NAVIER-STOKES-ALPHA EQUATION

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Introduction

A pervasive difficulty in dynamo simulations is that of dealing with the small scales beneath the attainable numerical resolution. Simulators have resorted to a number of ways of treating these *sub-grid scales* e.g., by enhancing the diffusivities with increasing wave number or by introducing hyperdiffusion, both of which suppress the small scale motions. Another method that has attracted attention is the replacement of the governing equations by a so-called "regularized" set that involves additional smoothed variables. Typical of these is the Navier–Stokes–α method, which involves not only the velocity $\mathbf{v}$ but also $\overline{\mathbf{v}}$. These velocities are related by

$$\mathbf{v} = (1 - \alpha^2 \nabla^2)\overline{\mathbf{v}}$$

where α is a constant length of the order of the attainable numerical resolution. The equations have attractive properties. For example Kelvin's theorem holds for the circulation of $\mathbf{v}$ round a circuit carried with velocity $\overline{\mathbf{v}}$.

Numerical experiments with the Navier–Stokes–α equations have been somewhat encouraging and have even led to speculations that the method provides a realistic way of dealing with turbulent flows. Several people, notably Darryl Holm, have tried to put the equations on a firm mathematical foundation. We describe difficulties we have encountered in his analysis.

Hybrid Eulerian–Lagrangian Description
HEL

Consider a wave riding on a mean flow, and envisage a magnetic field aligned with the flow and "frozen" to it. The wave moves the line and it may be good to label points on the line rather than use the original coordinates. The *Lagrangian description* is a mapping:

$$\mathbf{x} \mapsto \mathbf{x}^{\ell}(\mathbf{x}, t)$$

Unfortunately the mean flow generally moves $\mathbf{x}^{\ell}$ further and further away from its initial position $\mathbf{x}$. Since we are mainly interested in the location of the field line and not the actual position of the material particles on it, we may advantageously use another mapping:

$$\mathbf{x} \mapsto \mathbf{x}^{L}(\mathbf{x}, t)$$

such that $\mathbf{x}^{L}$ lies on the field line, not at $\mathbf{x}^{\ell}$ but at some other point conveniently close to $\mathbf{x}$. This is called the

HEL Description

We call $\mathbf{x}$ the **HEL Coordinate**

and $\mathbf{x}^{L}$ the **HEL Position**

Kinematics

The material derivative is defined by

$$\frac{\mathcal{D}}{\mathcal{D}t} = \partial_t + \mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\nabla}\,,$$

where $\mathbf{v}(\mathbf{x},t)$ is the velocity at $\mathbf{x}$ and time t. From a Lagrangian point of view

$$\mathbf{v}(\mathbf{x},t) = \frac{\mathcal{D}\mathbf{x}}{\mathcal{D}t} = \mathbf{0} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{x} = \mathbf{v}\,.$$

We define a *HEL operator* ^{L} by

$$\psi^L(\mathbf{x},t) \equiv \psi(\mathbf{x}^L(\mathbf{x},t),t)\,,$$

i.e., ψ^L is a function of the HEL coordinate $\mathbf{x}$ evaluated at the HEL position $\mathbf{x}^L(\mathbf{x},t)$.

From $\mathbf{v}^L$ we define a *'reference' velocity* $\mathbf{u}(\mathbf{x},t)$ and a *reference material derivative* by

$$\mathbf{v}^L(\mathbf{x},t) = \frac{D\mathbf{x}^L}{Dt}\,,$$

where

$$\frac{D}{Dt} = \partial_t + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}\,.$$

Examples of L operations

Gradient:

$$(\nabla\psi^L)_i = \frac{\partial\psi^L}{\partial x_i} = \frac{\partial x_j^L}{\partial x_i}\left(\frac{\partial\psi}{\partial x_j}\right)^L ,$$

$$\text{i.e.,}\quad \nabla\psi^L = (\nabla\mathbf{x}^L)\cdot(\nabla\psi)^L .$$

Time derivative:

$$\partial_t\psi^L = \partial_t\psi(\mathbf{x}^L,t) + \partial_t x_j^L \frac{\partial\psi(\mathbf{x}^L,t)}{\partial x_j^L} ,$$

$$\text{i.e.,}\quad \partial_t\psi^L = (\partial_t\psi)^L + \partial_t\mathbf{x}^L\cdot(\nabla\psi)^L .$$

Material derivative:

$$\begin{aligned}\frac{D\psi^L}{Dt} &= \partial_t\psi^L + (\mathbf{u}\cdot\nabla)\psi^L \\ &= (\partial_t\psi)^L + \partial_t\mathbf{x}^L\cdot(\nabla\psi)^L + \mathbf{u}\cdot(\nabla\mathbf{x}^L)\cdot(\nabla\psi)^L \\ &= (\partial_t\psi)^L + \mathbf{v}^L\cdot(\nabla\psi)^L = \left(\frac{\mathcal{D}\psi}{\mathcal{D}t}\right)^L .\end{aligned}$$

The result

$$\frac{D\psi^L}{Dt} = \left(\frac{\mathcal{D}\psi}{\mathcal{D}t}\right)^L$$

means that, if ψ is "frozen" to the flow $\mathbf{v}^L$ ($\mathcal{D}\psi/\mathcal{D}t = 0$), then $D\psi^L/Dt = 0$, which means that $\psi^L(\mathbf{x},t)$ ($= \psi(\mathbf{x}^L(\mathbf{x},t))$ is "frozen" to the reference flow $\mathbf{u}(\mathbf{x},t)$.

Momentum + pseudo-momentum = $\rho\mathbf{V}$

Whereas

$$\frac{D\mathbf{x}^L}{Dt} - \mathbf{v}^L = \mathbf{0}\,,$$

the gradient and contraction with $\mathbf{v}^L$ yields

$$\left(\frac{D}{Dt}(\nabla\mathbf{x}^L) - \nabla\mathbf{v}^L\right)\cdot\mathbf{v}^L = -(\nabla\mathbf{u})\cdot\mathbf{V}\,, \qquad \spadesuit$$

in which we have introduced

$$\mathbf{V} = (\nabla\mathbf{x}^L)\cdot\mathbf{v}^L\,.$$

In the language of tensor calculus

$$V_i = \frac{\partial x_j^L}{\partial x_i} v_j^L \quad \text{is a covariant vector}\,,$$
$$u_i = \frac{\partial x_i}{\partial x_j^L}(v_j^L - \partial_t x_j^L) \quad \text{is a contravariant vector}\,.$$

It is important to appreciate that both $\mathbf{u}$ and $\mathbf{V}$ refer to properties of the flow at the HEL position $\mathbf{x}^L$ and **not** at the HEL coordinate $\mathbf{x}$. ($\mathbf{x}^L = \mathbf{x}^L(\mathbf{x},t)$)

Euler's equation for a fluid of constant density ρ

Euler's equation may be written as

$$\boldsymbol{\mathcal{E}} \equiv \frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} + \boldsymbol{\nabla}\left(\frac{p}{\rho}\right) = \mathbf{0}, \qquad (\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} = 0)\,.$$

Since a gradient has a natural covariant transformation property

$$\boldsymbol{\nabla}\psi^L = (\boldsymbol{\nabla}\mathbf{x}^L)\boldsymbol{\cdot}(\boldsymbol{\nabla}\psi)^L\,,$$

the presence of the pressure gradient $\boldsymbol{\nabla}(p/\rho)$ strongly suggest that we should consider

$$\mathbf{E} = (\boldsymbol{\nabla}\mathbf{x}^L)\boldsymbol{\cdot}\boldsymbol{\mathcal{E}}^L\,.$$

It is convenient to write $\boldsymbol{\mathcal{E}}$ as

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{A}} + \boldsymbol{\nabla}\Pi\,,$$

where

$$\boldsymbol{\mathcal{A}} = \frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} + (\boldsymbol{\nabla}\mathbf{v})\boldsymbol{\cdot}\mathbf{v}, \qquad \Pi = \frac{p}{\rho} - \tfrac{1}{2}|\mathbf{v}|^2\,.$$

Then $\mathbf{E} = \mathbf{A} + \boldsymbol{\nabla}\Pi^L$, where

$$\mathbf{A} = (\boldsymbol{\nabla}\mathbf{x}^L)\boldsymbol{\cdot}\boldsymbol{\mathcal{A}}^L\,, \qquad \boldsymbol{\nabla}\Pi^L = (\boldsymbol{\nabla}\mathbf{x}^L)\boldsymbol{\cdot}(\boldsymbol{\nabla}\Pi)^L\,.$$

The value of $\mathbf{A}$ is

$$\begin{aligned}\mathbf{A} &= (\nabla \mathbf{x}^L)\cdot\left[\left(\frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t}\right)^L + (\nabla \mathbf{v}^L)\cdot\mathbf{v}^L\right] \\ &= (\nabla \mathbf{x}^L)\cdot\frac{D\mathbf{v}^L}{Dt} + (\nabla \mathbf{v}^L)\cdot\mathbf{v}^L \\ &= \frac{D\mathbf{V}}{Dt} + (\nabla \mathbf{u})\cdot\mathbf{V} \qquad \text{(by } \spadesuit\text{)} \\ &= \partial_t \mathbf{V} + \{\mathbf{u}, \mathbf{V}\}\,,\end{aligned}$$

where $\{\mathbf{u}, \mathbf{V}\} = \mathbf{u}\cdot\nabla\mathbf{V} + (\nabla \mathbf{u})\cdot\mathbf{V}$.

Thus the covariant transformation of

$$\boldsymbol{\mathcal{E}} \equiv \partial_t \mathbf{v} + \{\mathbf{v}, \mathbf{v}\} + \nabla\Pi$$

is

$$\mathbf{E} \equiv \partial_t \mathbf{V} + \{\mathbf{u}, \mathbf{V}\} + \nabla\Pi^L\,.$$

So

$$\boldsymbol{\mathcal{E}} = \mathbf{0} \qquad \Longleftrightarrow \qquad \mathbf{E} = \mathbf{0}\,.$$

Circulation

Note that

$$(d\mathbf{x}\cdot\mathbf{v})^L = (d\mathbf{x}^L)\cdot\mathbf{v}^L = (d\mathbf{x}\cdot\nabla\mathbf{x}^L)\cdot\mathbf{v}^L = d\mathbf{x}\cdot\mathbf{V}\,.$$

It is easily established that

$$\begin{aligned}\frac{\mathcal{D}}{\mathcal{D}t}(\mathbf{v}\cdot d\mathbf{x}) &= \boldsymbol{\mathcal{A}}\cdot d\mathbf{x} = -d\Pi\,,\\ \frac{D}{Dt}(\mathbf{V}\cdot d\mathbf{x}) &= \mathbf{A}\cdot d\mathbf{x} = -d\Pi^L\,.\end{aligned}$$

Kelvin's circulation theorem

$$\frac{d}{dt}\oint_{C^L}\mathbf{v}^{\mathbf{L}}\cdot d\mathbf{x}^{\mathbf{L}} = -\oint_{C^L} d\Pi^L = 0$$

becomes

$$\frac{d}{dt}\oint_{C}\mathbf{V}\cdot d\mathbf{x} = -\oint_{C} d\Pi^L = 0\,,$$

where

$$C \;\mapsto\; \mathcal{C}^L \quad \text{by} \quad \mathbf{x} \;\mapsto\; \mathbf{x}^L\,.$$

Averages

Re-express the mapping as $\mathbf{x} \;\mapsto\; \mathbf{x}^L(\mathbf{x},t) = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x},t)$, so that

$$\mathbf{v}^L(\mathbf{x},t) \equiv \frac{D\mathbf{x}^L}{Dt} = \mathbf{u} + \frac{D\boldsymbol{\xi}}{Dt}\,.$$

We assume that, for averages,

$$\overline{\mathbf{u}} = \mathbf{u}, \qquad \text{and} \qquad \overline{\boldsymbol{\xi}} = \mathbf{0}\,.$$

Then $\mathbf{v}^L = \overline{\mathbf{u}} + \partial_t\boldsymbol{\xi} + \overline{\mathbf{u}}\cdot\nabla\boldsymbol{\xi}$ and so $\overline{\mathbf{v}^L} = \overline{\mathbf{u}} = \mathbf{u}$, i.e.,

$\mathbf{u}$ is the average of $\mathbf{v}^L$.

Accordingly, the average of Euler's equation is

$$\overline{\mathbf{E}} = \partial_t\overline{\mathbf{V}} + \{\mathbf{u}, \overline{\mathbf{V}}\} + \nabla\overline{\Pi^L} = \mathbf{0}\,,$$

and Kelvin's theorem holds for $\overline{\mathbf{V}}$:

$$\frac{d}{dt}\int_C \overline{\mathbf{V}}\cdot d\mathbf{x} = 0\,.$$

Eulerian description for $\xi \equiv |\boldsymbol{\xi}| \ll 1$

The essential problem is that $\mathbf{u}$ and $\mathbf{v}$ define properties at $\mathbf{x}^L = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ but we would really like to know them at $\mathbf{x}$, i.e., we would like to use a Eulerian description. To achieve that goal, we use Taylor series:

$$\begin{aligned}\psi^L &= \psi(\mathbf{x} + \boldsymbol{\xi}, t) \\ &= \psi + (\boldsymbol{\xi} \cdot \nabla)\psi + \tfrac{1}{2}\xi_j \xi_k \nabla_j \nabla_k \psi \ldots .\end{aligned}$$

This may be inverted as

$$\psi(\mathbf{x}, t) = \psi^L - (\boldsymbol{\xi} \cdot \nabla)\psi^L + \mathrm{O}(\xi^2) .$$

Applying this to $\mathbf{v}^L(\mathbf{x}, t)$, we obtain

$$\begin{aligned}\mathbf{v}(\mathbf{x}, t) &= \left(\mathbf{u} + \frac{D\boldsymbol{\xi}}{Dt}\right) - (\boldsymbol{\xi} \cdot \nabla)\mathbf{u} + \mathrm{O}(\xi^2) \\ &= \mathbf{u} + (\partial_t \boldsymbol{\xi} + [\mathbf{u}, \boldsymbol{\xi}]) + \mathrm{O}(\xi^2) ,\end{aligned}$$

where $\quad [\mathbf{u}, \boldsymbol{\xi}] = (\mathbf{u} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{u} .$

So on splitting $\mathbf{v}$ into its mean and fluctuating parts,

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}' ,$$

we have

$$\overline{\mathbf{v}} = \mathbf{u} + \mathrm{O}(\xi^2), \qquad \mathbf{v}' = \partial_t \boldsymbol{\xi} + [\mathbf{u}, \boldsymbol{\xi}] + \mathrm{O}(\xi^2) .$$

Our variant of the Holm approach

We write $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$ where $\mathbf{v}'$ is defined by

$$\mathbf{v}' = \partial_t \boldsymbol{\zeta} + [\overline{\mathbf{v}}, \boldsymbol{\zeta}]. \qquad \clubsuit$$

For the time being this decomposition is arbitrary; at this stage $\overline{\mathbf{v}}$ is not necessarily the average of $\mathbf{v}$. We suppose however that

$$\overline{\mathbf{v}} = \mathbf{u} + \mathrm{O}(\xi^2), \qquad \boldsymbol{\zeta} = \boldsymbol{\xi} + \mathrm{O}(\xi^2)\,,$$

so that the representation agrees with HEL at $\mathrm{O}(\xi)$ but differs at $\mathrm{O}(\xi^2)$.

Variational methods are based on

$$\iint \left\{ \tfrac{1}{2}\rho|\mathbf{v}|^2 - p\left(\frac{\rho}{\rho_0} - 1\right)\right\} d\mathbf{x}\, dt\,.$$

Variations are normally made of two types:

$$\delta\mathbf{v} = \begin{cases} \delta\overline{\mathbf{v}} & \text{with } \mathbf{v}' \text{ fixed,} \\ \delta\mathbf{v}' & \text{with } \overline{\mathbf{v}} \text{ fixed.} \end{cases}$$

Holm uses $\delta\mathbf{v} = \delta\overline{\mathbf{v}}$ but holds $\boldsymbol{\zeta}$ fixed instead of $\mathbf{v}'$. By $\clubsuit$, this causes $\mathbf{v}'$ to vary simultaneously with $\delta\overline{\mathbf{v}}$ ($\delta\mathbf{v}' \neq \mathbf{0}$), i.e.,

$$\delta\mathbf{v} \;\neq\; \delta\overline{\mathbf{v}} \;\text{ when } \boldsymbol{\zeta} \text{ is fixed}\,.$$

Euler's equation

By analogy with $\mathbf{E}$ and $\mathbf{V}$, we write

with
$$\mathbf{E}^E = \boldsymbol{\mathcal{E}} + \{\boldsymbol{\zeta}, \boldsymbol{\mathcal{E}}\} \qquad \mathbf{V}^E = \mathbf{v} + \{\boldsymbol{\zeta}, \mathbf{v}\},$$
$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{A}} + \nabla\Pi, \qquad \boldsymbol{\mathcal{A}} = \partial_t\mathbf{v} + \{\mathbf{v}, \mathbf{v}\},$$
$$\mathbf{E}^E = \mathbf{A}^E + \nabla\Pi^E,$$

where

$$\mathbf{A}^E = \boldsymbol{\mathcal{A}} + \{\boldsymbol{\zeta}, \boldsymbol{\mathcal{A}}\}, \qquad \nabla\Pi^E = \nabla\Pi + \{\boldsymbol{\zeta}, \nabla\Pi\},$$

which holds because $\Pi^E = \Pi + \boldsymbol{\zeta}\cdot\nabla\Pi$.
The value of $\mathbf{A}^E$ is

$$\begin{aligned}\mathbf{A}^E &= (\partial_t\mathbf{v} + \{\boldsymbol{\zeta}, \partial_t\mathbf{v}\}) + \{\mathbf{v}, \mathbf{v}\} + \{\boldsymbol{\zeta}, \{\mathbf{v}, \mathbf{v}\}\} \\ &= (\partial_t\mathbf{V}^E - \{\partial_t\boldsymbol{\zeta}, \mathbf{v}\}) + \{\mathbf{v}, \mathbf{v}\} + \{\boldsymbol{\zeta}, \{\mathbf{v}, \mathbf{v}\}\} \\ &= \partial_t\mathbf{V}^E + \{[\overline{\mathbf{v}}, \boldsymbol{\zeta}], \mathbf{v}\} + \{\overline{\mathbf{v}}, \mathbf{v}\} + \{\boldsymbol{\zeta}, \{\mathbf{v}, \overline{\mathbf{v}}\}\}.\end{aligned}$$

By making use of a cunning identity, this can be rewritten as
$$\mathbf{A}^E = \partial_t\mathbf{V}^E + \{\overline{\mathbf{v}}, \mathbf{V}^E\} + \{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\}.$$

So $\boldsymbol{\mathcal{E}} = \mathbf{0}$ (Euler's equation) implies $\mathbf{E}^E = \mathbf{0}$ giving

$$\mathbf{A}^E \equiv \partial_t\mathbf{V}^E + \{\overline{\mathbf{v}}, \mathbf{V}^E\} + \{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\} + \nabla\Pi^E = \mathbf{0}.$$

Averages again

We now interpret the overbar as an average, and assume that

$$\overline{\boldsymbol{\zeta}} = \mathbf{0}, \quad \text{implying} \quad \mathbf{v}' = \mathbf{0}\,.$$

Then the average of $\mathbf{A}^E$ is

$$\partial_t \overline{\mathbf{V}^E} + \{\overline{\mathbf{v}}, \overline{\mathbf{V}^E}\} + \overline{\{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\}} + \nabla \overline{\Pi^E} = \mathbf{0}\,. \qquad (\bullet)$$

For material circuits C^E moving with velocity $\overline{\mathbf{v}}$ we therefore have

$$\frac{d}{dt} \oint_{C^E} \overline{\mathbf{V}^E} \cdot d\mathbf{x} = - \oint_{C^E} \overline{\{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\}} \cdot d\mathbf{x} \neq 0\,.$$

Kelvin's theorem does not hold for circuits moving with the mean velocity $\overline{\mathbf{v}}$.

Holm obtains an equation almost identical to $(\bullet)$ but lacking the term $\overline{\{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\}}$. He therefore obtains a Kelvin theorem.

The "momentum" $\rho\overline{\mathbf{V}^E}$

The averaged total momentum is $\rho\overline{\mathbf{V}^E}$:

$$\overline{\mathbf{V}^E} = \overline{\mathbf{v}} + \overline{\mathbf{v}^s} + \overline{\mathbf{v}^p}\,,$$

where $\quad \overline{\mathbf{v}^s} = \overline{(\boldsymbol{\zeta}\cdot\boldsymbol{\nabla})\mathbf{v}'}, \qquad \overline{\mathbf{v}^p} = \overline{(\boldsymbol{\nabla}\boldsymbol{\zeta})\cdot\mathbf{v}'}\,.$

The pseudo- or wave momentum is $\rho(\overline{\mathbf{V}^E} - \overline{\mathbf{v}})$:

$$\overline{\mathbf{V}^E} - \overline{\mathbf{v}} = \overline{\mathbf{v}^s} + \overline{\mathbf{v}^p}.$$

Taylor hypotheses

To close the equations makes a number of proposals that he calls "Taylor hypotheses". The principal one, on which he mainly focuses, is

$$\partial_t\boldsymbol{\zeta} + \overline{\mathbf{v}}\cdot\boldsymbol{\nabla}\boldsymbol{\zeta} = \mathbf{0} \quad \text{so that} \quad \mathbf{v}' = -\boldsymbol{\zeta}\cdot\boldsymbol{\nabla}\overline{\mathbf{v}}\,.$$

This implies that

$$\partial_t(\overline{\zeta_i\zeta_j}) + \overline{\mathbf{v}}\cdot\boldsymbol{\nabla}(\overline{\zeta_i\zeta_j}) = 0$$

with solution $\overline{\zeta_i\zeta_j} = \alpha^2\delta_{ij}$ where α is a scalar length advected with the mean flow $\overline{\mathbf{v}}$:

$$\partial_t\alpha + \overline{\mathbf{v}}\cdot\boldsymbol{\nabla}\alpha = 0$$

In consequence

$$\overline{\mathbf{v}^s} = -\overline{(\boldsymbol{\zeta}\cdot\boldsymbol{\nabla})^2}\overline{\mathbf{v}} = -\nabla_j(\overline{\zeta_j\zeta_k}\nabla_k\overline{\mathbf{v}}) + \overline{(\boldsymbol{\nabla}\cdot\boldsymbol{\zeta})\boldsymbol{\zeta}}\cdot\boldsymbol{\nabla}\overline{\mathbf{v}}\,.$$

Since $\boldsymbol{\nabla}\cdot\boldsymbol{\zeta} = 0$, this becomes $\overline{\mathbf{v}^s} = -\boldsymbol{\nabla}\cdot(\alpha^2\boldsymbol{\nabla}\overline{\mathbf{v}})$.

For constant α, this gives

$$\overline{\mathbf{V}^E} = \overline{\mathbf{v}} - \alpha^2\nabla^2\overline{\mathbf{v}} + \overline{\mathbf{v}^p}$$

This is the origin of the Holm α–term and the Navier–Stokes–α equations. In fact our analysis has isolated an additional part of $\overline{\mathbf{V}^E}$, namely

$$\overline{\mathbf{v}^p} = -(\nabla\overline{\boldsymbol{\zeta})\cdot(\boldsymbol{\zeta}}\cdot\nabla)\overline{\mathbf{v}}$$

or in components

$$\overline{v_i^p} = \overline{(\nabla_i\zeta_j)\zeta_k}\nabla_k\overline{v_j}\,.$$

For this there is no obvious simplification.

Holm actually derives the Navier–Stokes–α equations,

$$\partial_t\overline{\mathbf{V}^E}+\{\overline{\mathbf{v}},\overline{\mathbf{V}^E}\} + \nabla\overline{\Pi^E} = \mathbf{0}\,,$$
$$\overline{\mathbf{V}^E} = \overline{\mathbf{v}} - \alpha^2\nabla^2\overline{\mathbf{v}}$$

from a variational principle based on an averaged Lagrangian. The first of these equation lacks the term $\overline{\{\boldsymbol{\zeta},\{\mathbf{v}',\mathbf{v}\}\}}$ and, because of this omission, possesses a Kelvin's theorem.

Conclusions

- The classical mean field version of Euler's equation is

$$\partial_t \overline{\mathbf{v}} + (\overline{\mathbf{v}}\cdot\boldsymbol{\nabla})\overline{\mathbf{v}} + \boldsymbol{\nabla}\cdot(\overline{\mathbf{v}\mathbf{v}}) = -\boldsymbol{\nabla}(\overline{p}/\rho)\,.$$

- Holm's mean field version of Euler's equation should be

$$\partial_t \overline{\mathbf{V}^E} + \{\overline{\mathbf{v}}, \overline{\mathbf{V}^E}\} + \overline{\{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\}} = -\boldsymbol{\nabla}\overline{\Pi^E}\,.$$

- Holm's neglect of the term $\overline{\{\boldsymbol{\zeta}, \{\mathbf{v}', \mathbf{v}\}\}}$ is no better or no worse than the neglect of $\boldsymbol{\nabla}\cdot(\overline{\mathbf{v}\mathbf{v}})$ in the classical version.

- Obviously $\overline{\mathbf{V}^E}$ possesses no nice mean circulation properties whatever.

- The only correct way to preserve the mean circulation is via the HEL version of Euler's equation:

$$\partial_t \overline{\mathbf{V}} + \{\mathbf{u}, \overline{\mathbf{V}}\} = -\boldsymbol{\nabla}\overline{\Pi^L}\,.$$