Role of Cross-Helicity ①

Loosely speaking, this is $\underline{u} \cdot \underline{B}$

Local cross-helicity $\dfrac{\underline{u} \cdot \underline{B}}{|\underline{u}|\,|\underline{B}|}$

- alignment $\cos\theta$

Total cross-helicity $\dfrac{\int_V \underline{u} \cdot \underline{B}\, dV}{\left(\left(\int_V u^2 dV\right)^{1/2}\left(\int_V B^2 dV\right)\right)^{1/2}}$

Debatably a misnomer! (Due to D. Montgomery?)
Moffatt (1969) - physical interpretation gives sense to name.

Related to use of Elsasser variables

$\underline{\Lambda}_+ = \underline{u} + \dfrac{\underline{B}}{\sqrt{\mu_0 \rho}}$, $\underline{\Lambda}_- = \underline{u} - \dfrac{\underline{B}}{\sqrt{\mu_0 \rho}}$

$\underline{u} \cdot \underline{B} = \frac{1}{4}\sqrt{\mu_0 \rho}\,(\Lambda_+^2 - \Lambda_-^2)$

Equations (Alfvénic units) ②

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla p + \underline{j} \times \underline{B} + \underline{F} + \nu \nabla^2 \underline{u} \quad ①$$

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B} \quad ②$$

$$\nabla \cdot \underline{u} = 0, \quad \nabla \cdot \underline{B} = 0 \quad ③$$

or

$$\frac{\partial \underline{\Lambda}_+}{\partial t} = -\underline{\Lambda}_- \cdot \nabla \underline{\Lambda}_+ - \nabla\left(p + \tfrac{1}{2}B^2\right) + \underline{F} + \tfrac{1}{2}(\nu+\eta)\nabla^2 \underline{\Lambda}_+ + \tfrac{1}{2}(\nu-\eta)\nabla^2 \underline{\Lambda}_-$$

$$\frac{\partial \underline{\Lambda}_-}{\partial t} = -\underline{\Lambda}_+ \cdot \nabla \underline{\Lambda}_- - \nabla\left(p + \tfrac{1}{2}B^2\right) + \underline{F} + \tfrac{1}{2}(\nu-\eta)\nabla^2 \underline{\Lambda}_+ + \tfrac{1}{2}(\nu+\eta)\nabla^2 \underline{\Lambda}_+$$

$$\nabla \cdot \underline{\Lambda}_+ = 0, \quad \nabla \cdot \underline{\Lambda}_- = 0.$$

(note simplification if $\nu = \eta$; odd!)

$\underline{B} \cdot ① + \underline{u} \cdot ②$, $\int_V dV \Rightarrow \int_V \underline{u} \cdot \underline{B}\, dV$ conserved in ideal ($\nu = \eta = 0$) unforced ($\underline{F} = 0$) case (Woltjer 1958; HKM 1969)

$\Rightarrow$ ($\underline{F} \propto \nu$) cross-helicity evolves on diffusive timescale.

Archontis Dynamo

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(Archontis, 1999; Cameron + DJG 2006; Archontis, Dorch & Nordlund 2007)

2π-periodic cube, not scale separated

$\underline{F} = \nu(\sin z, \sin x, \sin y)$

[ABC without cosines]

motivation: $\underline{u} = (\sin z, \sin x, \sin y)$ gives (probably) fast kinematic dynamo.

N.B. without cosines, this $\underline{u}$ does not solve Navier-Stokes with $\underline{F} = \nu \equiv 0$.

Amazing fact: at high Rm & Re, adding a seed $\underline{B}$ gives steady dynamo solution with $\underline{u} = \underline{B} + O(\nu)$ [take $\nu = \eta$]

$= \frac{1}{2}(\sin z, \sin x, \sin y)$
+ small terms which don't tend to zero as $\nu \to 0$

[by symmetry, can also have $\underline{u} = -\underline{B} + O(\nu)$]

(Result discovered numerically; little analytic progress since)

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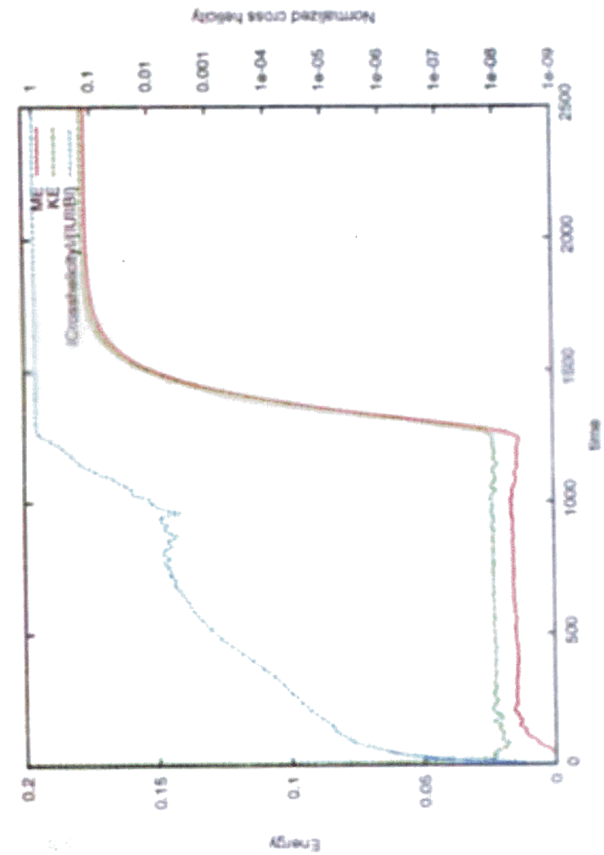

Figure 19. Evolution of kinetic and magnetic energies, as well as the cross helicity, as a function of time. The cross-helicity is shown using the logarithmic scale of the right-hand side of the graph whereas the energies use the linear scale on the left-hand side. The cross-helicity is normalized by $\sqrt{(KE)(ME)}/(8\pi^3)$; the simulation was started with a weak seed magnetic field.

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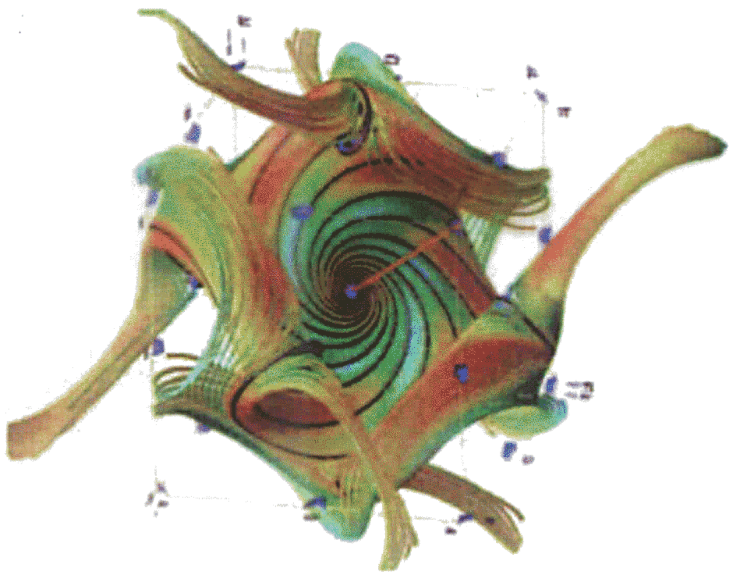

Figure 17. The structure of the 2D manifold, colour coded by $\Lambda_{-\,\mathrm{plane}}$. The relationship between the spiral arms and the heteroclinic orbits is apparent.

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Other examples found, eg for
$\frac{\nu}{\eta} = 4$, $\frac{\nu}{\eta} = \frac{1}{4}$; also modified
ABC force $\underline{F} = (\sin z + \epsilon \cos y, \sin x + \epsilon \cos z, \sin y + \epsilon \cos x)$
$0 \leq \epsilon \leq 1$

Lots of these seem to be stable (ie attracting)

"Most" Re "=" ∞ Euler flows are unstable – we're used to having to live with turbulence.

<u>All</u> $\underline{u} = \pm \underline{B}$ ideal unforced MHD solutions are <u>neutrally</u> stable (Friedlander and Vishik 1995)

$\Rightarrow$ as $\nu \to 0$ and $\eta \to 0$, maybe 50% of Alfvénic solutions are stable?

Perhaps the MHD universe is less turbulent than we think, if Elsasser states are attracting!