KITP 23/05/08

Rattleback Reversals

A Prototype of Chiral Dynamics

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I lectured here at KITP on Euler's Disk and its Finite Time Singularity in 2000

Euler's Disk led to the Rising Egg Problem – a table-top example of dissipative (slow, secular) instability
(with Y.Shimomura and Michal Branicki)

The Rising Egg led to the Rattleback – chiral dynamics, and a table-top 'model' of geomagnetic reversals

Toys provide surprising phenomena, dynamical insights
Once these are understood, …, and fun!

Wolfgang Pauli, Neils Bohr, and the tippe-top, ~ 1955
A distinguished precedent for the investigation of spinning objects!

Rattleback: otherwise known as the ***celt*** or the ***wobblestone***

A rigid body that exhibits the property of *spin asymmetry*: it can spin quite smoothly on a table in one sense, but when spun in the opposite sense, a *pitching instability* develops which extracts energy from the spin to such an extent that this spin actually reverses in sign.

Walker, G.T. (1896) On a dynamical top. *Q.J.Pure Appl.Math.* **28**, 175-184.

Walker used the word *celt* to describe such an object

Mathematics of the Rattleback

Let $N(t)$ = spin

$A(t)$ = amplitude of pitching instability

$B(t)$ = amplitude of rolling instability

Slow-time evolution equations:

$$dA/dt = (4N \; 0 \; \mathrm{p}_1)\, A \, ,$$

$$dB/dt = (0 \, N \; 0 \; \mathrm{p}_2)\, B \, ,$$

$$dN/dt = \; 0 \, 4A^2 + B^2 \; 0 \, \mathrm{p}_3 \, N \, ,$$

($\mathrm{p}_{1,\,2,\,3}$ are friction parameters)

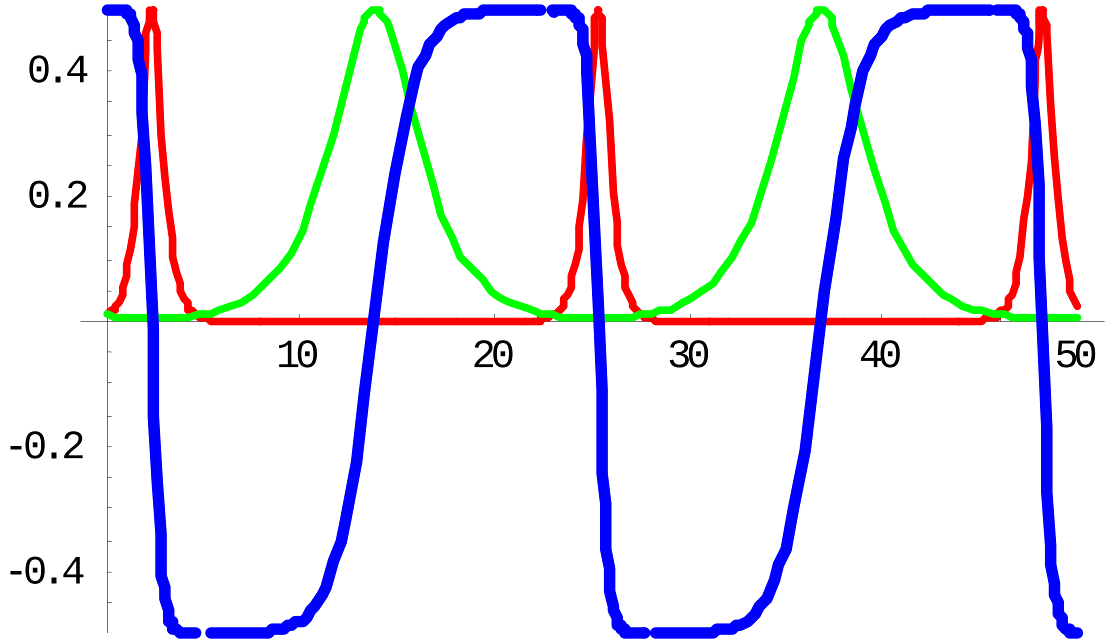


Figure 1. Solution of equations (7a,b, 8) with ○ = 4 and initial conditions *A*(0) = *B*(0) = 0.01, *N*(0) = 0.5. Blue shows *N*(w), red *A*(w) (pitching), and green *B*(w) (rolling). Rapid reversals of *N* (from positive to negative) are induced by the pitching instability, and slow reversals (from negative to positive) by the rolling instability.

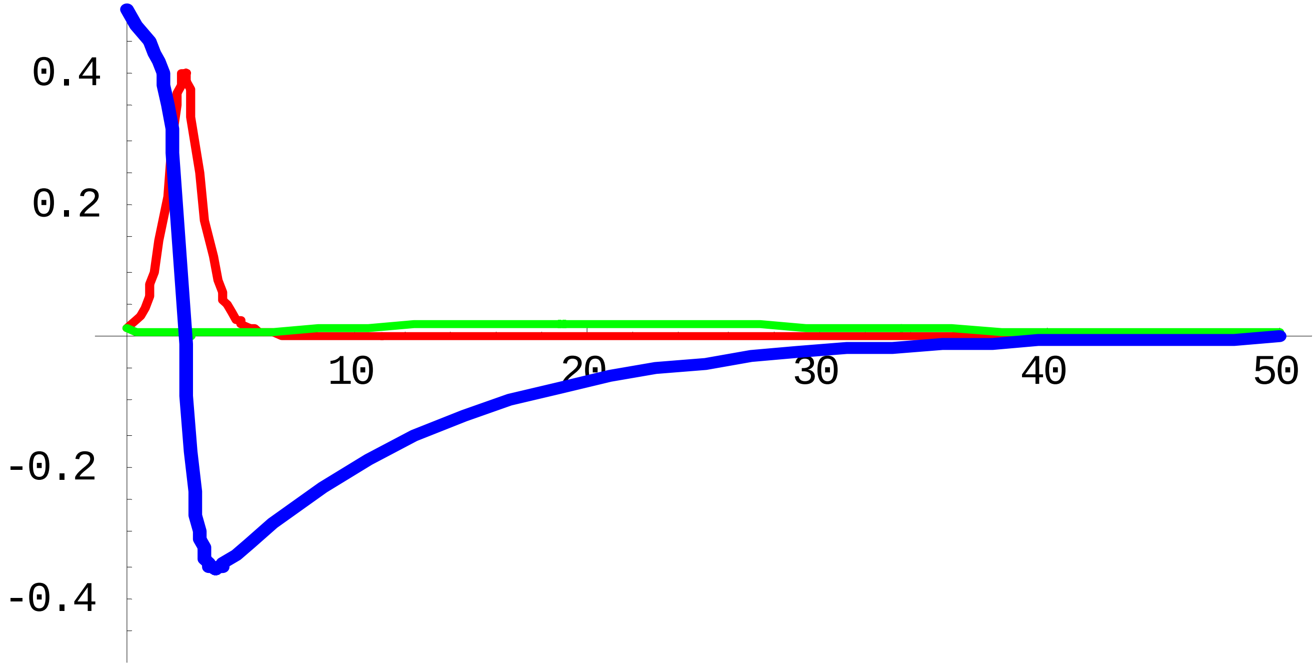


Figure 2. Solution of equations (9a,b, 10) with ○ = 4, dissipation parameters p_1 = 0.04, p_2 = 0.08, p_3 = 0.1; initial conditions and colour code as in figure 1. The pitching instability still induces a spin reversal, but the subsequent rolling instability is not sufficiently strong to induce a second reversal against the effects of dissipation.

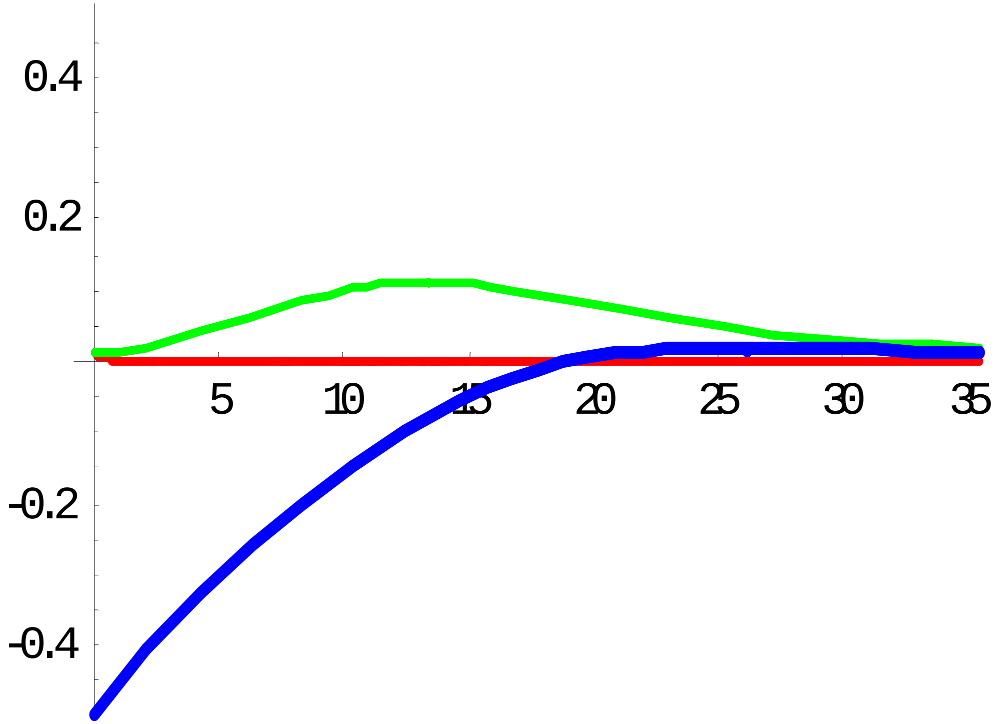
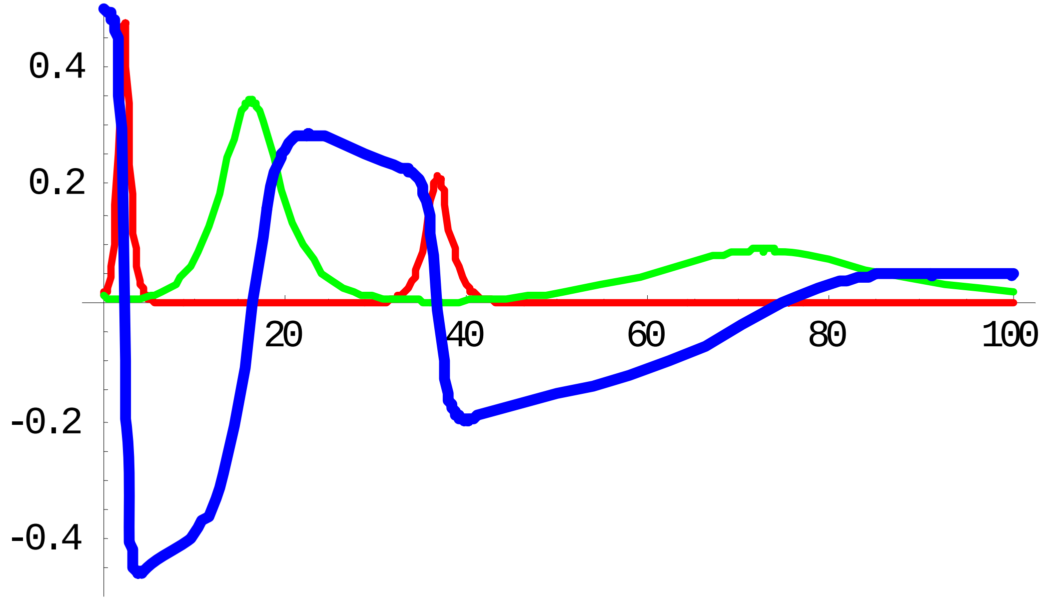


Figure 3. Same as figure 2, except that *N*(0) = 0 0.5. Here, a single weak reversal is induced at a late stage (w¬19) by the rolling instability, and the pitching instability is not excited.



An example with reduced friction, showing four reversals,
as observed with a carefully constructed Cambridge rattleback

($p_1 = 0.01$, $p_2 = 0.02$, $p_3 = 0.025$)

On the word *celt*

According to the ***Oxford English Dictionary*** :

“Though always confused with ‘Celt’ (pronounced kelt) as in Celtic peoples, it is a separate word, pronounced selt: an implement with chisel-shaped edge, of bronze or stone,…, found among the remains of prehistoric man.”

Stylo ferreo, et plumbi lamina, vel celte sculpantur in silice.

Job xix 24, Vulgate

Oh that my words were now written! Oh that they were printed in a book!
That they were graven with an iron pen and lead, in the rock for ever.

Job xix 23,24 Authorized Version

(An appropriate sentiment for those submitting papers to scientific Journals!)

1

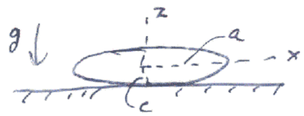
Pitching and Rolling

Consider a uniform solid ellipsoid, mass M

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a > b > c$$

Table $z = -c$

$\underset{\sim}{g} = (0, 0, -g)$



Use $M, c, \sqrt{\frac{c}{g}}$ as units of mass, length, time

[equivalently, $M = c = g = 1$]

Then P is $(0, 0, -1)$

Principal moments of inertia at P are

$$\alpha = \frac{b^2+6}{5}, \quad \beta = \frac{a^2+6}{5}, \quad \gamma = \frac{a^2+b^2}{5}$$

Near 1, contact surface is locally

$$z = -1 + \frac{x^2}{2a^2} + \frac{y^2}{2b^2}$$

If disturbed from rest, two oscillatory modes:

$\ddot{x} + \omega_1^2 x = 0 \qquad \omega_1^2 = \frac{5(a^2-1)}{a^2+6}$ Pitching

$\ddot{y} + \omega_2^2 y = 0 \qquad \omega_2^2 = \frac{5(b^2-1)}{b^2+6}$ Rolling

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Chiral distortion

Now imagine mass redistributed inside ellipsoid so that principal axes of inertia at P are rotated through a small angle about vertical relative to principal axes of curvature at P.

Let $Oxyz$ be princ. axes of inertia.

Then near P, surface is locally

$$z = -1 + \frac{x^2}{2a^2} + \chi\frac{xy}{a^2} + \frac{y^2}{2b^2}$$

χ = chirality parameter $\quad |\chi| \ll 1$

[Gk: χείρ = hand]

Chiral instability

The body can spin with angular velocity $(0, 0, n)$

Effect of chirality is to destabilize pitching or rolling, or both [H. Bondi, 1986, Proc. Roy. Soc. A405]

We suppose

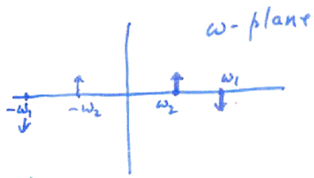
- $|n| \ll 1$ (if $n > O(1)$, other instabilities appear)
- $|\chi| \ll 1$ (small deformation)
- $a^2 \gg b^2 > 1$, $b-1 = O(1)$ (long rattleback)

Then Bondi's equations for the oscillatory modes can be arranged, at leading order in $a^2 \gg 1$ in the form

$$\ddot{x} + \omega_1^2 x + \underline{\chi \ddot{y}} = (\text{garbage})_1,$$

$$\ddot{y} + \omega_2^2 y + \underline{N \dot{x}} = (\text{garbage})_2$$

where $N = 5\left(1 + \frac{6}{b^2}\right) n$



Garbage terms (linear in x & y, and small) move ω_1 and ω_2 slightly along real axis

Only the terms $\chi \ddot{y}$ and $N\dot{x}$ are (jointly) responsible for instability.

So in fact, it is sufficient to consider the simplified system (dispose of the garbage!)

$$\ddot{x} + \omega_1^2 x + \underline{\chi \ddot{y}} = 0$$

$$\ddot{y} + \omega_2^2 y + \underline{N \dot{x}} = 0$$

So we retain:

- a small change in pitching mode due to chirality
- a small change in rolling mode due to Coriolis

[Cf: equations for $\alpha\omega$-dynamo

$$\frac{\partial A}{\partial t} = \underline{\alpha B} + \eta \nabla^2 A \qquad \text{chirality/helicity/}\alpha$$

$$\frac{\partial B}{\partial t} = \underline{(\nabla \wedge A \mathbf{e}_y) \cdot \nabla U} + \eta \nabla^2 B \qquad \text{diff}^{!} \text{ rotation}$$
]

Let $x(t), y(t) \sim e^{i\omega t}$
$$\begin{vmatrix} -\omega^2 + \omega_1^2 & -\chi\omega^2 \\ i\omega N & -\omega^2 + \omega_2^2 \end{vmatrix} = 0$$

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) + i N \chi \omega^3 = 0$$

for $|N\chi| \ll 1$, perturbed roots are

$$\omega_1 - i\sigma_1, \quad \omega_2 - i\sigma_2$$

where $\sigma_1 = \dfrac{N\chi\omega_1^2}{2(\omega_1^2 - \omega_2^2)}$, $\sigma_2 = \dfrac{-N\chi\omega_2^2}{2(\omega_1^2 - \omega_2^2)}$

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Growth rates

$$\sigma_1 = \frac{N\chi\omega_1^2}{2(\omega_1^2-\omega_2^2)}, \quad \sigma_2 = \frac{-N\chi\omega_2^2}{2(\omega_1^2-\omega_2^2)}$$

Corresponding eigenvectors are

$$(x_1, y_1) = \left(1, \frac{-iN}{\omega_1^2-\omega_2^2}\right) \quad \text{Pitching}$$

$$(x_2, y_2) = \left(\frac{\chi\omega_2^2}{\omega_1^2-\omega_2^2}, 1\right) \quad \text{Rolling}$$

- If $N\chi > 0$, then pitching $\sim e^{(i\omega_1+\sigma_1)t}$ is unstable
  rolling $\sim e^{(i\omega_2+\sigma_2)t}$ is stable
- If $N\chi < 0$, then pitching is stable
  rolling is unstable

This agrees with observed behaviour of long thin ~~celts~~ rattlebacks

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Two-timing

$$|\sigma_{1,2}| \sim |N\chi| \ll 1 \quad (|\omega_{1,2}| = O(1))$$

N is constant only in linearised analysis.

In fact, when say pitching is unstable, it extracts energy from the spin, so $|N|$ must decrease on the slow time-scale associated with σ_1. Let

$$\tau = \frac{|\chi|}{2(\lambda-1)}t \qquad \lambda = \left(\frac{\omega_1}{\omega_2}\right)^2$$

slow time variable

The amplitudes $A_0(t), B_0(t)$ of pitching and rolling evidently satisfy

$$\frac{dA_0}{dt} = \sigma_1 A_0, \quad \frac{dB_0}{dt} = \sigma_2 B_0$$

Take $\chi > 0$; then these become

$$\frac{dA_0}{d\tau} = \lambda N A_0, \quad \frac{dB_0}{d\tau} = -N B_0$$

where now $N(\tau)$ varies on the slow time-scale

Hence $A_0 B_0^\lambda = cst.$!

If we neglect dissipative effects (i.e. assume no-slip condition, neglect air friction etc) then energy (kinetic + potential) is also conserved

$$E_{pitching} = \tfrac{1}{2}\beta(\dot{x}^2 + \omega_1^2 x^2) = \tfrac{1}{2}\beta\omega_1^2 A_0^2$$

$$E_{rolling} = \tfrac{1}{2}\alpha(\dot{y}^2 + \omega_2^2 y^2) = \tfrac{1}{2}\alpha\omega_2^2 B_0^2$$

$$E_{spin} = \tfrac{1}{2}\gamma n^2 = \tfrac{1}{2}\gamma N^2 / 25(1+6/\delta^2)$$

Rescale A_0 and B_0 :

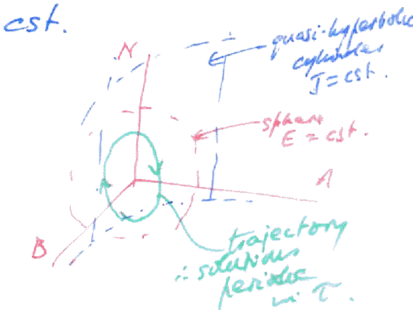
$$A = 5\left(1+\frac{6}{\delta^2}\right)\sqrt{\frac{\beta}{\gamma}}\, A_0 \quad , \quad B = 5\left(1+\frac{6}{\delta^2}\right)\sqrt{\frac{\alpha}{\gamma}}\, B_0$$

Then $E = E_p + E_r + E_s = cst.$ becomes

$$A^2 + B^2 + N^2 = cst.$$

So we have

- $\dfrac{dA}{d\tau} = \lambda N A$
- $\dfrac{dB}{d\tau} = -NB$
- $\dfrac{dN}{d\tau} = -\lambda A^2 + B^2$



with integrals $A^2 + B^2 + N^2 = E \quad cst.$

$AB^\lambda = J \quad cst.$