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Shell models for magnetostrophic turbulence?

KITP Discussion

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Shell models of turbulence

Gledzer–Ohkitani–Yamada (GOY) models provide a scalar analogue of the spectral Navier–Stokes equations.
E.g., the spectral domain is represented by N shells, of wavenumbers

$$k_n = k_0 2^n \,, \qquad n = 1, 2, \ldots, N \,.$$

The complex modes u_n satisfy

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -\nu k_n^2 u_n + f_n + i k_n \left(u_{n+2}^* u_{n+1}^* - \frac{1}{4} u_{n+1}^* u_{n-1}^* + \frac{1}{8} u_{n-1}^* u_{n-2}^* \right) \,.$$

Shell models of turbulence

Neglecting viscosity and forcing, this model conserves energy E and helicity H,

$$E = \frac{1}{2}\sum_{n=1}^{N} |u_n|^2 \ , \qquad H = \sum_{n=1}^{N} (-1)^n k_n |u_n|^2 \ .$$

exhibits the inertial range scaling of Kolmogorov,

$$E_n = \frac{1}{2}|u_n|^2 \sim k_n^{-5/3} \ ,$$

reproduces anomalous scalings of the structure functions S_p,

$$S_p(k_n) = \langle |u_n|^p \rangle \sim k_n^{-\zeta(p)} \ ,$$

and shows increasing intermittency with n, in the PDFs of u_n.

Use in heuristic dynamo models

Such shell models have been coupled to heuristic α-effect dynamo models, using mean-field type relations

$$\alpha \sim \frac{1}{3} \, \langle \tau \, \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle \sim \sum_{n=1}^{N} (-1)^n \, |u_n| \ ,$$

$$\beta \sim \frac{1}{3} \, \left\langle \tau \, |\mathbf{u}|^2 \right\rangle \sim \sum_{n=1}^{N} k_n^{-1} \, |u_n| \ .$$

E.g. Galactic disk dynamo (Sokoloff & Frick, 2003).

$$\frac{\mathrm{d}S}{\mathrm{d}t} = i k_L \alpha T - \beta S \ ,$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -i k_L \alpha S - \beta T \ .$$

Use in heuristic dynamo models

E.g. geomagnetic reversals (Ryan & Sarson, 2007).

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \alpha T - \beta S \ ,$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \omega S - \beta T \ ,$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \Gamma - \kappa\omega - \lambda_1 S T - \lambda_2 (S^2 + T^2) \ .$$

Interesting results perhaps due simply to effect of intermittent noise in α, rather than any inherent realism of the underlying turbulence (or low-order dynamo model).
But many extensions to more 'realistic' shell models exist...

MHD shell models

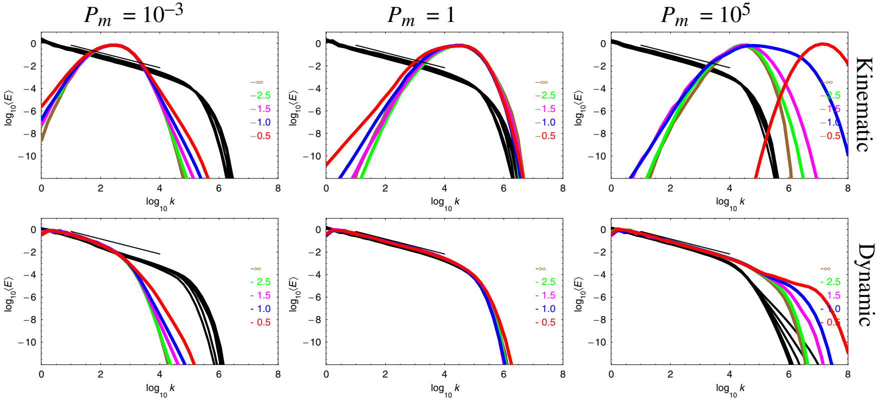
MHD shell models are relatively well studied. E.g. Frick & Sokoloff (1998):

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -\mathrm{Re}^{-1}k_n^2 u_n + f_n + ik_n \Big[\big(u^*_{n+2}u^*_{n+1} - b^*_{n+2}b^*_{n+1}\big) - \frac{1}{4}\big(u^*_{n+1}u^*_{n-1} - b^*_{n+1}b^*_{n-1}\big) + \frac{1}{8}\big(u^*_{n-1}u^*_{n-2} - b^*_{n-1}b^*_{n-2}\big)\Big] \, .$$

$$\frac{\mathrm{d}b_n}{\mathrm{d}t} = -\mathrm{Rm}^{-1}k_n^2 u_n + \frac{ik_n}{6}\Big[\big(u^*_{n+1}b^*_{n+2} - b^*_{n+1}u^*_{n+2}\big) + \big(u^*_{n-1}b^*_{n+1} - b^*_{n-1}u^*_{n+1}\big) + \big(u^*_{n-2}b^*_{n-1} - b^*_{n-2}u^*_{n-1}\big)\Big] \, .$$

MHD shell models

Plunian & Stepanov (2007) incorporate non-local shell interactions into a variant of this model.



Thermal convective turbulence

Similar models have incorporated thermal convection. E.g. Mingshun & Shida (1997):

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -\nu k_n^2 u_n - \alpha\theta_n$$

$$+ik_n \left(u_{n+2}^* u_{n+1}^* - \frac{1}{4} u_{n+1}^* u_{n-1}^* + \frac{1}{8} u_{n-1}^* u_{n-2}^* \right) \; .$$

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = -\kappa k_n^2 u_n + f_n$$

$$+\frac{ik_n}{2} \left(u_{n+1}^* \theta_{n+2}^* + u_{n+2}^* \theta_{n+1}^* + u_{n-1}^* \theta_{n+1}^* - u_{n-1}^* \theta_{n-2}^* \right) \; .$$

Relative scaling exponents consistent with She & Leveque (1994):

$$S_p(k_n) \sim k_n^{-\zeta(p)} \; , \qquad \zeta(p) = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right] \; .$$

Rotating turbulence

And other models have introduce rotation. E.g. Hattori, Rubinstein & Ishizawa (2004):

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -\nu k_n^2 u_n + f_n + i\Omega_n(t) u_n$$

$$+ik_n \left(u_{n+2} u_{n+1}^* + \frac{1}{2} u_{n+1} u_{n-1}^* - \frac{1}{2} u_{n-1} u_{n-2} \right) ,$$

$$\Omega_n = \Omega_c + \Omega_n' , \qquad \frac{\mathrm{d}\Omega_n'}{\mathrm{d}t} = -\frac{\Omega_n'}{T} + \frac{\tilde{\Omega}_n(t)}{T} ,$$

$$\frac{\mathrm{d}\tilde{\Omega}_n}{\mathrm{d}t} = -\frac{\tilde{\Omega}_n}{\tau} + \frac{g_n(t)}{\tau} , \qquad \langle g_n(t) g_n(s) \rangle = \sigma_n \delta(t-s) .$$

For large Ω_c, the energy spectrum changes to $E_n \sim k_n^{-2}$.
A similar approach has also been used for coherent large-scale fields $\mathbf{B}_0$ in MHD turbulence (Hattori & Ishizawa, 2001).

Magnetostrophic turbulence?

- Can similar models be developed for magnetostrophic turbulence? (Or at least, moving in that direction, for magnetoconvective or rotating MHD turbulence.)
  - Via simple shell models, as described here?
  - Via vector or hierarchical shell models?
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- Such models need be clearly constrained.
  - Scalings expected from theory? (E.g. Moffatt, 2006.)
  - Scalings observed in DNS simulations?
  - Scalings observed in laboratory dynamos?
  - Scalings observed in geomagnetic data?