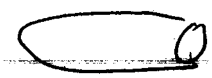
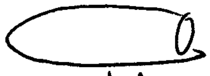
gauge theory on $M_4\times\mathbb{R}_+$

Good version is the 6d (0,2) theory on ~~M_4~~ $\times$  W

M_x

$M_3\times S^1\times$  W.

$\downarrow$ $S^1\times S^1$ fiber (generic)

$M^3\times L$