Scaling laws for dynamos – from planets to rapidly rotating stars

Ulrich Christensen

Max-Planck-Institute for Solar System Research, Katlenburg-Lindau, Germany

Collaborators: Julien Aubert, Peter Olson, Andreas Tilgner, Ansgar Reiners, Volkmar Holzwarth

Christensen & Tilgner, Nature, 429, 169 (2004)

Christensen & Aubert, Geophys J Int., 166, 97 (2006)

Olson & Christensen, Earth Planet Sci Lett, 250, 561 (2006)

Christensen, Holzwarth & Reiners, in prep.

Questions

- For convection-driven dynamos in rotating spheres, how do characteristic properties (heat flow, velocity, magnetic field strength) vary with control parameters?
- Does the dynamo regime change between parameter values accessible in numerical models and planetary values ?
- Do planetary dynamos and (some) stellar dynamos follow the same rules ?

Hypothesis

Diffusive processes, described by the

- kinematic viscosity ν
- thermal diffusivity κ
- magnetic diffusivity η

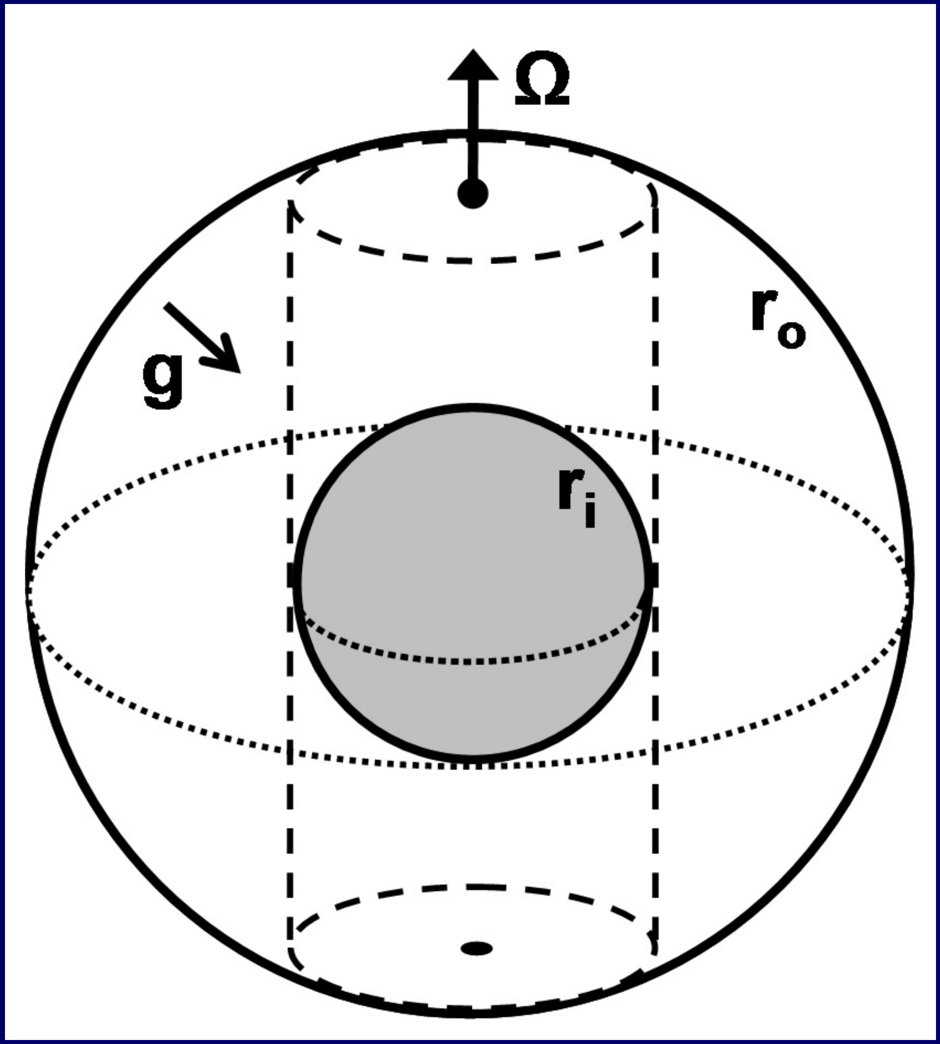
don't play first-order role in planetary dynamos

and

geodynamo models are close to this regime.

Outline of dynamo models

- Boussinesq equations for convection-driven MHD flow
- Rigid inner and outer boundary
- $r_i / r_o = 0.35$
- Fixed temperature contrast, no internal heat sources



Control parameters

• Ekman number	$E = \nu/(\Omega D^2)$	10^{-6} .... 10^{-3}
• Prandtl number	$Pr = \nu/\kappa$	0.1 10
• Magnetic Prandtl #	$Pm = \nu/\eta$	0.06 20
• Modified Rayleigh #	$Ra^* = \alpha g_o \Delta T/\Omega^2 D$	5 .... 200 × Ra_{crit}

Modified Rayleigh number is independent of diffusivity.

$Ra^* = Ro_c^2$ (convected Rossby number)

Diagnostic numbers

Use non-dimensional measures for velocity, magnetic field and heat transport efficiency that or independent of diffusivities.

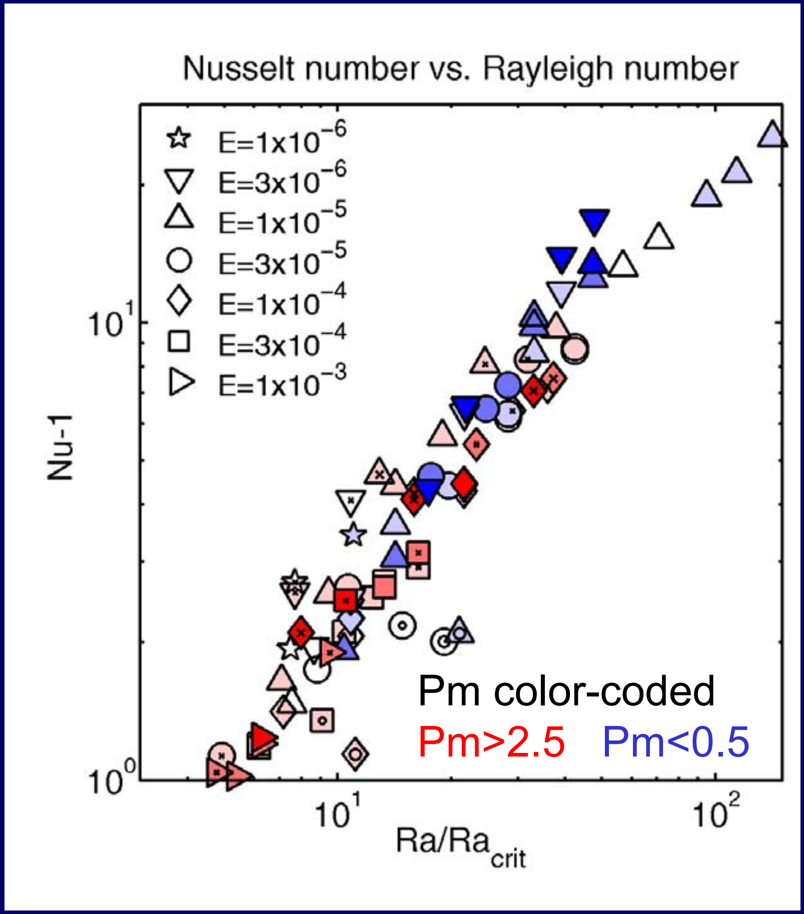
- Ro Rossby number $Ro = U/ \Omega D$
- Lo Lorentz number $Lo = B / (\rho\mu)^{1/2}\Omega D$
- Nu* Modified Nusselt number $Nu^* = Q_{adv}/(4\pi r_o r_i \rho c_p \Delta T \Omega D)$

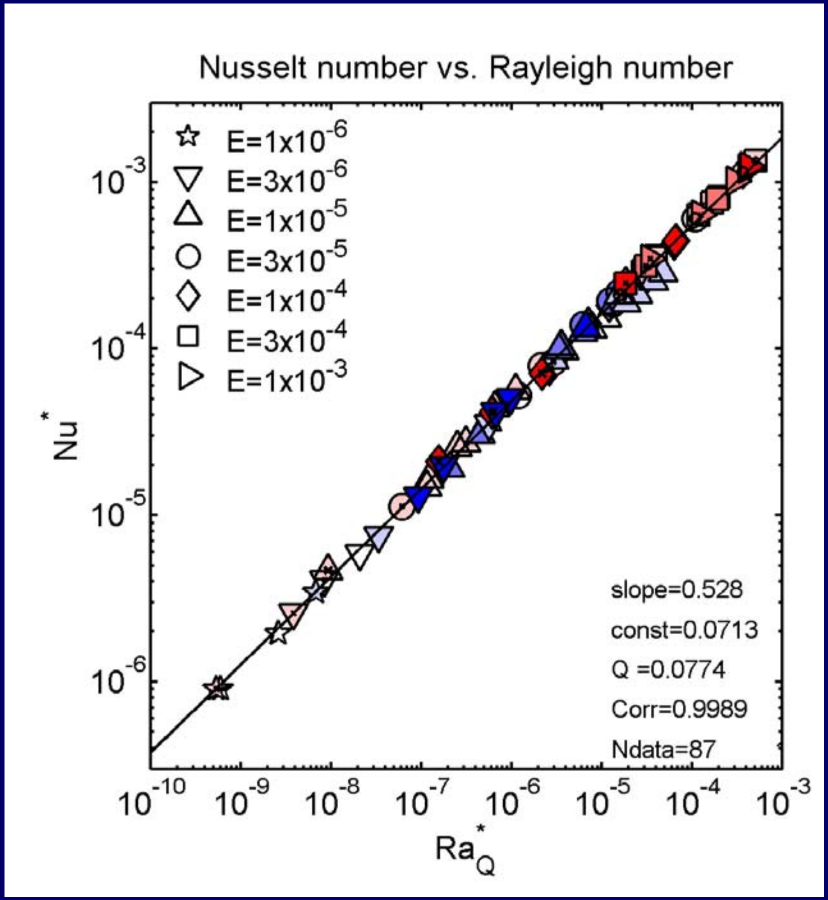
Modified flux Rayleigh number

$$Ra^*_Q = Ra^* Nu^* = Ra\ E^{-3}\ Pr^{-2}$$

Ra^*_Q is a measure for the work by buoyancy forces

Scaling of Nusselt number

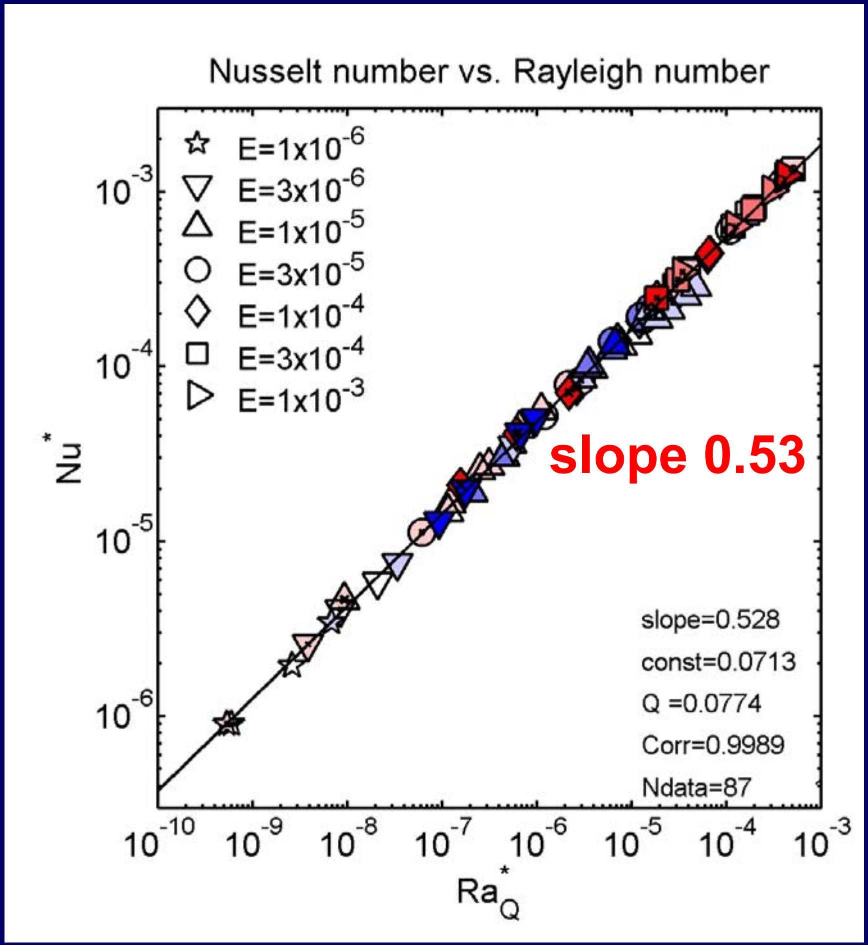




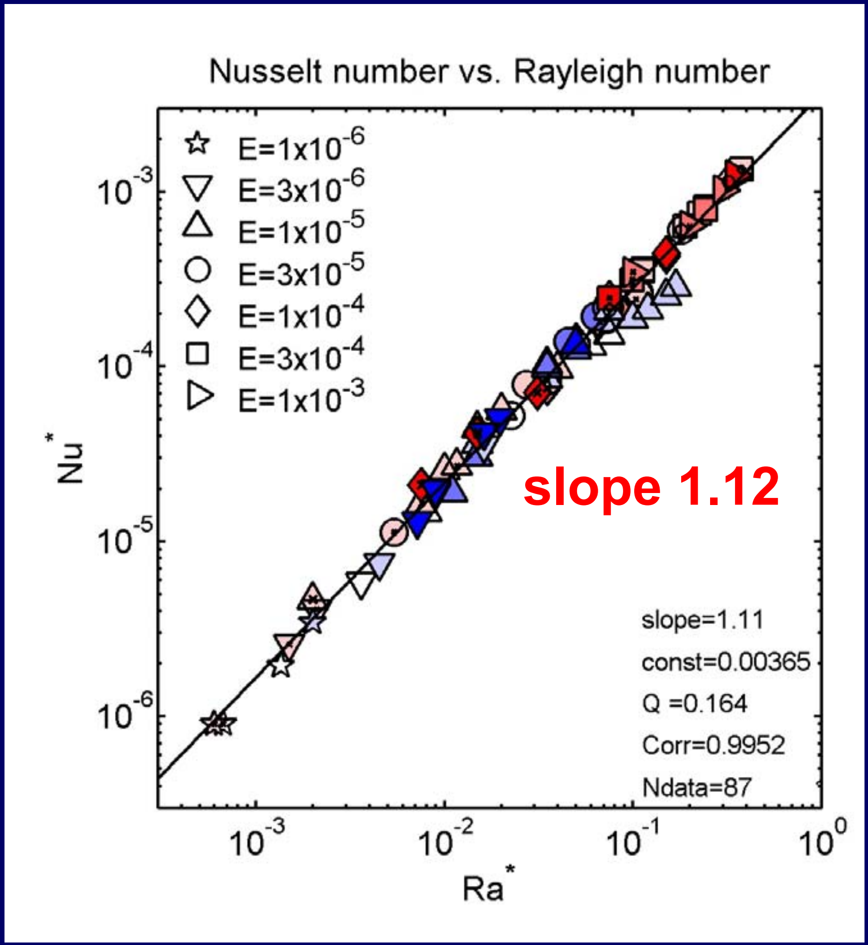
Use of modified „diffusionless“ parameters allows to collapse the data and express the dependence by a single power-law.

Compared to non-rotating convection, the exponent is very large (≈ 0.53).

Scaling of Nusselt number

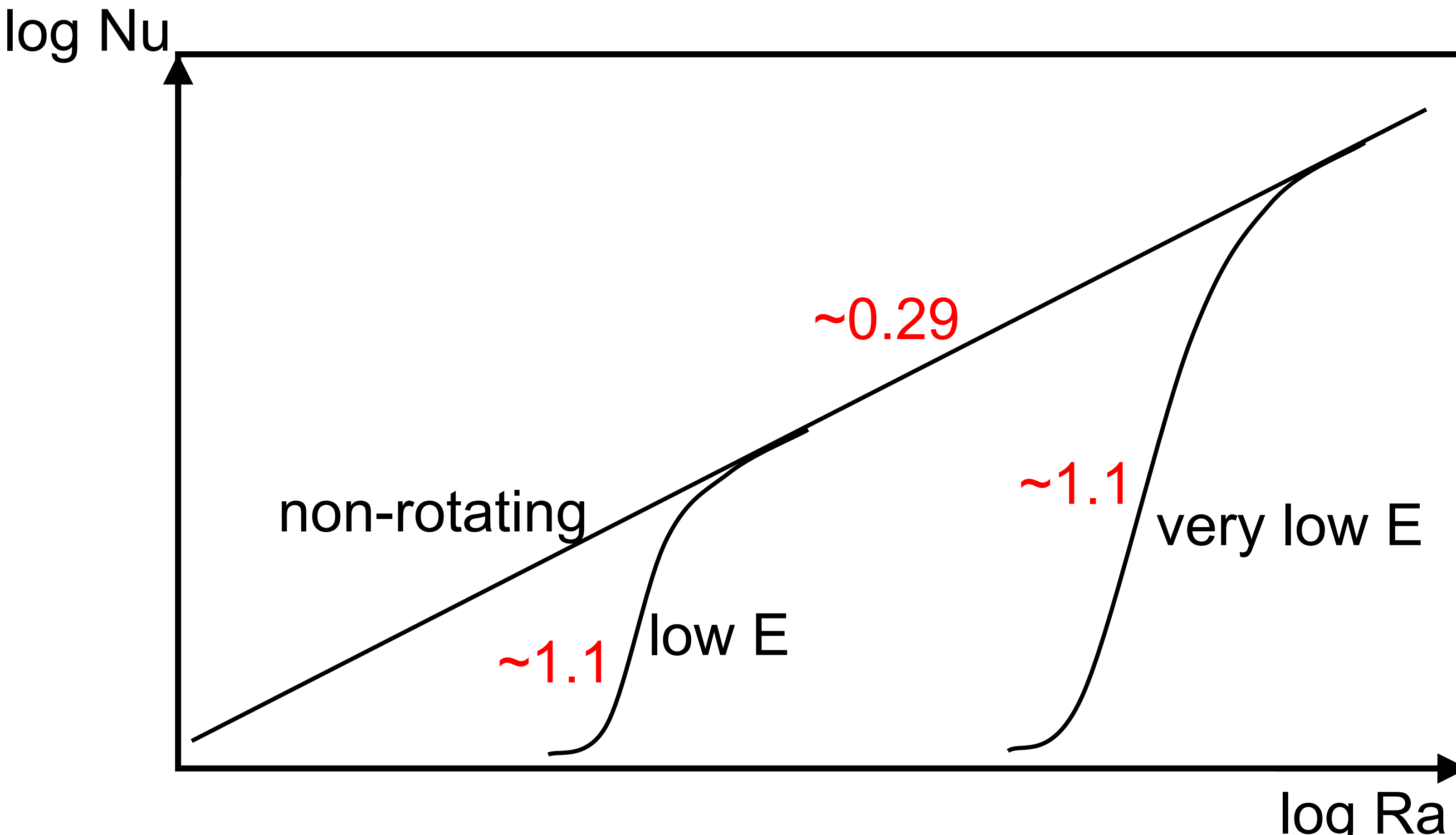


Flux Rayleigh #



Rayleigh # based on ΔT

Nusselt number scaling



See Poster by Eric King

Force balance

Balance in vorticity equation:

$$\nabla\times(\omega\times u) \quad \sim \quad \alpha g\, \nabla\times T e_r \quad \sim \quad 2\Omega\, \partial u/\partial z$$

(1) Assume mixing length ℓ=L, balance inertia ~ buoyancy

$U^2/\ell^2 \sim \alpha g \delta T/\ell$ $\quad q = \rho c_p U_r \delta T$ $\quad$ (q: advected heat flux)

$$U \sim [\, q\, \ell \,/\, \rho\, H_T]^{1/3} \quad \text{or} \quad Ro \sim Ra_Q{}^{*1/3}$$

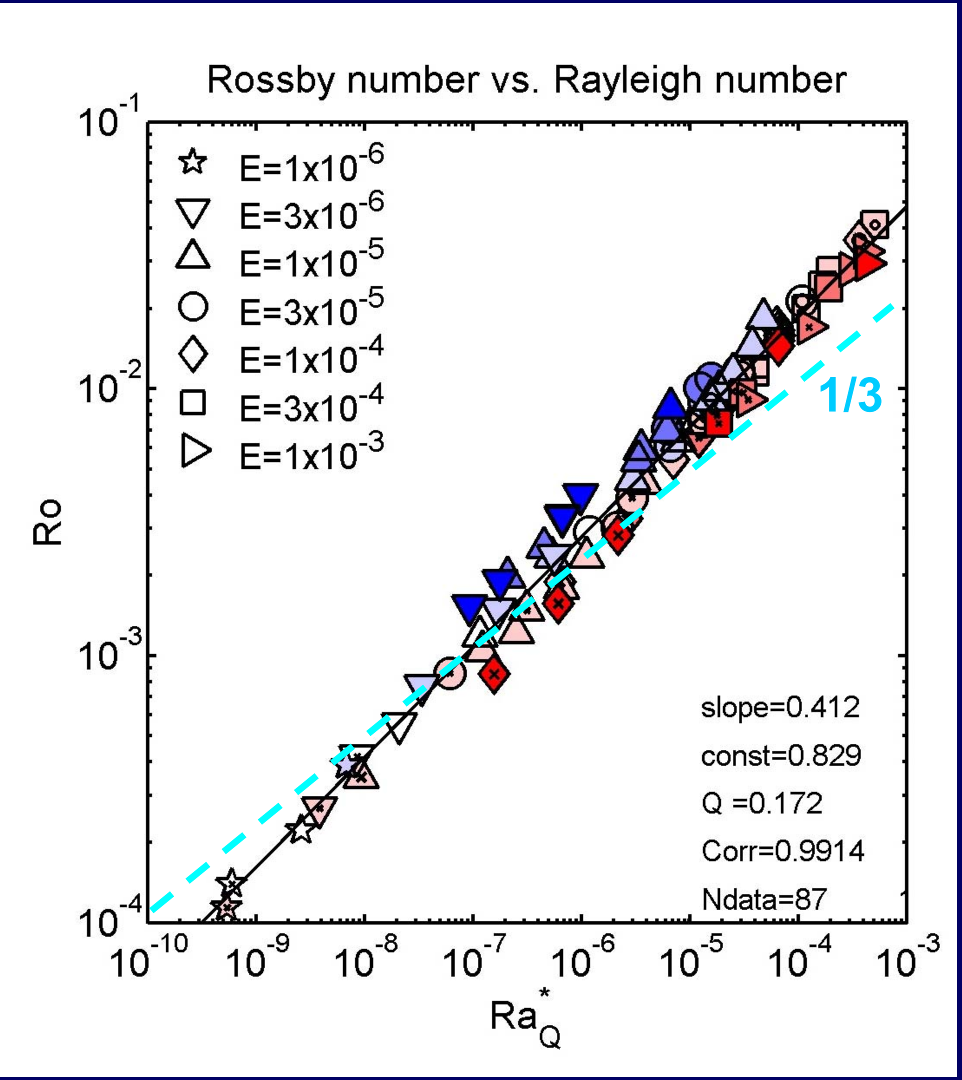
(2) Triple force balance, determine ℓ from Coriolis ~ Inertia

$U^2/\ell^2 \sim \Omega U/L \quad \Rightarrow \quad \ell \sim (UL/\Omega)^{1/2}$ $\quad$ (L: „global“ length scale)

$$U \sim (q \,/\, [\rho H_T])^{2/5}\, (L/\Omega)^{1/5} \quad \text{or} \quad Ro \sim Ra_Q{}^{*2/5}$$

(with density stratification, $L = H_\rho$, else L = shell thickness)

Velocity Scaling

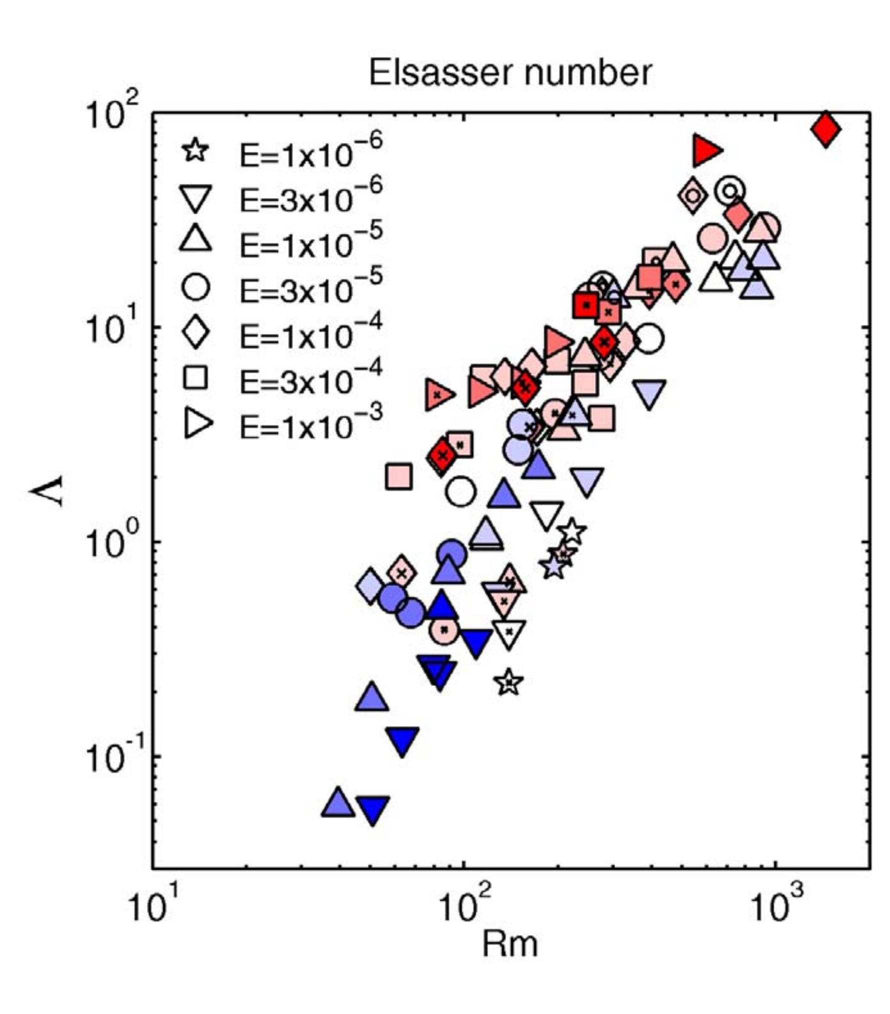


$$Ro \sim Ra^*_Q{}^{0.41}$$

Agrees well with prediction from triple force balance

Small effect of Pm

What controls the strength of the magnetic field?



Paradigm:

Magnetostrophic balance
Elsasser number

$\Lambda = B^2/(\mu\eta\rho\Omega) \sim O(1)$.

In the numerical models, the Elsasser number varies in the range 0.06 – 100.

Either force balance not magnetostrophic, or Λ not a good measure for magnetostrophy.

Alternative: Field strength controlled by available power ?

Power-limited magnetic field strength

Work done by buoyancy: $P \sim \rho g \alpha U_r \delta T \sim q / H_T$

Ohmic dissipation: $D_{ohm} = f_{ohm} P$

Dissipation time: $\tau_{ohm} = E_{mag}/D_{ohm} \sim \tau_\eta Rm^{-1} \sim L/U$

$B^2/2\mu_o = f_{ohm} P \tau_{ohm} \sim f_{ohm} (q/H_T) (L/U)$

(1) Mixing length theory ($U \sim q^{1/3}$):

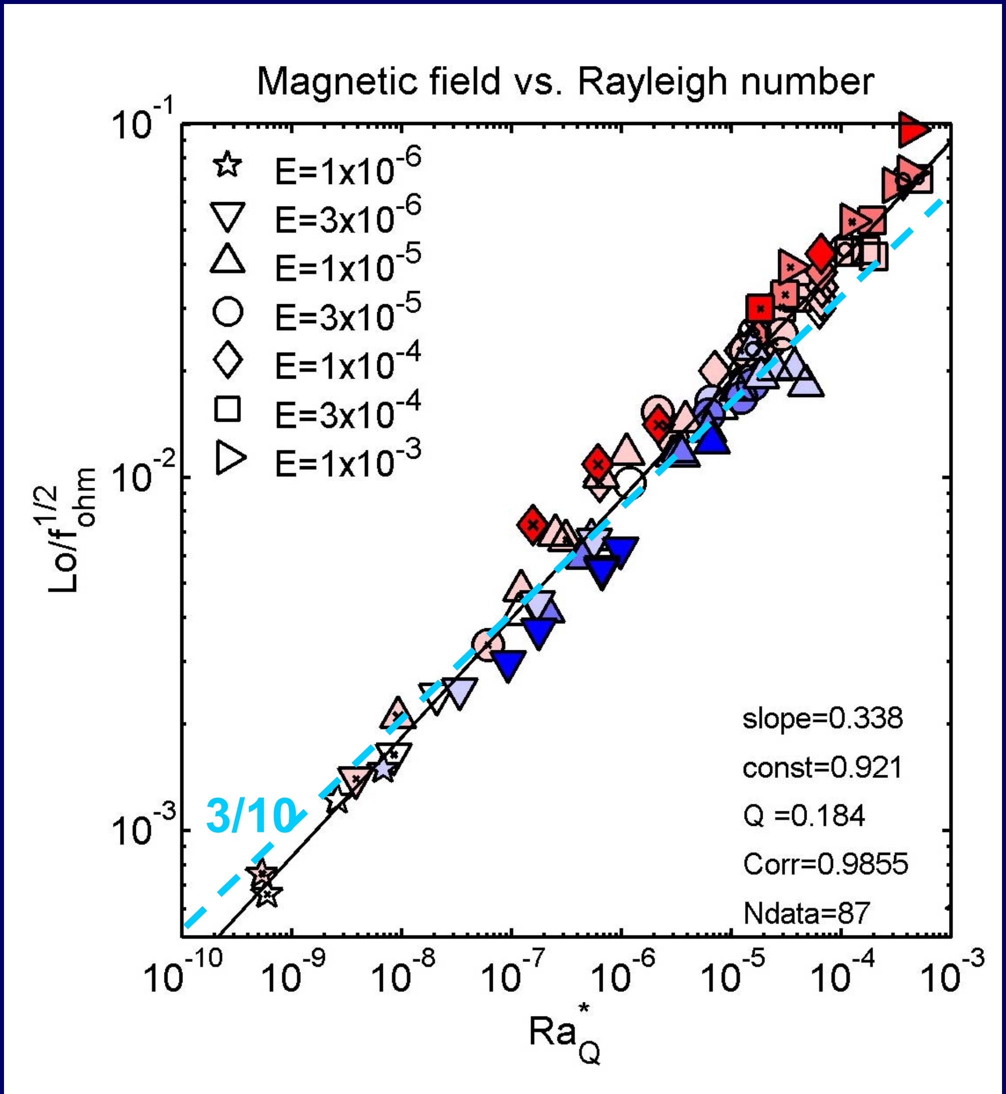
$$B^2/2\mu_o \sim f_{ohm} \rho^{1/3} (qL/H_T)^{2/3}$$

(2) Triple force balance ($U \sim q^{2/5}$):

$$B^2/2\mu_o \sim f_{ohm} \rho^{2/5} (q/H_T)^{3/5} L^{4/5} \Omega^{1/5}$$

Non-dimensional: $Lo/f_{ohm}{}^{1/2} \sim Ra_Q{}^{*1/3}$ or $Ra_Q{}^{*3/10}$

Magnetic Field Scaling



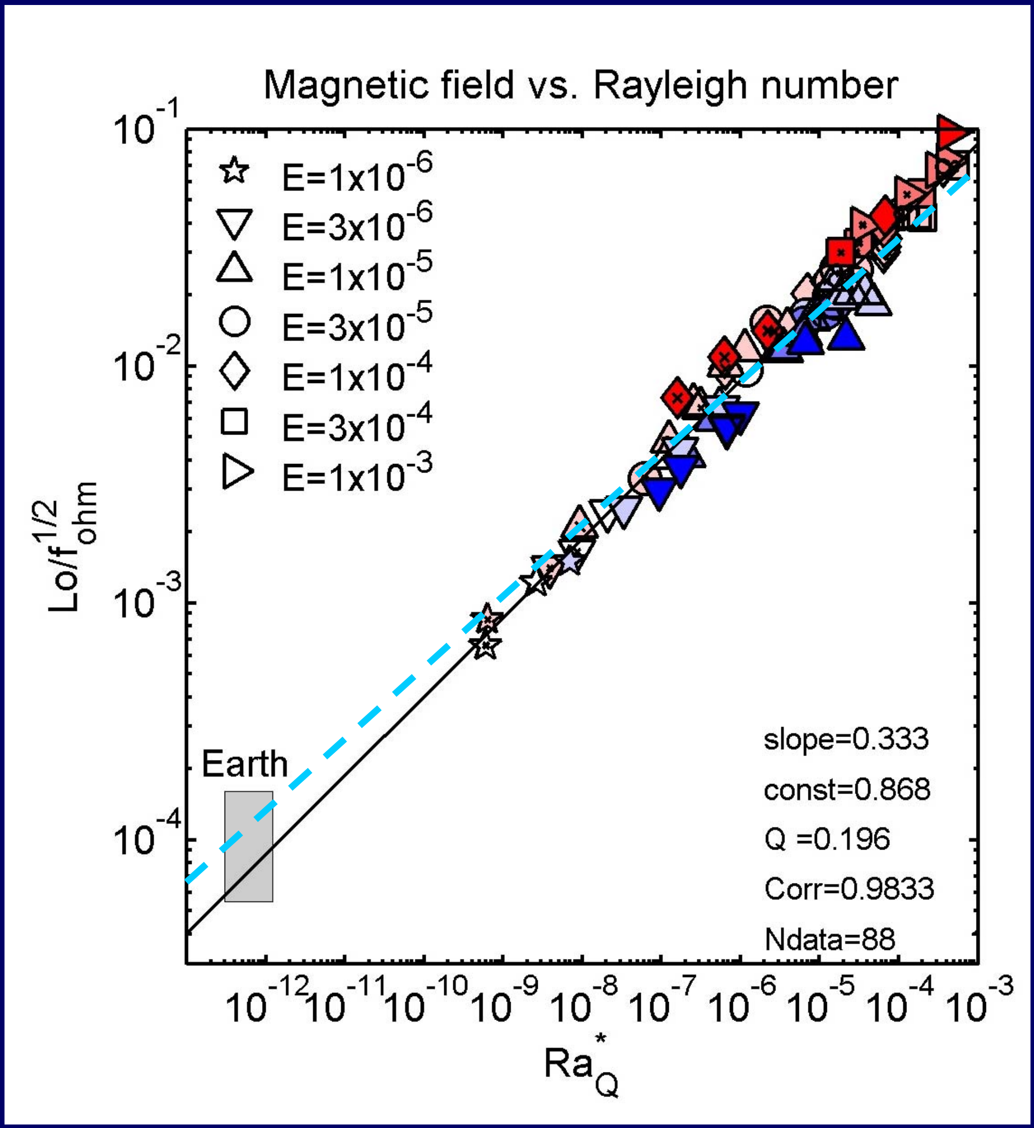
$$Lo \sim Ra^{*}_{Q}{}^{1/3}$$

(B ~ heat flux$^{1/3}$ and independent of rotation)

Fit is marginal for 3/10 exponent.

Problem: 1/3-scaling for B should go along with 1/3-scaling for U

Comparison with Earth



Assume for Earth‘s core:

$f_{ohm} \approx 1$

$B_{rms} \approx 1 – 3$ mT

$Q \approx$ 2 – 8 TW

(effective value: superadiabatic heat flux plus effect of compositional convection)

Geodynamo fits on correlation line

Energy partitioning

$E_{mag} / E_{kin} = Lo^2/Ro^2$

$\sim 1.5\ Ra_Q^{* -2/15}$

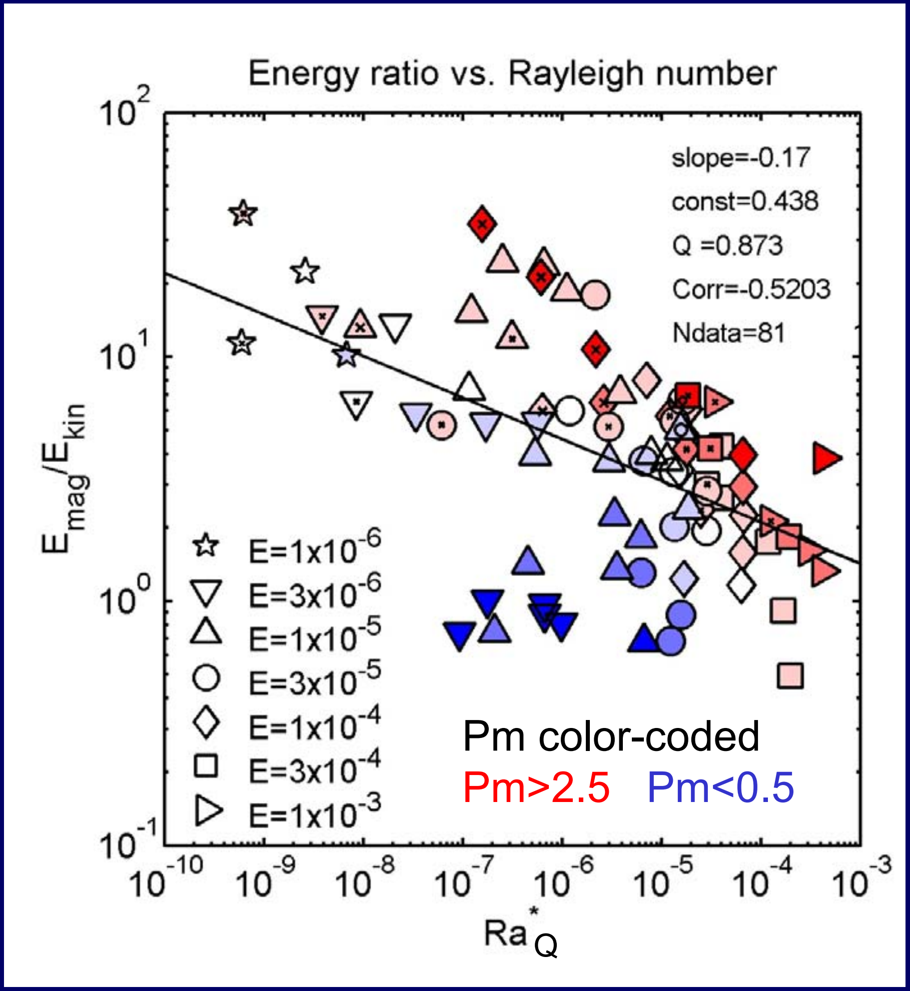
$\sim\ Ro_c^{-3/5}$

Earth's core:

$Ra_Q^* \approx 10^{-13}$

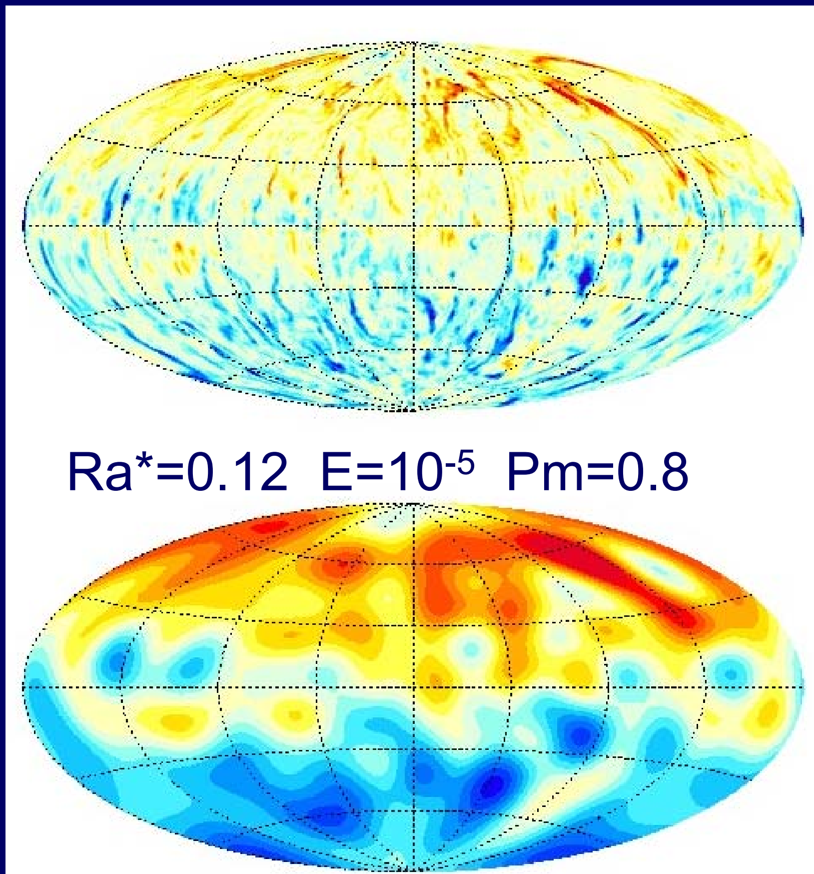
$\Rightarrow$

$E_{mag}/E_{kin} \approx 100$

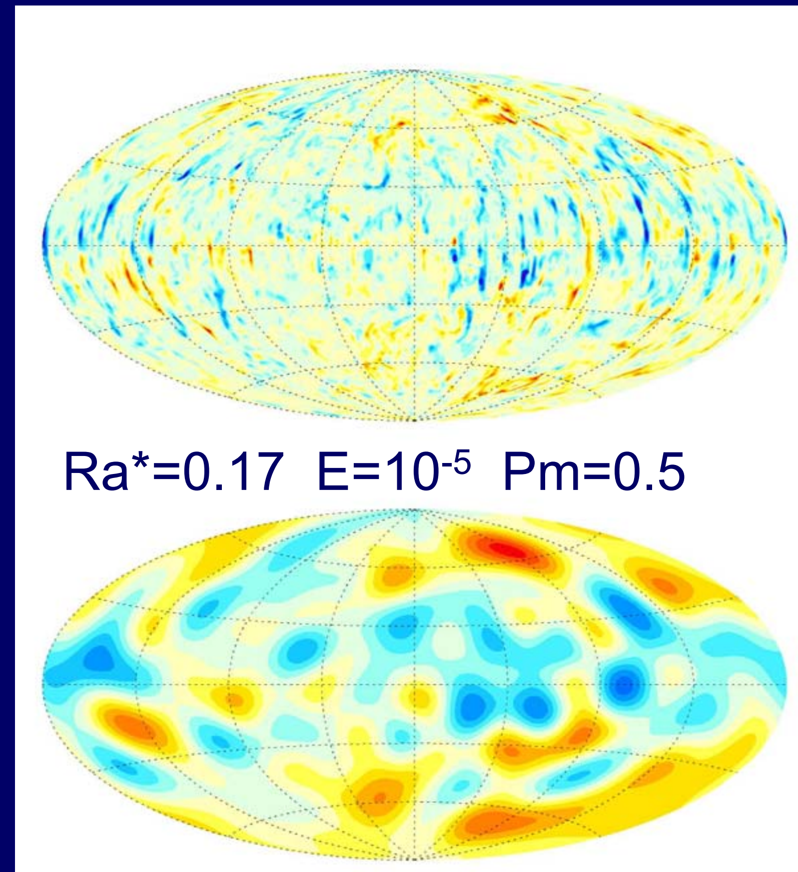


Role of magnetic Prandtl # ?

Field topology



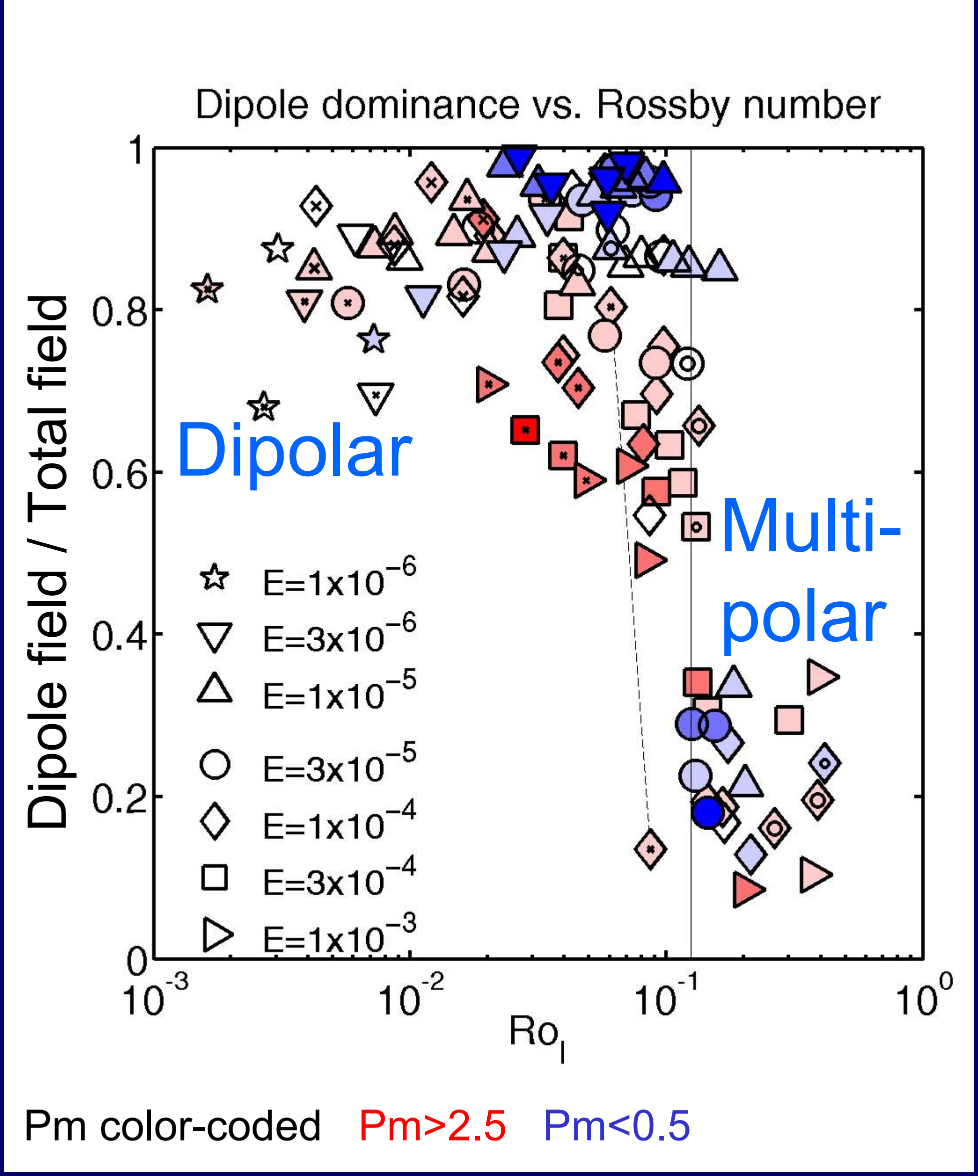
Dipolar



Multipolar

Scaling laws so far restricted to dipolar dynamos

Role of rotation



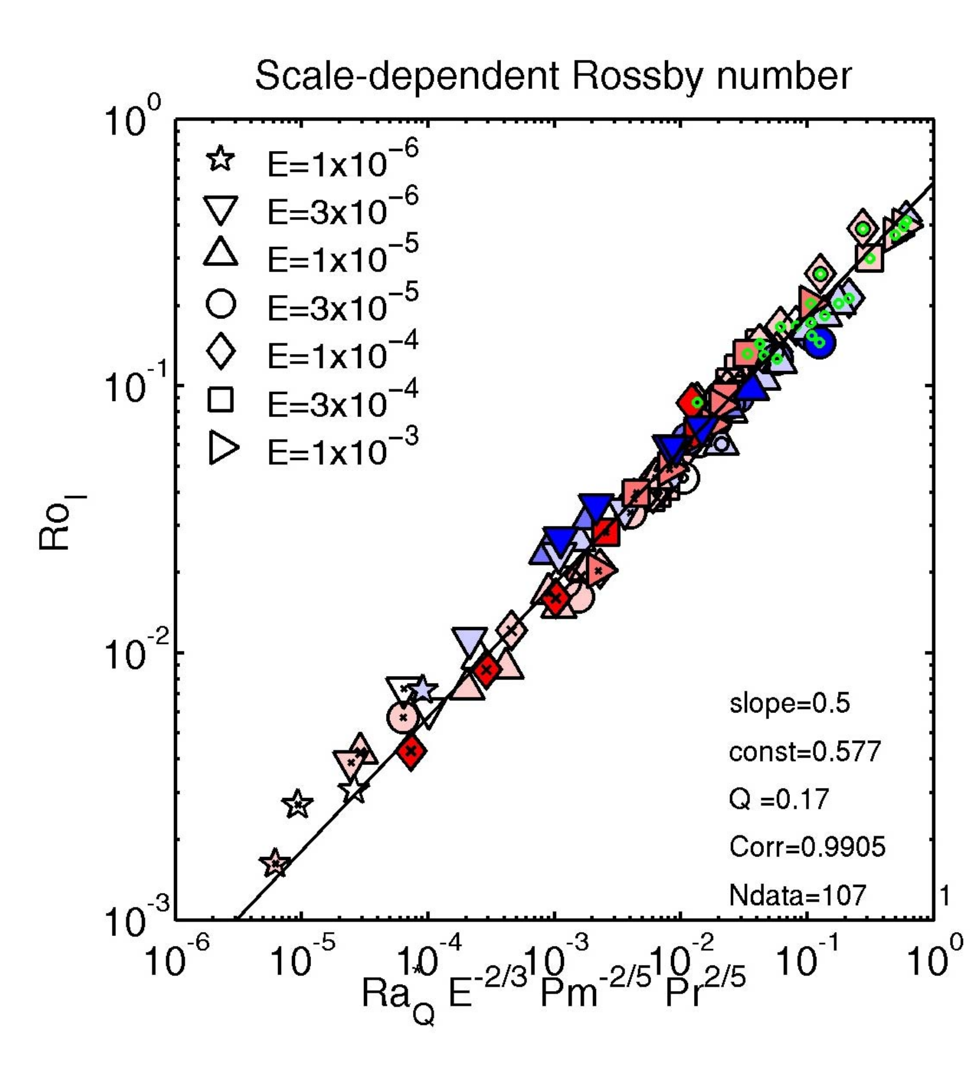
Inertial vs. Coriolis force:

Rossby number Ro_ℓ calculated with mean length scale ℓ in the kinetic energy spectrum

$$Ro_\ell = U/\Omega\ell$$

Regime boundary at $Ro_\ell \approx 0.12$ (depends on heating mode, b.c.)

Ro_ℓ vs. control parameters



Fit involves all four control parameters

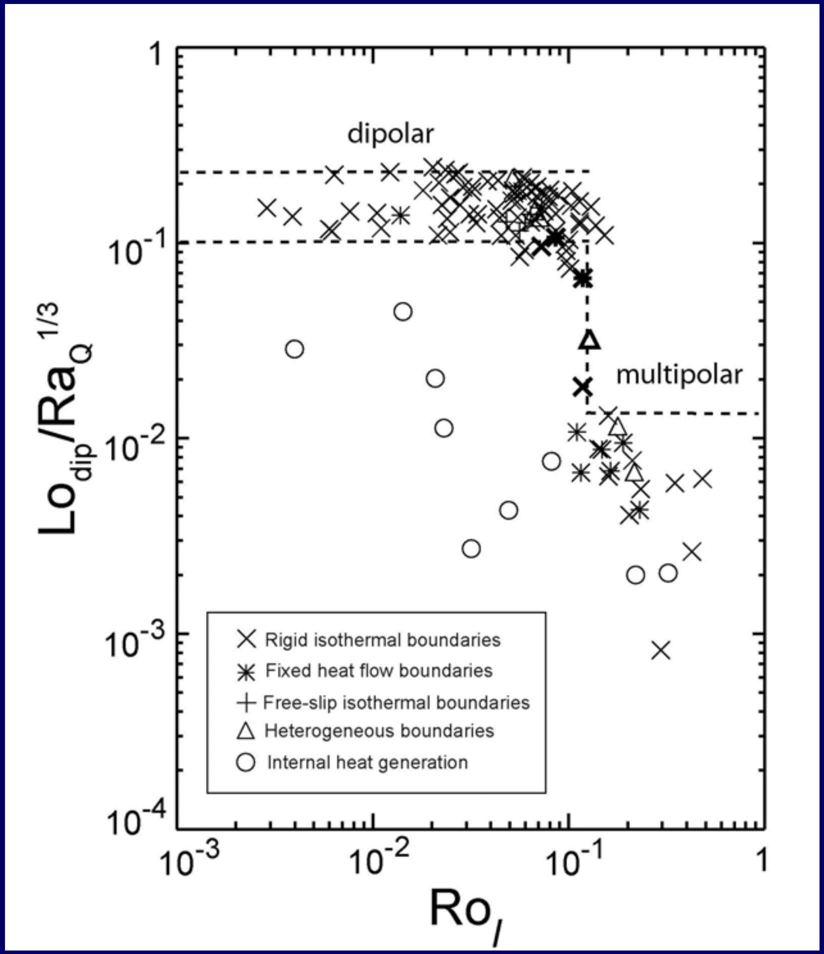
$$Ro_\ell \sim$$

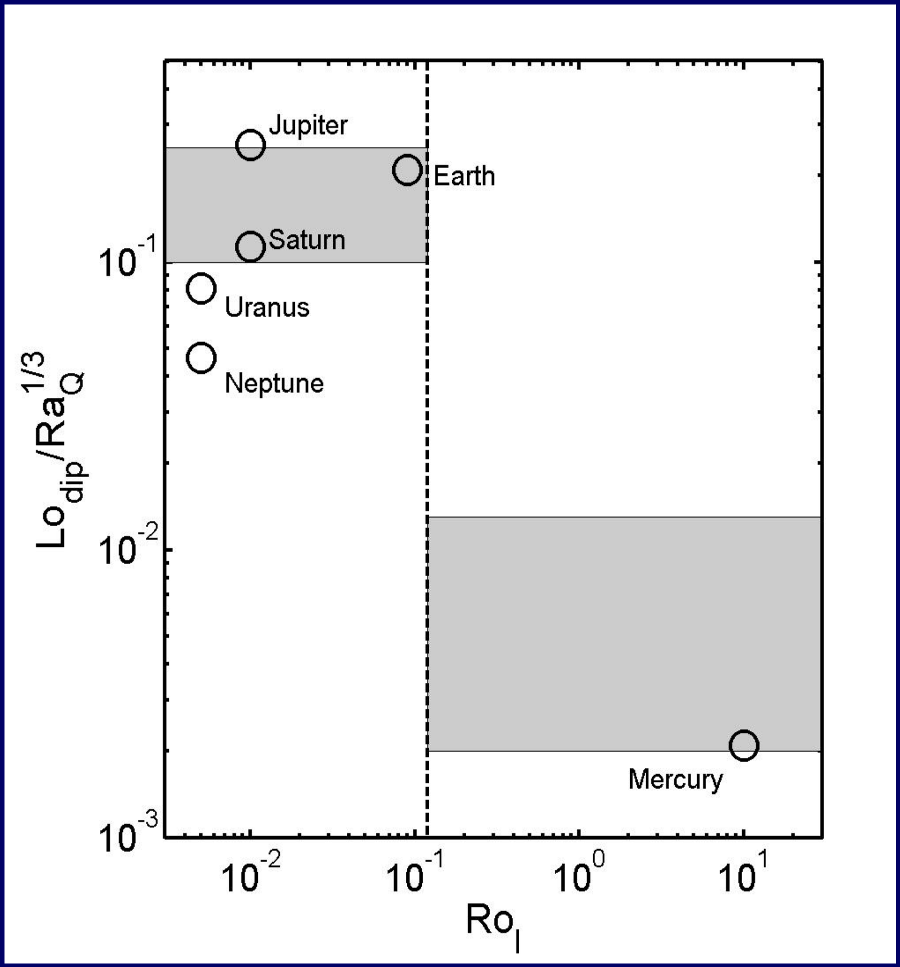
$$Ra_Q^{*1/2}\, E^{-1/3}\, Pm^{-1/5}\, Pr^{1/5}$$

Wild extrapolation to obtain Earth value:

$$Ro_\ell \approx 0.1$$

Dipole moment scaling





Earth predicted to lie close to transition dipolar - multipolar

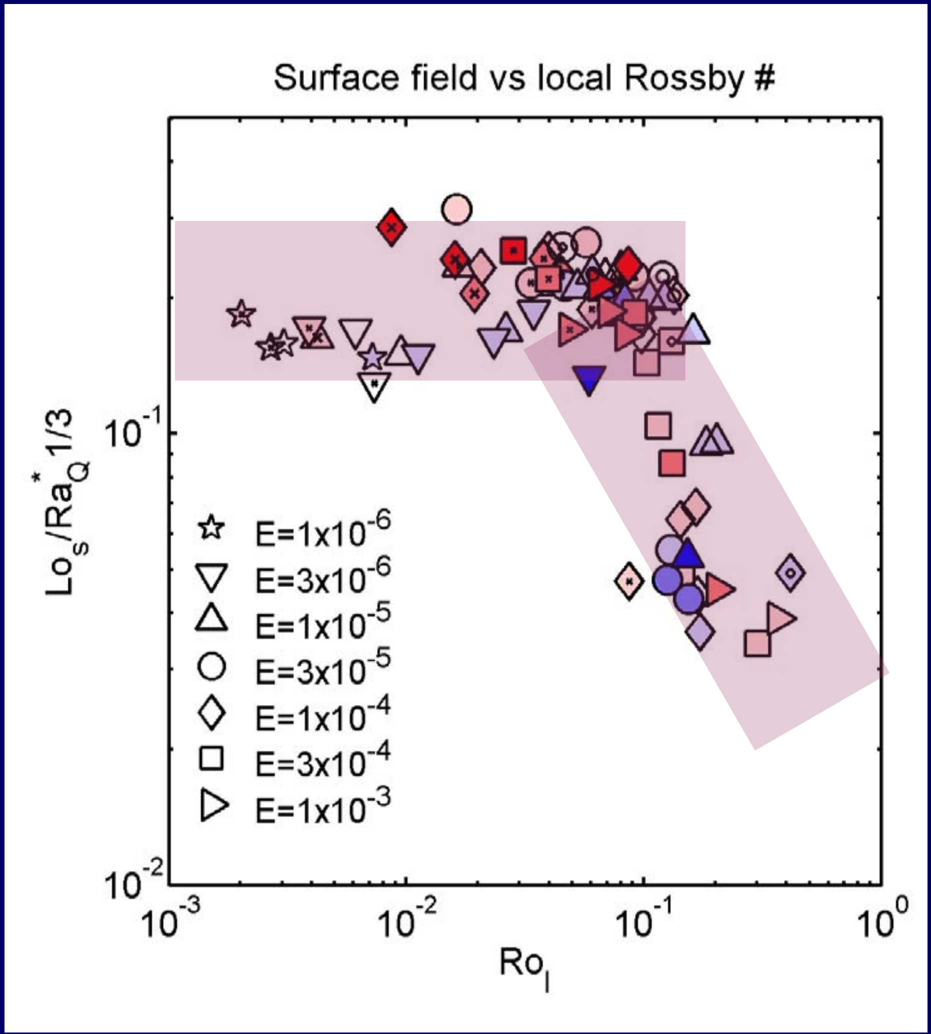
Same scaling of B for stars ?

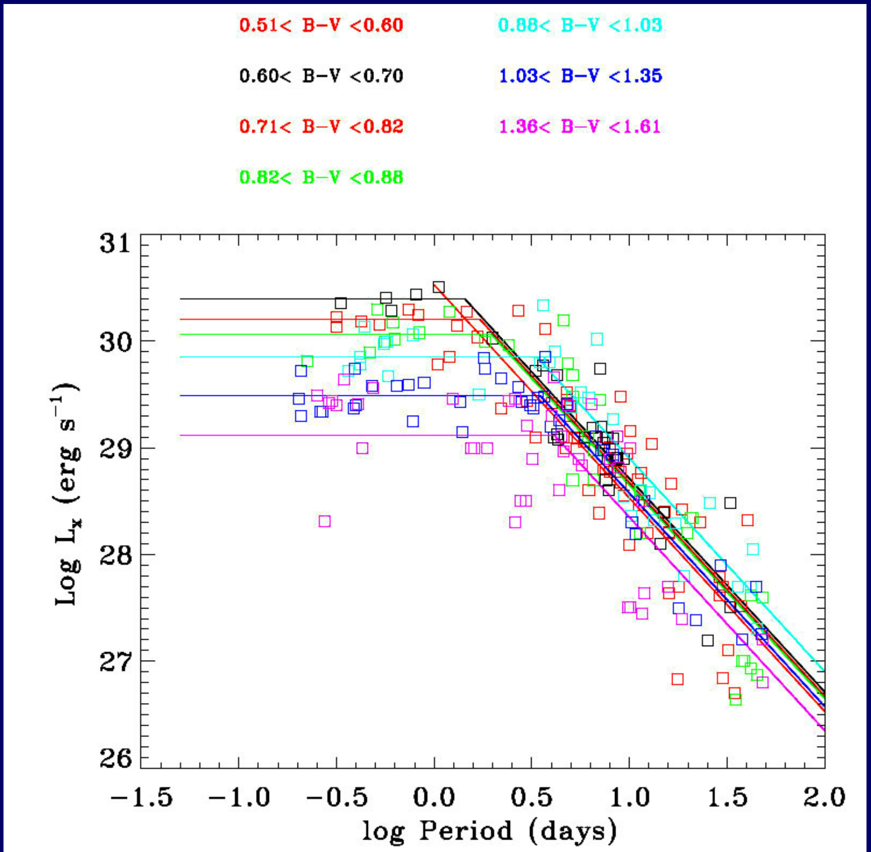
Problems with sun:

- Tachocline may lead to fundamentally different dynamo
- Large field scales (dipole) not dominant
- Rotation plays lesser role than in planets

What about fully convective (low mass) and rapidly rotating stars ?

Surface field vs. rotation



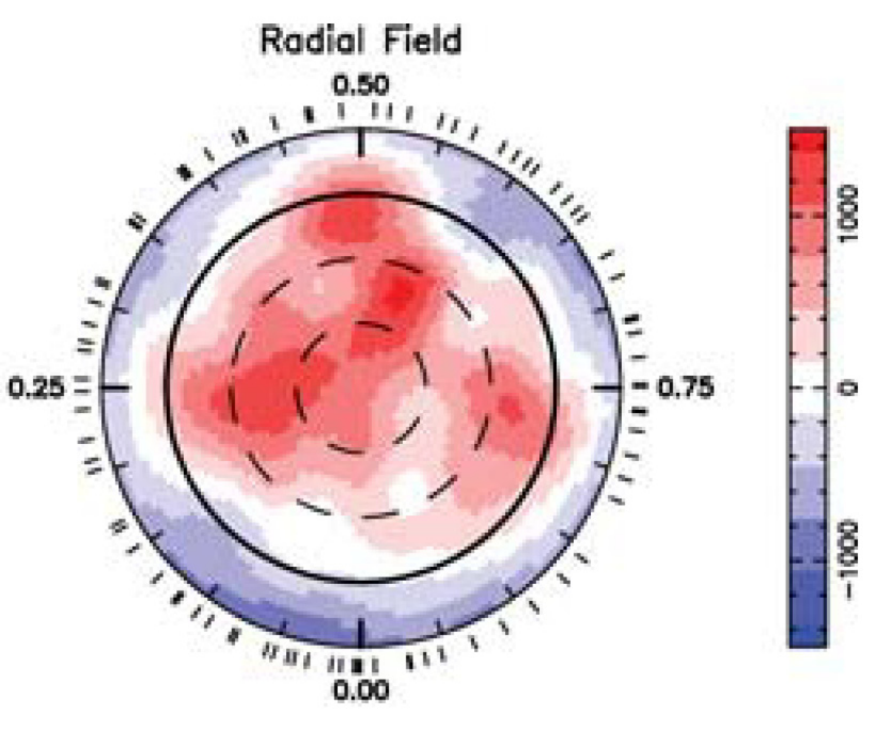


Pizzolato et al. (2003)

Increase with rotation rate

Saturation at threshold

V374 Peg



V374 Peg (Donati et al. 2006)

- $M \approx 0.28\ M_{☉}$
- $T_{rot} = 0.44$ days

Zeeman-Doppler imaging

- Dipolar field
- ≈ 0.1 T (1 kG)
- Little diff. rotation

Scaling B for planets and stars

- Theoretical result with $U \sim q^{1/3}$ scaling:

$$B^2/2\mu_o = c\, f_{ohm}\, \rho^{1/3}\, (qL/H_T)^{2/3}$$

- Determine constant c=0.63 from geodynamo models, where L=D, ρ = const and $H_T^{-1} \sim r$.
- Density stratification: assume $L(r) = \min(H_\rho, D)$
- Stellar model: $\rho(r)$, $q_{bol}(r)$, $q_{rad}(r)$, $H_T(r)$, $H_\rho(r)$
- Take volume average (q_o is bolometric surface flux):

$$< B^2 > / 2\mu_o = c\, F\, f_{ohm} <\rho>^{1/3} q_o^{2/3}$$

- Radial dependencies condensed into efficiency factor F

F - factors

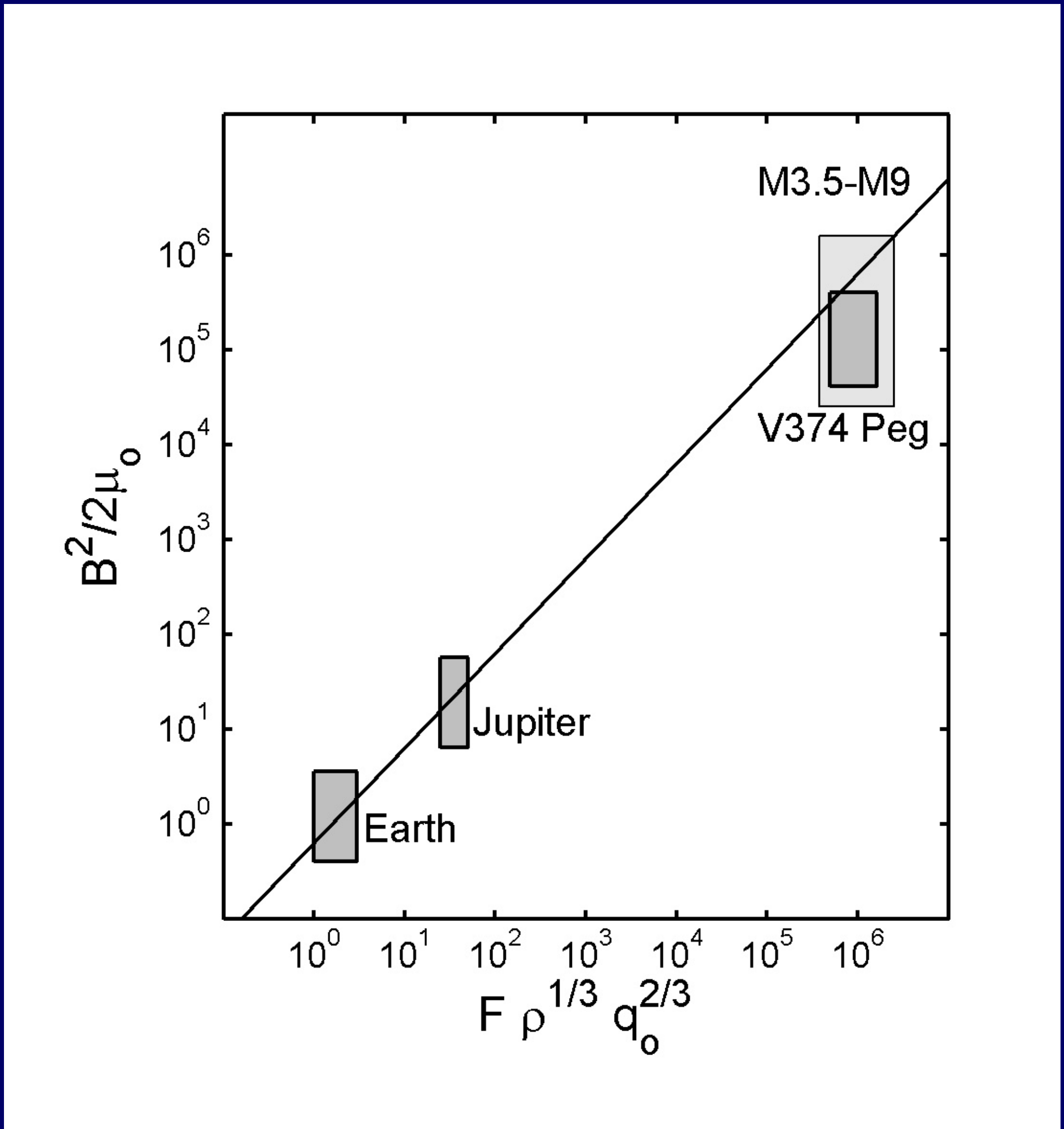
- Earth‘s core $F \approx 0.6$
- Jupiter $F \approx 1.0$
- 0.25 $M_{☉}$ star $F \approx 1.3$

Ratio internal vs surface field

From numerical models with a dipolar field:

- $\langle B^2 \rangle_{int} \sim (2.5 – 5) \times \langle B^2 \rangle_{surf}$
- $\langle B^2 \rangle_{int} \sim (4 – 10) \times \langle B^2 \rangle_{dipole}$

From planets to stars



V374 Peg (dipole field) + 13 rapidly rotating M-stars from Reiners & Basri (2006) with mean flux determined from FeH-line spectroscopy

Planets and rapidly rotating low-mass stars seem to follow the same scaling law

Conclusions

- „Diffusionless“ scaling in terms of Ra* explains properties of dipole-dominated numerical dynamos within the accessible range of control parameters
- In dipole-dominant dynamos (rapid rotation), magnetic field strength is controlled by the available power and is independent of rotation
- For slow rotation ($Ro_{local} > 0.1$) the field is weaker and multipolar
- The predictions of the scaling law for B agrees with Earth‘s and Jupiter‘s field strength and with assumptions how to generalize it for strong density stratification also with the field strength of rapidly rotating low-mass stars